Department of
Mathematics
Göteborg

Numerical Linear Algebra, TMA265

Solutions to the examination 23 October 2008

1a) Symmetric: $A^T = A$. Indefinite: $x^T Ax > 0$ for some x and $y^T Ay < 0$ for some y.
1b) See text book or lecture notes; pivots of size 1×1 or 2×2.

2a) See text book or lecture notes; $q \approx \sqrt{M/3}$.
2b) "Matrix times matrix" and "solving triangular systems with many right hand sides".

3a) $A = U\Sigma V^T$, U and V with orthonormal columns and Σ diagonal $r \times r$, where $r = rank(A)$.
3b) $A^T = V\Sigma U^T$ with Σ and U "regular" matrices. Thus, $Col(A^T) = Col(V)$, so the columns of V make up an orthonormal basis for $Col(A^T) = Row(A)$ and the projection is then by the matrix VV^T.

4a) For the construction of H we have $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow u = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$,

$H = I - 2uu^T$. Here u is orthogonal to columns 2-4 of A. Thus, $HA = \begin{bmatrix} 2 & 3 & 0 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 3 & 1 & 0 \\ 0 & -1 & 1 & -2 \\ 0 & 3 & 2 & 0 \end{bmatrix}$.

For Z we get $\tilde{u} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} \Rightarrow u = \frac{1}{\sqrt{20}} \begin{bmatrix} 0 \\ 2 \\ 0 \\ -4 \end{bmatrix}$, $Z = I - 2uu^T$

and $HA \times Z$ can by computed row by row, $HAZ = \begin{bmatrix} 2 & 5 & 0 & 0 \\ 0 & -1 & 1 & 2 \\ 0 & 1.8 & 1 & 2.4 \\ 0 & -2.2 & 1 & 0.4 \\ 0 & 1.8 & 2 & 2.4 \end{bmatrix}$.

5a) $Ax = \lambda x, \quad x \neq 0$ right eigenvector.
$y^* A = \lambda y^*, \quad y \neq 0$ left eigenvector.
5b) $\kappa(\lambda) = \frac{1}{|y^* x|}$, where x and y are normed left and right eigenvectors corresponding to λ.
5c) Let $\{u_i\}_{i=1}^n$ be ON-basis in R^n with $u_1 = u$. Then $Hu_1 = u_1 - 2u_1 u_1^T u_1 = -u_1$, since $u_1^T u_1 = 1$, so u_1 is right eigenvector and $\lambda_1 = -1$. Further, $Hu_j = u_j - 2u_1 u_1^T u_j = u_j, \quad j \neq 1$, since $u_1^T u_j = 0, \quad j \neq 1$, so $u_j, \quad j = 2, ..., n$ are right eigenvectors and $\lambda_j = 1, \quad j = 2, ..., n$. H is symmetric, $H^T = H$, so left and right eigenvectors are the same. The condition numbers are then $\kappa(\lambda_j) = \frac{1}{u_j^T u_j} = 1, \ j = 1, ..., n$.

6) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 4c - 3s & c & -2s \\ 4s + 3c & s & 2c \end{bmatrix}.$$
Choose $c = 4/5, \ s = -3/5$ to zero-out the (3,1)-element.
$$\begin{bmatrix} 2 & 4 & 3 \\ 5 & 4/5 & 6/5 \\ 0 & -3/5 & 8/5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 0 \\ 5 & 34/25 & 12/25 \\ 0 & 12/25 & 41/25 \end{bmatrix}.$$

7) Assume μ is not an eigenvalue of A, otherwise the result is trivial.
The matrix $A + E - \mu I$ is then singular and so is the matrix
$X^{-1}(A + E - \mu I)X = D - \mu I + X^{-1}EX$.
Then there exists a $z \neq 0$ such that $(D - \mu I)z = -X^{-1}EXz$.
Solving for z in this equation gives
$z = -(D - \mu I)^{-1}X^{-1}EXz$ and taking norms gives
$\|z\|_p = \|(D - \mu I)^{-1}X^{-1}EXz\|_p \leq$
$\|(D - \mu I)^{-1}\|_p \|X^{-1}\|_p \|E\|_p \|X\|_p \|z\|_p = \|(D - \mu I)^{-1}\|_p \kappa_p(X) \|E\|_p \|z\|_p$. (*)
But $D - \mu I$ is diagonal so $\|(D - \mu I)^{-1}\|_p = \frac{1}{\min_{1 \leq i \leq n} |\lambda_i - \mu|}$ for any norm p. Recall that $D = diag(\lambda_1, \lambda_2, ..., \lambda_n)$
Divide the inequality (*) by $\|z\|_p$ to obtain the result.

8) See text book or lecture notes.