Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2017–05–30 f SB

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20p–29p, 4: 30p–39p, 5: 40p–, G: 20p–34p, VG: 35p–

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1. *The trace theorem.* Let Ω be the unit square with boundary Γ and let $\gamma : C^1(\bar{\Omega}) \to C(\Gamma)$ be the trace operator. Show that γ may be extended to $\gamma : H^1(\Omega) \to L^2(\Gamma)$, which defines the trace $\gamma v \in L^2(\Gamma)$ for any $v \in H^1(\Omega)$, and show that $\|\gamma v\|_{L^2(\Gamma)} \leq C\|v\|_{H^1(\Omega)}$.

2. Consider the Robin problem on a bounded domain Ω with boundary Γ: find u such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ -\partial_n u = \kappa u, & \text{on } \Gamma, \end{cases}$$

where $f \in L^2(\Omega)$, $0 < \kappa$ is a constant, and ∂_n denotes the normal derivative.

(a) Write the equation on weak form with test and trial space $H^1(\Omega)$.
(b) Show that the bilinear form is coercive and bounded on $H^1(\Omega)$. *Hint: You are allowed to use that* $\|v\|_{L^2(\Omega)} \leq C(\|\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Gamma)})$ *without proof.*
(c) Show $\|u\|_{H^1(\Omega)} \leq C_{\kappa,f}$, where $C_{\kappa,f}$ depends on κ and $\|f\|_{L^2(\Omega)}$.

3. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary Γ. Consider the eigenvalue problem:

$$\begin{cases} -\Delta u + Vu = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \end{cases}$$

where $0 \leq V(x) \leq C_V$ is a bounded function.

(a) Show that eigenfunctions corresponding to different eigenvalues are orthogonal and that the eigenvalues are real and positive.
(b) Let $\lambda_1 = \min_{0 \neq u \in H_0^1(\Omega)} \frac{(\nabla u, \nabla u) + (Vu, u)}{(u,u)}$. Show that there is a constant C such that $0 < C \leq \lambda_1$.

4. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, with boundary Γ. Consider the homogeneous heat equation with homogeneous Dirichlet boundary conditions and initial condition $u(\cdot, 0) = v \in L^2(\Omega)$:

$$\dot{u} - \Delta u = 0, \qquad \text{in } \Omega \times (0, T).$$

(a) Show that $\|u\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)}$.
(b) Formulate the backward Euler-Galerkin method for this problem with time step k.
(c) Show that the L^2 norm of the discrete solution $\|U^n\|_{L^2(\Omega)}$ decays with increasing n.

5. Let $\Omega \subset \mathbb{R}^2$ be a domain, with boundary Γ. Let $f(0) = 0$, $|f'(z)| \leq B$ for $z \in \mathbb{R}$, and let u solve,

$$\begin{cases} \dot{u} - \Delta u = f(u), & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \Gamma \times (0, T), \\ u(\cdot, 0) = v, & \text{in } \Omega, \end{cases} \tag{1}$$

Further, let V_h be a finite element space and $k = T/N$ the timestep. An alternative to the Backward Euler Galerkin method is the IMEX method: find $\{U_h^n\}_{n=1}^N \subset V_h$ such that,

$$(U_h^n, w) + k(\nabla U_h^n, \nabla w) = (U_h^{n-1}, w) + k(f(U_h^{n-1}), w), \quad \forall w \in V_h, \quad 1 \leq n \leq N,$$

where $U_h^0 = v_h \in V_h$ is an approximation of v.

(a) How does IMEX differ from Backward Euler? Why is it computationally cheaper?
(b) Given $v_h \in V_h$, show existence and uniqueness of the iterates $\{U_h^n\}_{n=1}^N$.
(c) Show that $\|U_h^n\|_{L^2(\Omega)} \leq C(T, B)$ for all $1 \leq n \leq N$.

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