Department of
Mathematics
Göteborg

EXAMINATION FOR NUMERICAL LINEAR ALGEBRA, TMA265/MMA600 2009-10-23

DATE: Friday 23 October TIME: 8.30 - 12.30 PLACE: V

Examiner:	Ivar Gustafsson, tel: 772 10 94
Teacher on duty:	Ivar Gustafsson
Solutions:	Will be announced at the end of the exam on the board nearby room MVF21
Result:	Will be sent to you by November 6 at the latest Your marked examination can be received at the student's office at Mathematics Department, daily 12.30-13
Grades:	To pass requires 13 point, including bonus points from homework assignments Grades are evaluated by a formula involving also the computer exercises
Aids:	None (except dictionaries)

Instructions:
- State your methodology carefully. Motivate your statements clearly.
- Only write on one page of the sheet. Do not use a red pen. Do not answer more than one question per page.
- Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover.

GOOD LUCK!

Question 1.
Consider the least squares problem $min_x\|Ax-b\|_2$ with full rank and perturbation δb to the right hand side $b \in R^m$ and no perturbation to the matrix $A \in R^{m\times n}$ with $m > n$. Show that the corresponding error δx in the solution x is bounded by $\frac{\|\delta x\|_2}{\|x\|_2} \le \kappa(A)\frac{1}{\cos\theta}\frac{\|\delta b\|_2}{\|b\|_2}$ where θ is the angle between b and Ax. **(3p)**
Hint: Apply the perturbation theory for linear systems to the normal equations. $\kappa(A)$ is the condition number $\|A\|_2\|A^+\|_2$, where A^+ is the generalized inverse.

Question 2.
Describe on principle how Gaussian elimination on block form with delayed updating can be carried out with basicly BLAS-3 routines and just a small amount of computations on BLAS-2 level. **(3p)**

Question 3.
a) Show how the problem:
$$\begin{cases} min_x\|x\|_2 \\ subject\ to\ Ax=b \end{cases},$$
where $A \in R^{m\times n}$, $m < n$, $rank(A) = m$ and $b \in R^m$, can be solved by the so called normal equations of second kind: $AA^Ty = b$. **(2p)**
b) Prove that the matrix AA^T in a) is symmetric and positive definite. **(2p)**.

Question 4.

Transform the matrix $\begin{bmatrix} 0 & -1 & 1 \\ 4 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$ to upper Hessenberg form by a similar transformation based on a Givens rotation. **(3p)**

Question 5.

a) Let $G \in R^{m \times n}, \quad m > n$ with $rank(G) = r < n$. Define the full SVD of G. **(1p)**

b) Let $A \in R^{n \times n}$, i.e. A is square, and let $H = \begin{bmatrix} O & A^T \\ A & O \end{bmatrix}$, where O is the zero-matrix. Use SVD to prove that if $\sigma_i, \; v_i, \; u_i, \; i = 1, \; \ldots \; n$ are singular values, right and left singular vectors of A, respectively, then $\pm\sigma_i$ and $\frac{1}{\sqrt{2}} \begin{bmatrix} v_i \\ \pm u_i \end{bmatrix}$ are eigenvalues and eigenvectors of H, respectively. **(2p)**

Question 6.

Perform the first step in the bidiagonalization of the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ i.e. find Householder reflections H and Z such that $A_1 = HAZ$ has the form $A_1 = \begin{bmatrix} x & x & 0 & 0 \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}$.

(3p)

Question 7.

a) Descibe briefly all steps in the method: Householder explicit shift QR iteration, for computing eigenvalues of a real unsymmetric matrix. **(2p)**

b) Prove that the Hessenberg form is preserved during the iterations in a). **(1p)**

Question 8.

a) Define the concept **congruent transformation** for a symmetric matrix and state a property regarding eigenvalues being preserved by such a transformation. **(1p)**

b) For the factorization $(A - \sigma I) = LDL^T$ of the matrix $A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 3 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$ we have the following number π of positive elements in the diagonal matrix D, depending on the value of σ:

σ	0	1	1.5	2	2.5	3	3.5	4	4.5	5
π	4	3	3	2	2	2	2	1	1	0

We know that the eigenvalues of A are integers. Determine the eigenvalues of A using a technique based on the property in a). **(1p)**

c) What can you get from Gerschgorins theorem regarding the eigenvalues of A in b)? **(1p)**