SOLUTIONS: FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS (CTH[TMA285]&GU[MAM695])

August 28, 2007, morning (4 hours), v
No aids.
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Each problem is worth 3 points.

1. Suppose $X = 3W(\frac{1}{3}) - W(\frac{1}{2}) + 2W(\frac{3}{4})$. (a) Find a deterministic function $f(t)$, $0 \leq t \leq 1$ such that $X = \int_0^1 f(t)dW(t)$. (b) Compute $\mathrm{Var}(X)$.

Solution. (a) We have

$$X = 3W(\frac{1}{3}) - W(\frac{1}{2}) + 2W(\frac{3}{4}) = \frac{1}{3}W(\frac{1}{3}) + W(\frac{1}{2}) + 2(W(\frac{3}{4}) - W(\frac{1}{2}))$$

$$= \frac{4}{3}W(\frac{1}{3}) + (W(\frac{1}{2}) - W(\frac{1}{3})) + 2(W(\frac{3}{4}) - W(\frac{1}{2})) = \int_0^1 f(t)dW(t)$$

where

$$f(t) = \begin{cases} \frac{4}{3}, & 0 \leq t < \frac{1}{3} \\ 1, & \frac{1}{3} \leq t < \frac{1}{2} \\ 2, & \frac{1}{2} \leq t < \frac{3}{4} \\ 0, & \frac{3}{4} \leq t \leq 1. \end{cases}$$

(b) Since $E[X] = 0$,

$$\mathrm{Var}(X) = \int_0^1 f^2(t)dt = \frac{16}{9} \cdot \frac{1}{3} + 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{4} = \frac{95}{54}.$$

2. (The Vasicek interest rate model) Suppose

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t),\ 0 \leq t \leq T$$

where α, β, and σ are positive constants. Find the distribution of the average rate $\frac{1}{T}\int_0^T R(t)dt$.

Solution. Set $R(t) = X(t) + c(t)$ so that

$$dX(t) + dc(t) = (\alpha - \beta X(t) - \beta c(t))dt + \sigma dW(t)$$

and, furthermore, assume

$$dc(t) = (\alpha - \beta c(t))dt \text{ and } c(0) = 0.$$

From this

$$c(t) = \frac{\alpha}{\beta}(1 - e^{-\beta t})$$

and

$$dX(t) = -\beta X(t)dt + \sigma dW(t).$$

Hence

$$X(t) = e^{-\beta t}R(0) + \sigma e^{-\beta t}\int_0^t e^{\beta u}dW(u)$$

and it follows that

$$E\left[X(t)\right] = e^{-\beta t}R(0)$$

and

$$\mathrm{Cov}(X(s), X(t)) = \sigma^2 e^{-\beta(s+t)}E\left[\int_0^s e^{\beta u}dW(u)\int_0^t e^{\beta u}dW(u)\right]$$

$$= \sigma^2 e^{-\beta(s+t)}\int_0^{\min(s,t)} e^{2\beta u}du = \frac{\sigma^2 e^{-\beta(s+t)}}{2\beta}(e^{2\beta\min(s,t)} - 1).$$

We now set

$$Y = \int_0^T X(s)ds.$$

Clearly,

$$E\left[Y\right] = \frac{R(0)}{\beta}(1 - e^{-\beta T})$$

and

$$\mathrm{Var}(Y) = \mathrm{Cov}(\int_0^T X(s)ds, \int_0^T X(t)dt) = \int_0^T\int_0^T \mathrm{Cov}(X(s), X(t))dsdt$$

$$= \frac{\sigma^2}{2\beta}\int_0^T\int_0^T e^{-\beta(s+t)}(e^{2\beta\min(s,t)} - 1)dsdt$$

$$= \frac{\sigma^2}{2\beta^3}(2\beta T - 3 + 4e^{-\beta T} - e^{-2\beta T}).$$

Since

$$\int_0^T c(t)dt = \frac{\alpha}{\beta}(T + \frac{1}{\beta}e^{-\beta T} - \frac{1}{\beta})$$

we get that the random variable $\frac{1}{T}\int_0^T R(t)dt$ is Gaussian with expectation

$$\frac{R(0)}{\beta T}(1 - e^{-\beta T}) + \frac{\alpha}{\beta}(1 + \frac{1}{\beta T}e^{-\beta T} - \frac{1}{\beta T})$$

and variance

$$\frac{\sigma^2}{2\beta^3 T^2}(2\beta T - 3 + 4e^{-\beta T} - e^{-2\beta T}).$$

3. Let $S_f(t)$ denote the price in foreign currency of a foreign stock and $Q(t)$ the exchange rate, which gives units of domestic currency per unit of foreign currency. Moreover, suppose K and T are strictly positive real numbers and consider a European derivative which pays the amount

$$Y = (S_f(T) - K)^+$$

in domestic currency at maturity T. The domestic and foreign interest rates are constant and denoted by r and r_f, respectively. Derive the price $\Pi_Y(0)$ in domestic currency of the derivative at time 0 under the following condition:

There exists a Brownian motion $W = (W_1, W_2)$ in the plane such that

$$\begin{cases} dS_f(t) = S_f(t)(\alpha_{S_f}dt + \sigma_{11}dW_1(t) + \sigma_{12}dW_2(t)) \\ dQ(t) = Q(t)(\alpha_Q dt + \sigma_{21}dW_1(t) + \sigma_{22}dW_2(t)) \end{cases}$$

where the volatility matrix $(\sigma_{ik})_{1 \le i,k \le 2}$ is deterministic and invertible and α_{S_f} and α_Q are real constants.

Solution. Introducing $\sigma_i = [\sigma_{i1}\ \sigma_{i2}]$, $i = 1, 2$, we have

$$\begin{cases} dS_f(t) = S_f(t)(\alpha_{S_f}dt + \sigma_1 \cdot dW(t)) \\ dQ(t) = Q(t)(\alpha_Q dt + \sigma_2 \cdot dW(t)) \end{cases}$$

Let $B_f(t) = B_f(0)e^{r_f t}$ be the the foreign bond price at time t and, for simplicity, assume $B_f(0) = 1$. Set $S(t) = Q(t)S_f(t)$ and $U(t) = Q(t)B_f(t)$, $t \geq 0$. Then $(S(t))_{0\leq t\leq T}$ and $(U(t))_{0\leq t\leq T}$ are the price processes of domestic traded assets and

$$\begin{cases} dS(t) = S(t)(()dt + (\sigma_1 + \sigma_2)\cdot dW(t)) \\ dU(t) = U(t)(()dt + \sigma_2 \cdot dW(t)). \end{cases}$$

for appropriate drift coefficients which need not be specified here.

Changing to Brownian motion $\tilde{W}$ under the domestic risk-neutral measure, we get

$$\begin{cases} dS(t) = S(t)(rdt + (\sigma_1 + \sigma_2)\cdot d\tilde{W}(t)) \\ dU(t) = U(t)(rdt + \sigma_2 \cdot d\tilde{W}(t)). \end{cases}$$

Moreover, since

$$Y = (B_f(T)\frac{S(T)}{U(T)} - K)^+$$

we have

$$\Pi_Y(0) = e^{-rT}\tilde{E}\left[(\frac{B_f(T)S(T)}{U(T)} - K)^+\right].$$

Here

$$\frac{S(t)}{U(t)} = S_f(0)e^{\frac{1}{2}(|\sigma_2|^2 - |\sigma_1+\sigma_2|^2)t + \sigma_1\cdot\tilde{W}(t)}$$

and we get

$$\Pi_Y(0) = e^{-rT}\tilde{E}\left[(B_f(T)S_f(0)e^{\frac{1}{2}(-|\sigma_1|^2 - 2\sigma_1\cdot\sigma_2)T + \sigma_1\cdot\tilde{W}(T)} - K)^+\right]$$

$$= e^{-rT}\tilde{E}\left[(B_f(T)e^{-(r+\sigma_1\cdot\sigma_2)T}S_f(0)e^{(r-\frac{1}{2}|\sigma_1|^2)T + \sigma_1\cdot\tilde{W}(T)} - K)^+\right].$$

Thus, if

$$c(0,s,K,T) = s\Phi(\frac{\ln\frac{s}{K} + (r + \frac{|\sigma_1|^2}{2})T}{\mid\sigma_1\mid\sqrt{T}}) - Ke^{-rT}\Phi(\frac{\ln\frac{s}{K} + (r - \frac{|\sigma_1|^2}{2})T}{\mid\sigma_1\mid\sqrt{T}}).$$

it follows that

$$\Pi_Y(0) = c(0, B_f(T)e^{-(r+\sigma_1\cdot\sigma_2)T}S_f(0), K, T)$$

$$= c(0, e^{(r_f - r - \sigma_1\cdot\sigma_2)T}S_f(0), K, T).$$

4. Let $W = (W(t))_{t \geq 0}$ be a Brownian motion and let $(\mathcal{F}(t))_{t \geq 0}$ be a filtration for this Brownian motion. Show that W is a Markov process.

5. Use the Itô-Doeblin formula to derivate the Black-Scholes-Merton partial differential equation for the price of a European call option.