Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2016–08–26 f M

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20p–29p, 4: 30p–39p, 5: 40p–, G: 20p–34p, VG: 35p–

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1. Consider the Poisson equation $-\Delta u = f$ in $\mathbb{R}^3$.

(a) Show that the fundamental solution $U(x) = \frac{1}{4\pi|x|}$.
Sol. We remember that Laplace operator in spherical coordinates $-\Delta U = -r^{-2}(r^2 U_r')_r'$. We note for $r \neq 0$ that,

$$-\Delta U = -r^{-2}(r^2 U_r')_r' = -r^{-2}(-(4\pi)^{-1})_r' = 0.$$

For any $\phi \in C_0^\infty(\mathbb{R}^d)$ we have,

$$\begin{aligned}\int_{|x|>\epsilon} U(-\Delta\phi)\,dx &= \int_{|x|>\epsilon} (-\Delta U)\phi\,dx + \int_{|x|=\epsilon} (\phi\partial_n U - U\partial_n\phi)\,ds \\ &= + \int_{|x|=\epsilon} (\phi\partial_n U - U\partial_n\phi)\,ds.\end{aligned}$$

Note that $\partial_n U = -U_r'$. We get for the first term $\partial_n U|_{|x|=\epsilon} = -U_r'|_{r=\epsilon} = 4\pi\epsilon^{-2}$ and therefore,

$$\int_{|x|=\epsilon} \phi\partial_n U\,ds = \frac{1}{4\pi\epsilon^2}\int_{|x|=\epsilon} \phi\,ds \to \phi(0),$$

as $\epsilon \to 0$.
We also have

$$\left|\int_{|x|=\epsilon} \partial_n\phi U\,ds\right| = (4\pi\epsilon)^{-1}\left|\int_{|x|=\epsilon} \partial_n\phi\right| \le \epsilon\|\nabla\phi\|_{C(\mathbb{R}^d)} \to 0,$$

as $\epsilon \to 0$. We conclude,

$$\int_{|x|>\epsilon} U(-\Delta\phi)\,dx \to \phi(0),$$

as $\epsilon \to 0$.

(b) Show that $u(x) = (U * f)(x) = \int_{\mathbb{R}^3} U(x-y)f(y)\,dy$.
Sol. We have

$$\begin{aligned}(f,\phi) &= \int_{\mathbb{R}^d} \phi(y)f(y)\,dy \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U(x-y)\mathcal{A}\phi(x)\,dx f(y)\,dy \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U(x-y)f(y)\,dy\mathcal{A}\phi(x)\,dx \\ &= (u,\mathcal{A}\phi) \\ &= (\mathcal{A}u,\phi),\end{aligned}$$

for all $\phi \in C_0^\infty(\mathbb{R}^d)$. Integration by parts is possible since $D_iD_ju = D_iD_j(U*f) = (D_iU * D_jf)(x) \in C^2(\mathbb{R}^d)$, see proof of Theorem 3.4 in Larsson-Thomeé. We conclude $\mathcal{A}u = f$.

2. Consider the Neumann problem, find u such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \partial_n u = g, & \text{on } \Gamma, \end{cases}$$

where $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$.

(a) Under what additional assumption on f and g do we have existence of solution?
Sol. We derive the weak form, find $u \in H^1(\Omega)$ such that, $(\nabla u, \nabla v) = (f, v) + (g, v)_\Gamma$ for all $v \in H^1(\Omega)$. We let $v = 1 \in H^1(\Omega)$ and conclude $\int_\Omega f\,dx + \int_\Gamma g\,ds = 0$.

(b) Show that a solution u can not be unique.
Sol. Given a solution $u + C$ is also a solution for any constant C.

(c) What is the smallest eigenvalue of the corresponding eigenvalue problem, where $g = 0$ and f is replaced by λu?
Sol. We first note that the Rayleigh quotient $\frac{(\nabla v, \nabla v)}{(v,v)}$ is non-negative since the bilinear form is symmetric. It is minimized by letting $v = C \in \mathbb{R}$ and the minimum is 0 i.e. the smallest eigenvalue is zero.

3. Consider the following abstract elliptic problem in weak form: find $u \in H_0^1(\Omega)$ such that,

$$a(u, v) = l(v),$$

where a is a bilinear form, l is a linear functional, and Ω is a bounded domain in $\mathbb{R}^3$.

(a) Show that $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$. The trace theorem for functions in $H^1(\Omega)$ can be used without proof.
Sol. Let $\{v_i\}_{i=1}^\infty \in H_0^1(\Omega)$ be a sequence with limit $v \notin H_0^1(\Omega)$ i.e. $\|\gamma v\|_{L^2(\Gamma)} = \delta > 0$. For any $\epsilon > 0$ there exists an n such that,

$$C\|v_i - v\|_{H^1(\Omega)} \leq \epsilon.$$

Using the trace theorem we get,

$$\delta = \|\gamma v\|_{L^2(\Gamma)} = \|\gamma(v - v_i)\|_{L^2(\Gamma)} \leq C\|v_i - v\|_{H^1(\Omega)} \leq \epsilon,$$

for all $i > n$. By choosing $\epsilon < \delta$ we get a contradiction i.e. $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$ and therefore a Hilbert space.

(b) Give sufficient assumptions on a and l so that the problem has a unique solution in $H_0^1(\Omega)$.
Sol. a should be coercive and bounded and l should be bounded.

(c) Give an example of a linear functional l that violates the conditions in (b).
Sol. Let $l = \delta$. Then $\|l\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{|v(x)|}{\|v\|_{H^1(\Omega)}} = \infty$ since $H^1(\Omega)$ are not in general pointwise defined in $\mathbb{R}^3$.

4. Let $\Omega \subset \mathbb{R}^d$ be a convex domain, with boundary Γ. Consider the heat equation,

$$\begin{cases} \dot{u} - \Delta u = 0, & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \Gamma \times (0, T), \\ u(\cdot, 0) = v, & \text{in } \Omega. \end{cases}$$

(a) Let $v \in L^2(\Omega)$. Show that $\|\nabla u(t)\|_{L^2(\Omega)} \leq Ct^{-1/2}\|v\|_{L^2(\Omega)}$, for $t > 0$.
Sol. Let $\{\phi_i\}$ be the set of eigenfunctions (orthogonal w.r.t. $(\nabla\cdot, \nabla\cdot)$) spanning $L^2(\Omega)$ with corresponding eigenvalues λ_i. Let $u(t) = \sum_{i=1}^\infty \alpha_i(t)\phi_i$. Inserting it into the equation yields $\alpha_i(t) = e^{-\lambda_i t}(v, \phi_i)$. Therefore,

$$|u(\cdot, t)|_{H^1(\Omega)}^2 = \sum_{i=1}^\infty \lambda_i e^{-2\lambda_i t}(v, \phi_i)^2 \leq Ct^{-1}\|v\|_{L^2(\Omega)}^2$$

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(b) Let $v \in H_0^1(\Omega)$. Show that $\|\nabla u(t)\|_{L^2(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)}$, for $t > 0$.
Sol. $|u(\cdot, t)|_{H^1(\Omega)}^2 = \sum_{i=1}^\infty \lambda_i e^{-2\lambda_i t}(v, \phi_i)^2 \leq \|\nabla v\|_{L^2(\Omega)}^2$.

(c) Formulate the Crank-Nicolson Galerkin finite element method for this problem.
Sol. The Crank-Nicolson Galerkin approximation at time $t_n = kn$, $U^n \in V_h$, with time step size k fulfills,

$$(U^n, w) + \frac{1}{2}k(\nabla U^n, \nabla w) = (U^{n-1}, w) - \frac{1}{2}k(\nabla U^{n-1}, \nabla w), \quad \forall w \in V_h,$$

with $(U^0, w) = (v, w)$ for all $w \in V_h$.

5. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with smooth boundary Γ. Consider the wave equation,

$$\begin{cases} \ddot{u} - \Delta u = f, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, \quad \dot{u}(\cdot, 0) = w, & \text{in } \Omega. \end{cases}$$

Let u_h be the semi-discrete (in space) Galerkin approximation of u using v_h and w_h as approximations for the initial conditions. Prove for $t \geq 0$ that,

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} \leq C\left(|v_h - R_h v|_{H^1(\Omega)} + \|w_h - R_h w\|\right) + Ch^2\left(\|u(t)\|_{H^2(\Omega)} + \int_0^t \|u_{tt}\|_{H^2(\Omega)}\, ds\right),$$

where R_h is the Ritz projection.
Sol. See Theorem 13.1.

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