Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2018–08–25 f SB

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40– (Chalmers) G: 20–34p, VG: 35– (GU).

1.

(a) The weak derivative of $|x|$ is -1 for $-1 < x < 0$ and 1 for $0 < x < 1$.

(b) We use polar coordinates and note that $v'(r) = \frac{2}{r\log(r^2)} = \frac{1}{r\log(r)}$. We change variables $y = \log(r)$ to get,

$$|v|^2_{H^1(\Omega)} = 2\pi \int_0^{1/2} \frac{1}{r\log(r)^2}\,dr = 2\pi \int_{-\infty}^{-\log(2)} y^{-2}e^{-y}e^y\,dy = 2\pi \int_{-\infty}^{-\log(2)} y^{-2}\,dy < \infty.$$

In $L^2(\Omega)$ norm the singularity is weaker so it is also bounded by similar calculation.

(c) $\||x|^\lambda\|^2_{L^2(\Omega)} = C_d \int_0^1 r^{2\lambda}r^{d-1}\,dr < \infty$ if $2\lambda + d - 1 > -1$ i.e. $\lambda > -\frac{d}{2}$.

2.

(a) Find $u \in H^1_0(\Omega)$ such that, $(\nabla u, \nabla v) = (f, v)$, for all $v \in H^1_0(\Omega)$. Let $V_h \subset H^1_0(\Omega)$ be the space of continuous piecewise linear functions defined on a triangulation of Ω. The finite element method reads: find $u_h \in V_h$ such that, $(\nabla u_h, \nabla v) = (f, v), \quad \forall v \in V_h$.

(b) Let $w \in H^1_0(\Omega)$ be arbitrary. We get $J(u+w) = J(u)+\int_\Omega |\nabla w|^2\,dx+\int_\Omega \nabla u\cdot\nabla w\,dx-\int_\Omega fw\,dx = J(u) + \int_\Omega |\nabla w|^2\,dx \geq J(u)$.

3.

(a) Let $-\Delta\phi_i = \lambda_i\phi_i$ with $\phi_i|_\Gamma = 0$. Then we insert $u = \sum_{i=1}^\infty \alpha_i(t)\phi_i$ into the equation to get

$$\sum_{i=1}^{\infty}(\alpha_i'(t) - \lambda_i\alpha_i(t))\phi_i = 0.$$

Orthogonality gives $\alpha_i(t) = \alpha_i(0)e^{-\lambda_i t}$ and initial value that $\alpha_i(0) = (v, \phi_i)$. We get $u(t) = \sum_{i=1}^\infty (v, \phi_i)e^{-\lambda_i t}\phi_i$.

(b) $\|u\|^2_{L^2(\Omega)} = \sum_{i=1}^\infty (v, \phi_i)^2 e^{-2\lambda_i t} \leq e^{-2\lambda_1 t}\|v\|^2_{L^2(\Omega)}$.

(c) Let $V_h \subset H^1_0(\Omega)$ be the space of continuous piecewise linear functions defined on a triangulation of Ω. The BE-FEM approximation fulfills: find $u^n_h \in V_h$ such that,

$$(u^n_h - u^{n-1}_h, v) + k(\nabla u^n_h, \nabla v) = 0, \quad \forall v \in V_h,$$

with k being the time-step, $u^0_h = P_h v$ and $P_h : L^2(\Omega) \to V_h$ isthe L^2-projection.

4.

(a) We have $\|f(u) - f(w)\|_{L^2(\Omega)} \leq \|u - w\|_{L^2(\Omega)} + \|u^2 + uv + v^2\|_{L^3(\Omega)}\|u - w\|_{L^6(\Omega)} \leq (1 + C2R^2)\|u - w\|_{H^1(\Omega)}$.

(b) We have

$$\begin{aligned}\|Su(t)\|_{H^1(\Omega)} &\leq \|v\|_{H^1(\Omega)} + \int_0^t (t-s)^{-1/2}\|f(u(s))\|_{L^2(\Omega)}\,ds \\ &\leq R_0 + C\tau^{1/2}R,\end{aligned}$$

since $\|f(u)\|_{L^2(\Omega)} \leq \|f(u) - f(0)\|_{L^2(\Omega)} \leq C\|u\|_{H^1(\Omega)} \leq C\cdot R$. Let $R = 2R_0$ and τ be small enough for $C\tau^{1/2}R \leq R_0$.

(c) We have,

$$\|Su - Sw\|_{H^1(\Omega)} \le \int_0^t (t-s)^{-1/2} \|f(u(s)) - f(w(s))\|_{L^2(\Omega)}\, ds$$
$$\le C(R)\tau^{1/2} \max_{0\le t\le\tau} \|u(t) - w(t)\|_{H^1(\Omega)}.$$

By taking $\max_{0\le t\le\tau}$ and choosing τ small enough for $\tau^{1/2}C(R) < 1$ we prove that S is a contraction.

5. See the proof of Theorem 5.4 in *Partial differential equations with numerical methods* by Thomée and Larsson.

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