Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2015–08–28 f V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20p–29p, 4: 30p–39p, 5: 40p–, G: 20p–34p, VG: 35p–

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1. Consider the unit square, $\Omega = [0,1] \times [0,1]$.

(a) Compute $\|x_1^2\|_{L^2(\Omega)}$, where x_1 is the first component of an element in $\mathbb{R}^2$.
Sol. $\|x_1^2\|_{L^2(\Omega)}^2 = \int_0^1 \int_0^1 x_1^4 \, dx_1 \, dx_2 = \frac{1}{5}$ i.e. $\|x_1^2\|_{L^2(\Omega)} = \frac{1}{\sqrt{5}}$.

(b) Show that $\|v\|_{L^2(\Omega)} \le \|\nabla v\|_{L^2(\Omega)}$, for all $v \in H_0^1(\Omega)$.
Sol. See proof of Theorem A.6.

(c) Show that $\|v\|_{H^{-1}(\Omega)} \le \|v\|_{L^2(\Omega)}$, for all $v \in L^2(\Omega)$.
Sol. $\|v\|_{H^{-1}(\Omega)} = \sup_{w \in H_0^1(\Omega)} \frac{\int vw\,dx}{\|\nabla w\|_{L^2(\Omega)}} \le \sup_{w \in H_0^1(\Omega)} \frac{\|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}}{\|\nabla w\|_{L^2(\Omega)}} \le \|v\|_{L^2(\Omega)}$.

2. Let $\Omega \subset \mathbb{R}^d$ be convex, with boundary Γ. Let $b \in \mathbb{R}^d$ be a constant vector and consider,

$$\begin{cases} -\Delta u + b \cdot \nabla u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma. \end{cases}$$

(a) Show that the corresponding weak form is coercive.
Sol. We note that $(b \cdot \nabla v, v) = ((\nabla \cdot b)v, v) - (v, b \cdot \nabla v) = -(v, b \cdot \nabla v) = 0$. Therefore $\|\nabla v\|_{L^2(\Omega)}^2 \le (\nabla v, \nabla v) + (b \cdot \nabla v, v)$ for all $v \in H_0^1(\Omega)$.

(b) Let $V_h \subset H_0^1(\Omega)$ be the space of continuous piecewise linear functions. Derive the finite element method using the space V_h.
Sol. Find $U \in V_h$ such that $(\nabla U, \nabla v) + (b \cdot \nabla U, v) = (f, v)$ for all $v \in V_h$.

(c) Derive an a priori bound for the error in the finite element approximation. Express explicitly the dependency on b in the bound.
Sol. We have

$$\begin{aligned} \|\nabla(u-U)\|_{L^2(\Omega)}^2 &= (\nabla(u-U), \nabla(u-U)) + (b \cdot \nabla(u-U), u-U) \\ &= (\nabla(u-U), \nabla(u-\pi u)) + (b \cdot \nabla(u-U), u-\pi u) \\ &\le \|\nabla(u-U)\|_{L^2(\Omega)}\|\nabla(u-\pi u)\|_{L^2(\Omega)} + |b|\|\nabla(u-U)\|_{L^2(\Omega)}\|u-\pi u\|_{L^2(\Omega)} \\ &\le \frac{1}{4}\|\nabla(u-U)\|_{L^2(\Omega)}^2 + \|\nabla(u-\pi u)\|_{L^2(\Omega)} + \frac{1}{4}\|\nabla(u-U)\|_{L^2(\Omega)}^2 + |b|^2\|u-\pi u\|_{L^2(\Omega)}^2. \end{aligned}$$

We get,

$$\|\nabla(u-U)\|_{L^2(\Omega)}^2 \le 2\|\nabla(u-\pi u)\|_{L^2(\Omega)}^2 + 2|b|^2\|u-\pi u\|_{L^2(\Omega)}^2.$$

3. Consider the eigenvalue problem: find $u \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$\begin{cases} -\Delta u + cu = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \end{cases}$$

where $c \in L^\infty(\Omega)$ is positive.

(a) Show that the eigenvalues λ are real and positive.
Sol. Let λ be an eigenvalue with corresponding eigenfunction u. The Rayleigh quotient gives $\lambda = \frac{(\nabla u, \nabla u) + (cu, u)}{(u,u)} = \frac{\|\nabla u\|_{L^2(\Omega)}^2 + \|\sqrt{c}u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2} \in \mathbb{R}_+$.

(b) Show that eigenfunctions corresponding to different eigenvalues are orthogonal both with respect to the $L^2(\Omega)$ scalar product and to the energy scalar product induced by the problem, $(\nabla v, \nabla w) + (cv, w)$.
Sol. Let (λ, u) and (μ, v) be eigenpairs with $\lambda \neq \mu$. We have $\lambda(u, v) = (\nabla u, \nabla v) + (cu, v) = (\nabla v, \nabla u) + (cv, u) = \mu(v, u)$ i.e. $(u, v) = 0$ and $(\nabla v, \nabla u) + (cv, u) = 0$ they are orthogonal.

(c) Bound the smallest eigenvalue in terms of the smallest eigenvalue to the Laplacian $-\Delta$ on Ω and the bounded function c.
Sol. Let $-\Delta v = \mu v$ be the lowest eigenvalue of the Laplacian with corresponding normalized eigenfunction $\|v\|_{L^2(\Omega)} = 1$. We have,

$$\lambda = \inf_{w \in H_0^1(\Omega)} \frac{(\nabla w, \nabla w) + (cw, w)}{(w, w)} \leq (\nabla v, \nabla v) + (cv, v) \leq \mu + \|c\|_{L^\infty(\Omega)}.$$

4. Let $\Omega \subset \mathbb{R}^d$ be a convex domain, with boundary Γ. Consider the heat equation,

$$\begin{cases} \dot{u} - \Delta u = 0, & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \Gamma \times (0, T), \\ u(\cdot, 0) = v, & \text{in } \Omega. \end{cases}$$

(a) Let $v \in L^2(\Omega)$. Show that $\|\nabla u(t)\|_{L^2(\Omega)} \leq Ct^{-1/2}\|v\|_{L^2(\Omega)}$, for $t > 0$.
Sol. Let $\{\phi_i\}$ be the set of eigenfunctions (orthogonal w.r.t. $(\nabla \cdot, \nabla \cdot)$) spanning $L^2(\Omega)$ with corresponding eigenvalues λ_i. Let $u(t) = \sum_{i=1}^\infty \alpha_i(t)\phi_i$. Inserting it into the equation yields $\alpha_i(t) = e^{-\lambda_i t}(v, \phi_i)$. Therefore,

$$|u(\cdot, t)|_{H^1(\Omega)}^2 = \sum_{i=1}^\infty \lambda_i e^{-2\lambda_i t}(v, \phi_i)^2 \leq Ct^{-1}\|v\|_{L^2(\Omega)}^2$$

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(b) Let $v \in H_0^1(\Omega)$. Show that $\|\nabla u(t)\|_{L^2(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)}$, for $t > 0$.
Sol. $|u(\cdot, t)|_{H^1(\Omega)}^2 = \sum_{i=1}^\infty \lambda_i e^{-2\lambda_i t}(v, \phi_i)^2 \leq \|\nabla v\|_{L^2(\Omega)}^2$.

(c) Formulate the Crank-Nicolson Galerkin finite element method for this problem.
Sol. The Crank-Nicolson Galerkin approximation at time $t_n = kn$, $U^n \in V_h$, with time step size k fulfills,

$$(U^n, w) + \frac{1}{2}k(\nabla U^n, \nabla w) = (U^{n-1}, w) - \frac{1}{2}k(\nabla U^{n-1}, \nabla w), \quad \forall w \in V_h,$$

with $(U^0, w) = (v, w)$ for all $w \in V_h$.

5. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with smooth boundary Γ. Consider the wave equation,

$$\begin{cases} \ddot{u} - \Delta u = 0, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, \quad \dot{u}(\cdot, 0) = w, & \text{in } \Omega, \end{cases}$$

where v and w are smooth. Show that the total energy of u is constant in time.
Sol. See Theorem 11.2.

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