# Optimering under osäkerhet

Föreläsning 2b, lp1 2003

Ann-Brith Strömberg

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## Linear programming (Linjärprogrammering)

- Continuous variables $\mathbf{x} \in \Re^n$
- Linear cost- and constraint functions

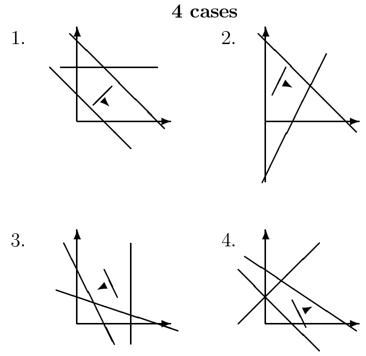
$$\begin{array}{lllcl} \text{(LP)} & \min & \mathbf{c}^T\mathbf{x} & | & \mathbf{c} \in \Re^n \\ & \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} & | & \mathbf{A} \in \Re^{m\times n} \\ & & \mathbf{x} \geq \mathbf{0} & | & \mathbf{b} \in \Re^m \end{array}$$

*One* of four cases holds for any (LP):

1. *Unique, bounded* solution (in extreme point)
2. *Alternative, bounded* solutions ($\infty$ many, convex combinations of extreme points)
3. *No feasible* solutions
4. *Unbounded* solution (no restricting constraints)



## 4 cases






## 4 cases

$$\begin{array}{llllll} \max z = & x_1 & - & x_2 & & \\ \text{då} & x_1 & + & x_2 & \geq & 1 \\ & x_1 & + & x_2 & \leq & 3 \\ & & & x_2 & \leq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{array}{llllll} \min z = & -4x_1 & + & 2x_2 & & \\ \text{då} & x_1 & + & x_2 & \leq & 3 \\ & 2x_1 & - & x_2 & \leq & 2 \\ & x_1 & & & \geq & 0 \end{array}$$

$$\begin{array}{llllll} \min z = & 2x_1 & + & x_2 & & \\ \text{då} & x_1 & + & 3x_2 & \leq & 3 \\ & 2x_1 & + & x_2 & \leq & 2 \\ & x_1 & & & \geq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{array}{llllll} \max z = & 2x_1 & + & x_2 & & \\ \text{då} & x_1 & + & x_2 & \geq & 1 \\ & 2x_1 & + & 3x_2 & \leq & 6 \\ & -x_1 & + & x_2 & \leq & 1 \end{array}$$



## Basic feasible solutions

- If an optimal solution to (LP) exists then there exists an *extreme point* to the feasible set that is optimal.

$\Rightarrow$ Search among the extreme points = basic feasible solutions

- Partition into basic ($\mathbf{x}_B$) and non-basic ($\mathbf{x}_N$) variables:

$$\begin{array}{rclll} \mathbf{A} = (\mathbf{B}, \mathbf{N}) & | & \text{(LP)} & \min & \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N \\ \mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N) & | & & \text{s.t.} & \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b} \\ \mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N) & | & & & \mathbf{x}_B, \mathbf{x}_N \geq \mathbf{0} \end{array}$$

$\mathbf{B} \in \Re^{m\times m}$ invertible ($\mathbf{B}^{-1}\exists$), $\quad \mathbf{N} \in \Re^{m\times(n-m)}$

- $\mathbf{x}_B = \mathbf{B}^{-1}(\mathbf{b} - \mathbf{N}\mathbf{x}_N)$
- $\mathbf{x}_N = \mathbf{0} \Rightarrow \mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ (basic solution)
- $\mathbf{x}_B \geq \mathbf{0}$ (feasible solution)



## Another representation of (LP)

$$\begin{array}{llll} \text{(LP)} & \min & \mathbf{c}_B^T\mathbf{B}^{-1}(\mathbf{b} - \mathbf{N}\mathbf{x}_N) + \mathbf{c}_N^T\mathbf{x}_N & \\ & \text{s.t.} & \mathbf{B}^{-1}(\mathbf{b} - \mathbf{N}\mathbf{x}_N) \geq \mathbf{0} & (\Leftrightarrow \mathbf{x}_B \geq \mathbf{0}) \\ & & \mathbf{x}_N \geq \mathbf{0} & \end{array}$$

$$\Updownarrow$$

$$\begin{array}{lll} \text{(LP)} & \min & \left(\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}\right)\mathbf{x}_N + \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} \\ & \text{s.t.} & \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \leq \mathbf{B}^{-1}\mathbf{b} \\ & & \mathbf{x}_N \geq \mathbf{0} \end{array}$$

Reduced costs: $\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}$

If negative, increasing $\mathbf{x}_N$ yields lower objective value



## Simplexalgoritmen

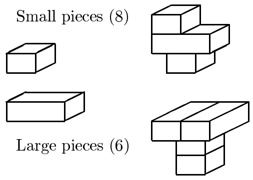
- Starta i en tillåten extrempunkt
- Vandra mellan närliggande extrempunkter, minska målfunktionsvärdet i varje steg (välj negativ reducerad kostnad)
- Alla reducerade kostnader $\geq 0 \Leftrightarrow$ optimal bas $\Leftrightarrow$ optimal extrempunkt



## LEGO-example

Small pieces (8) — Chair, $x_2$ (1000:–)

Large pieces (6) — Table, $x_1$ (1600:–)

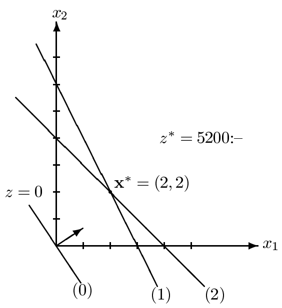


$$\begin{array}{lllllllll} \max & z & = & 1600x_1 & + & 1000x_2 & & & \\ \text{s.t.} & & & 2x_1 & + & x_2 & \leq & 6 \\ & & & 2x_1 & + & 2x_2 & \leq & 8 \\ & & & & & x_1,\ x_2 & \geq & 0 \end{array}$$

(0) $x_1^* = 2$

(1) $x_2^* = 2$

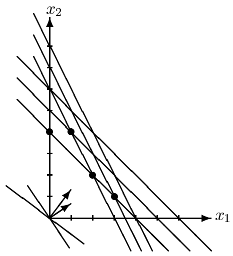
(2) $z^* = 5200$






## Sensitivity analysis

If data changes, an optimal solution might not be feasible any longer!






## Linear programming duality

| Primal problem (P): | | | | Dual problem (D): | | | |
|---|---|---|---|---|---|---|---|
| max | $z$ | $=$ | $\mathbf{c}^T\mathbf{x}$ | min | $w$ | $=$ | $\mathbf{b}^T\mathbf{y}$ |
| då | $\mathbf{Ax}$ | $\leq$ | $\mathbf{b}$ | då | $\mathbf{A}^T\mathbf{y}$ | $\geq$ | $\mathbf{c}$ |
| | $\mathbf{x}$ | $\geq$ | $\mathbf{0}$ | | $\mathbf{y}$ | $\geq$ | $\mathbf{0}$ |

- **Weak duality:** If $\mathbf{x}$ is feasible in (P) and $\mathbf{y}$ is feasible in (D) then $z \leq w$.
- **Strong duality:** If both (P) and (D) have feasible solutions then both have optimal solutions ($\mathbf{x}^*$ and $\mathbf{y}^*$) and $z^* = \mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^* = w^*$.
- **Complementarity:** If $\mathbf{x}^*$ and $\mathbf{y}^*$ are optimal solutions to (P) and (D), respectively, then
$$(\mathbf{y}^*)^T(\mathbf{Ax}^* - \mathbf{b}) = 0 \quad \text{and} \quad (\mathbf{x}^*)^T(\mathbf{A}^T\mathbf{y}^* - \mathbf{c}) = 0.$$



## LEGO-dual

- $y_1, y_2$ = the amount of money that another producer is willing to pay for each of our large/small pieces
- $w$ = the total amount that the other producer is willing to pay for all our pieces (6 large + 8 small)
- The bid must give us at least 1600:–/table and 1000:–/chair, otherwise we reject it

$$\begin{array}{llrcrcll}
\min & w = & 6y_1 & + & 8y_2 & & & \text{(total bid)} \\
\text{s.t.} & & 2y_1 & + & 2y_2 & \geq & 1600 & \text{(tables)} \\
 & & y_1 & + & 2y_2 & \geq & 1000 & \text{(chairs)} \\
 & & & & y_1,\ y_2 & \geq & 0 & \text{(prices)}
\end{array}$$

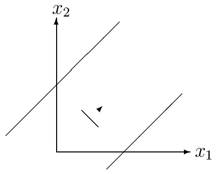
Optimum: $y_1^* = 600$:–, $y_2^* = 200$:–, $w^* = 5200$:– Complementarity?

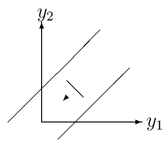


## Unbounded primal ⇒ infeasible dual

[P] $\max z = x_1 + x_2$
$$\begin{array}{lrcl}
\text{s.t.} & x_1 - x_2 & \leq & 2 \\
 & -x_1 + x_2 & \leq & 2 \\
 & x_1, x_2 & \geq & 0
\end{array}$$

[D] $\min w = 2y_1 + 2y_2$
$$\begin{array}{lrcl}
\text{s.t.} & y_1 - y_2 & \geq & 1 \\
 & -y_1 + y_2 & \geq & 1 \\
 & y_1, y_2 & \geq & 0
\end{array}$$






## Infeasible primal and dual

[P] $\min z = x_1 - 2x_2$
$$\begin{array}{lrcl}
\text{s.t.} & x_1 - x_2 & \geq & 1 \\
 & -x_1 + x_2 & \geq & 1 \\
 & x_1, x_2 & \geq & 0
\end{array}$$

[D] $\max w = y_1 + y_2$
$$\begin{array}{lrcl}
\text{s.t.} & y_1 - y_2 & \leq & 1 \\
 & -y_1 + y_2 & \leq & -2 \\
 & y_1, y_2 & \geq & 0
\end{array}$$