Linear programming (Linjärprogrammering)

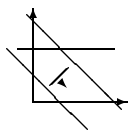
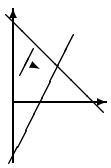
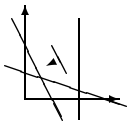
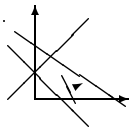
- Continuous variables $\mathbf{x} \in \Re^n$
- Linear cost- and constraint functions

$$\begin{array}{lllll} \text{(LP)} & \min & \mathbf{c}^T\mathbf{x} & | & \mathbf{c} \in \Re^n \\ & \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} & | & \mathbf{A} \in \Re^{m\times n} \\ & & \mathbf{x} \geq \mathbf{0} & | & \mathbf{b} \in \Re^m \end{array}$$

One of four cases holds for any (LP):

1. *Unique, bounded* solution (in extreme point)
2. *Alternative, bounded* solutions (∞ many, convex combinations of extreme points)
3. *No feasible* solutions
4. *Unbounded* solution (no restricting constraints)

4 cases

1. 
2. 
3. 
4. 

4 cases

$$\begin{array}{lrcrcl} \max z = & x_1 & - & x_2 & & \\ \text{då} & x_1 & + & x_2 & \geq & 1 \\ & x_1 & + & x_2 & \leq & 3 \\ & & & x_2 & \leq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{array}{lrcrcl} \min z = & -4x_1 & + & 2x_2 & & \\ \text{då} & x_1 & + & x_2 & \leq & 3 \\ & 2x_1 & - & x_2 & \leq & 2 \\ & x_1 & & & \geq & 0 \end{array}$$

$$\begin{array}{lrcrcl} \min z = & 2x_1 & + & x_2 & & \\ \text{då} & x_1 & + & 3x_2 & \leq & 3 \\ & 2x_1 & + & x_2 & \leq & 2 \\ & x_1 & & & \geq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{array}{lrcrcl} \max z = & 2x_1 & + & x_2 & & \\ \text{då} & x_1 & + & x_2 & \geq & 1 \\ & 2x_1 & + & 3x_2 & \leq & 6 \\ & -x_1 & + & x_2 & \leq & 1 \end{array}$$

Basic feasible solutions

- If an optimal solution to (LP) exists then there exists an *extreme point* to the feasible set that is optimal.

$\Rightarrow$ Search among the extreme points = basic feasible solutions

- Partition into basic ($\mathbf{x}_B$) and non-basic ($\mathbf{x}_N$) variables:

$$\begin{array}{lll} \mathbf{A} = (\mathbf{B}, \mathbf{N}) & | & \text{(LP)} \quad \min \quad \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N \\ \mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N) & | & \text{s.t.} \quad \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b} \\ \mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N) & | & \mathbf{x}_B, \mathbf{x}_N \geq \mathbf{0} \end{array}$$

$\mathbf{B} \in \Re^{m\times m}$ invertible ($\mathbf{B}^{-1}\exists$), $\mathbf{N} \in \Re^{m\times(n-m)}$

- $\mathbf{x}_B = \mathbf{B}^{-1}(\mathbf{b} - \mathbf{N}\mathbf{x}_N)$
- $\mathbf{x}_N = \mathbf{0} \Rightarrow \mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ (basic solution)
- $\mathbf{x}_B \geq \mathbf{0}$ (feasible solution)

Another representation of (LP)

$$\begin{array}{lll} \text{(LP)} \quad \min & \mathbf{c}_B^T\mathbf{B}^{-1}(\mathbf{b} - \mathbf{N}\mathbf{x}_N) + \mathbf{c}_N^T\mathbf{x}_N & \\ \text{s.t.} & \mathbf{B}^{-1}(\mathbf{b} - \mathbf{N}\mathbf{x}_N) \geq \mathbf{0} & (\Leftrightarrow \mathbf{x}_B \geq \mathbf{0}) \\ & \mathbf{x}_N \geq \mathbf{0} & \end{array}$$

$$\Updownarrow$$

$$\begin{array}{lll} \text{(LP)} \quad \min & \left(\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}\right)\mathbf{x}_N + \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} \\ \text{s.t.} & \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \leq \mathbf{B}^{-1}\mathbf{b} \\ & \mathbf{x}_N \geq \mathbf{0} \end{array}$$

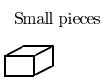
Reduced costs: $\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}$
If negative, increasing $\mathbf{x}_N$ yields lower objective value

Simplexalgoritmen

- Starta i en tillåten extrempunkt
- Vandra mellan närliggande extrempunkter, minska målfunktionsvärdet i varje steg (välj negativ reducerad kostnad)
- Alla reducerade kostnader $\geq 0 \Leftrightarrow$ optimal bas $\Leftrightarrow$ optimal extrempunkt

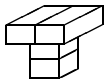
LEGO-example

Small pieces (8)



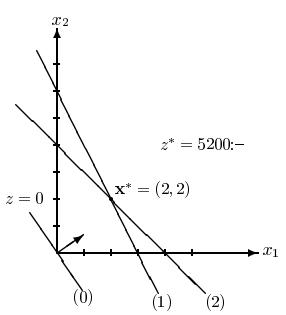
Chair, x_2 (1000:–)

Large pieces (6)



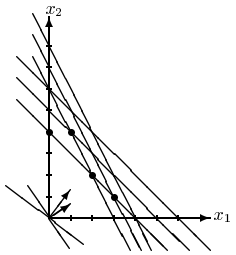
Table, x_1 (1600:–)

$$\begin{array}{lrcrcll} \max & z & = & 1600x_1 + & 1000x_2 & & (0) \quad x_1^* = 2 \\ \text{s.t.} & & & 2x_1 + & x_2 \leq 6 & & (1) \quad x_2^* = 2 \\ & & & 2x_1 + & 2x_2 \leq 8 & & (2) \quad z^* = 5200 \\ & & & & x_1,\ x_2 \geq 0 & & \end{array}$$



Sensitivity analysis

If data changes, an optimal solution might not be feasible any longer!



Linear programming duality

Primal problem (P):				Dual problem (D):			
max	$z = \mathbf{c}^T\mathbf{x}$			min	$w = \mathbf{b}^T\mathbf{y}$		
då	$\mathbf{Ax}$	$\leq$	$\mathbf{b}$	då	$\mathbf{A}^T\mathbf{y}$	$\geq$	$\mathbf{c}$
	$\mathbf{x}$	$\geq$	$\mathbf{0}$		$\mathbf{y}$	$\geq$	$\mathbf{0}$

- **Weak duality:** If $\mathbf{x}$ is feasible in (P) and $\mathbf{y}$ is feasible in (D) then $z \leq w$.
- **Strong duality:** If both (P) and (D) have feasible solutions then both have optimal solutions ($\mathbf{x}^*$ and $\mathbf{y}^*$) and $z^* = \mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^* = w^*$.
- **Complementarity:** If $\mathbf{x}^*$ and $\mathbf{y}^*$ are optimal solutions to (P) and (D), respectively, then
$$(\mathbf{y}^*)^T(\mathbf{Ax}^* - \mathbf{b}) = 0 \quad \text{and} \quad (\mathbf{x}^*)^T(\mathbf{A}^T\mathbf{y}^* - \mathbf{c}) = 0.$$

LEGO-dual

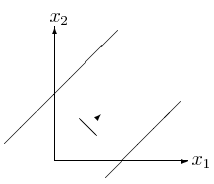
- y_1, y_2 = the amount of money that another producer is willing to pay for each of our large/small pieces
- w = the total amount that the other producer is willing to pay for all our pieces (6 large + 8 small)
- The bid must give us at least 1600:–/table and 1000:–/chair, otherwise we reject it

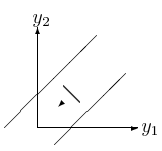
min	w	=	$6y_1$	+	$8y_2$		(total bid)
s.t.			$2y_1$	+	$2y_2$	$\geq$ 1600	(tables)
			y_1	+	$2y_2$	$\geq$ 1000	(chairs)
					$y_1,\ y_2$	$\geq$ 0	(prices)

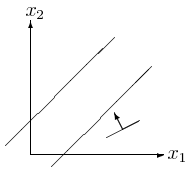
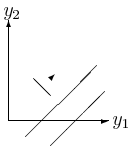
Optimum: $y_1^* = 600$:–, $y_2^* = 200$:–, $w^* = 5200$:– Complementarity?

Unbounded primal $\Rightarrow$ infeasible dual

[P]					[D]			
max	$z = x_1 + x_2$				min	$w = 2y_1 + 2y_2$		
s.t.	$x_1 - x_2$	$\leq$	2		s.t.	$y_1 - y_2$	$\geq$	1
	$-x_1 + x_2$	$\leq$	2			$-y_1 + y_2$	$\geq$	1
	x_1, x_2	$\geq$	0			y_1, y_2	$\geq$	0





Infeasible primal and dual

[P]					[D]			
min	$z = x_1 - 2x_2$				max	$w = y_1 + y_2$		
s.t.	$x_1 - x_2$	$\geq$	1		s.t.	$y_1 - y_2$	$\leq$	1
	$-x_1 + x_2$	$\geq$	1			$-y_1 + y_2$	$\leq$	-2
	x_1, x_2	$\geq$	0			y_1, y_2	$\geq$	0