Stochastic Calculus Financial Derivatives and PDE's

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Chapter 1

Probability spaces

1.1 σ-algebras and information

We begin with some notation and terminology. The symbol Ω denotes a generic non-empty set; the **power of Ω**, denoted by 2^Ω, is the set of all subsets of Ω. If the number of elements in the set Ω is $M \in \mathbb{N}$, we say that Ω is **finite**. If Ω contains an infinite number of elements and there exists a bijection $\Omega \leftrightarrow \mathbb{N}$, we say that Ω is **countably infinite**. If Ω is neither finite nor countably infinite, we say that it is **uncountable**. An example of uncountable set is the set $\mathbb{R}$ of real numbers. When Ω is finite we write $\Omega = \{\omega_1, \omega_2, \dots, \omega_M\}$, or $\Omega = \{\omega_k\}_{k=1,\dots,M}$. If Ω is countably infinite we write $\Omega = \{\omega_k\}_{k\in\mathbb{N}}$. Note that for a finite set Ω with M elements, the power set contains 2^M elements. For instance, if $\Omega = \{\heartsuit, 1, \$\}$, then

$$2^\Omega = \{\emptyset, \{\heartsuit\}, \{1\}, \{\$\}, \{\heartsuit, 1\}, \{\heartsuit, \$\}, \{1, \$\}, \{\heartsuit, 1, \$\} = \Omega\},$$

which contains $2^3 = 8$ elements. Here $\emptyset$ denotes the **empty set**, which by definition is a subset of all sets.

Within the applications in probability theory, the elements $\omega \in \Omega$ are called **sample points** and represent the possible outcomes of a given experiment (or trial), while the subsets of Ω correspond to **events** which may occur in the experiment. For instance, if the experiment consists in throwing a dice, then $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $A = \{2, 4, 6\}$ identifies the event that the result of the experiment is an even number. Now let $\Omega = \Omega_N$,

$$\Omega_N = \{(\gamma_1, \dots, \gamma_N),\ \gamma_k \in \{H, T\}\} = \{H, T\}^N, \tag{1.1}$$

where H stands for "head" and T stands for "tail". Each element $\omega = (\gamma_1, \dots, \gamma_N) \in \Omega_N$ is called a **N-toss** and represents a possible outcome for the experiment "tossing a coin N consecutive times". Evidently, Ω_N contains 2^N elements and so 2^{Ω_N} contains 2^{2^N} elements. We show in Appendix 1.A at the end of the present chapter that Ω_∞—the sample space for the experiment "tossing a coin infinitely many times"—is uncountable.

A collection of events, e.g., $\{A_1, A_2, \dots\} \subset 2^\Omega$, is also called **information**. To understand the meaning of this terminology, suppose that the experiment has been performed and we observe that the events $A_1, A_2, \dots$ have occurred. We may then use this information to restrict the possible outcomes of the experiment. For instance, if we are told that in a 5-toss the following two events have occurred:

1. there are more heads than tails
2. the first toss is a tail

then we may conclude that the result of the 5-toss is one of

$$(T, H, H, H, H),\ (T, T, H, H, H),\ (T, H, T, H, H),\ (T, H, H, T, H),\ (T, H, H, H, T).$$

If in addition we are given the information that

3. the last toss is a tail,

then we conclude that the result of the 5-toss is (T, H, H, H, T).

The power set of the sample space provides the **total accessible information** and represents the collection of all the events that can be **resolved** (i.e., whose occurrence can be inferred) by knowing the outcome of the experiment. For an uncountable sample space, the total accessible information is huge and it is typically replaced by a subclass of events $\mathcal{F} \subset 2^{\Omega}$, which is imposed to form a σ-algebra.

Definition 1.1. *A collection $\mathcal{F} \subseteq 2^{\Omega}$ of subsets of Ω is called a* σ-**algebra** *(or* σ-**field***) on Ω if*

(i) $\emptyset \in \mathcal{F}$;

(ii) $A \in \mathcal{F} \Rightarrow A^c := \{\omega \in \Omega : \omega \notin A\} \in \mathcal{F}$;

(iii) $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$, *for all* $\{A_k\}_{k\in\mathbb{N}} \subseteq \mathcal{F}$.

If $\mathcal{G}$ is another σ-algebra on Ω and $\mathcal{G} \subset \mathcal{F}$, we say that $\mathcal{G}$ is a **sub-σ-algebra** *of $\mathcal{F}$.*

Exercise 1.1. *Let $\mathcal{F}$ be a σ-algebra. Show that $\Omega \in \mathcal{F}$ and that $\cap_{k\in\mathbb{N}} A_k \in \mathcal{F}$, for all countable families $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ of events.*

Exercise 1.2. *Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ be the sample space of a dice roll. Which of the following sets of events are σ-algebras on Ω?*

1. $\{\emptyset, \{1\}, \{2, 3, 4, 5, 6\}, \Omega\}$,
2. $\{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}, \{3, 4, 5, 6\}, \Omega\}$,
3. $\{\emptyset, \{2\}, \{1, 3, 4, 5, 6\}, \Omega\}$.

Exercise 1.3 (•). *Prove that the intersection of any number of σ-algebras (including uncountably many) is a σ-algebra. Show with a counterexample that the union of two σ-algebras is not necessarily a σ-algebra.*

Remark 1.1 (Notation)**.** The letter A is used to denote a generic event in the σ-algebra. If we need to consider two such events, we denote them by A, B, while N generic events are denoted $A_1, \dots, A_N$.

Let us comment on Definition 1.1. The empty set represents the "nothing happens" event, while A^c represents the "A does not occur" event. Given a finite number $A_1, \dots, A_N$ of events, their union is the event that at least one of the events $A_1, \dots, A_N$ occurs, while their intersection is the event that all events $A_1, \dots, A_N$ occur. The reason to include the countable union/intersection of events in our analysis is to make it possible to "take limits" without crossing the boundaries of the theory. Of course, unions and intersections of infinitely many sets only matter when Ω is not finite.

The smallest σ-algebra on Ω is $\mathcal{F} = \{\emptyset, \Omega\}$, which is called the **trivial σ-algebra**. There is no relevant information contained in the trivial σ-algebra. The largest possible σ-algebra is $\mathcal{F} = 2^\Omega$, which contains the full amount of accessible information. When Ω is countable, it is common to pick 2^Ω as σ-algebra of events. However, as already mentioned, when Ω is uncountable this choice is unwise. A useful procedure to construct a σ-algebra of events when Ω is uncountable is the following. First we select a collection of events (i.e., subsets of Ω), which for some reason we regard as fundamental. Let $\mathcal{O}$ denote this collection of events. Then we introduce the smallest σ-algebra containing $\mathcal{O}$, which is formally defined as follows.

Definition 1.2. *Let $\mathcal{O} \subset 2^\Omega$. The* **$\sigma$-algebra generated by** *$\mathcal{O}$ is*

$$\mathcal{F}_{\mathcal{O}} = \bigcap \left\{ \mathcal{F} : \mathcal{F} \subset 2^\Omega \text{ is a } \sigma\text{-algebra and } \mathcal{O} \subseteq \mathcal{F} \right\},$$

i.e., $\mathcal{F}_{\mathcal{O}}$ is the smallest σ-algebra on Ω containing $\mathcal{O}$.

Recall that the intersection of any number of σ-algebras is still a σ-algebra, see Exercise 1.3, hence $\mathcal{F}_{\mathcal{O}}$ is a well-defined σ-algebra. For example, let $\Omega = \mathbb{R}^d$ and let $\mathcal{O}$ be the collection of all open balls:

$$\mathcal{O} = \{B_x(R)\}_{R>0, x\in\mathbb{R}^d}, \quad \text{where } B_x(R) = \{y \in \mathbb{R}^d : |x-y| < R\}.$$

The σ-algebra generated by $\mathcal{O}$ is called **Borel σ-algebra** and denoted $\mathcal{B}(\mathbb{R}^d)$. The elements of $\mathcal{B}(\mathbb{R}^d)$ are called **Borel sets**.

Remark 1.2 (Notation)**.** The Borel σ-algebra $\mathcal{B}(\mathbb{R})$ plays an important role in these notes, so we shall use a specific notation for its elements. A generic event in the σ-algebra $\mathcal{B}(\mathbb{R})$ will be denoted U; if we need to consider two such events we denote them by U, V, while N generic Borel sets of $\mathbb{R}$ will be denoted $U_1, \dots U_N$. Recall that for general σ-algebras, the notation used is the one indicated in Remark 1.1.

The σ-algebra generated by $\mathcal{O}$ has a particular simple form when $\mathcal{O}$ is a partition of Ω.

Definition 1.3. *Let $I \subseteq \mathbb{N}$. A collection $\mathcal{O} = \{A_k\}_{k\in I}$ of non-empty subsets of Ω is called a* **partition** *of Ω if*

(i) the events $\{A_k\}_{k\in I}$ are **disjoint***, i.e., $A_j \cap A_k = \emptyset$, for $j \neq k$;*

(ii) $\bigcup_{k \in I} A_k = \Omega$.

If I is a finite set we call $\mathcal{O}$ a **finite partition** *of Ω.*

Note that any countable sample space $\Omega = \{\omega_k\}_{k \in \mathbb{N}}$ is partitioned by the **atomic events** $A_k = \{\omega_k\}$, where $\{\omega_k\}$ identifies the event that the result of the experiment is exactly ω_k.

Exercise 1.4. *Show that when $\mathcal{O}$ is a partition, the σ-algebra generated by $\mathcal{O}$ is given by the set of all subsets of Ω which can be written as the union of sets in the partition $\mathcal{O}$ (plus the empty set, of course).*

Exercise 1.5. *Find the partition of $\Omega = \{1, 2, 3, 4, 5, 6\}$ that generates the σ-algebra 2 in Exercise 1.2.*

1.2 Probability measure

To any event $A \in \mathcal{F}$ we want to associate a probability that A occurred.

Definition 1.4. *Let $\mathcal{F}$ be a σ-algebra on Ω. A* **probability measure** *is a function*

$$\mathbb{P} : \mathcal{F} \to [0, 1]$$

such that

(i) $\mathbb{P}(\Omega) = 1$*;*

(ii) for any countable collection of disjoint events $\{A_k\}_{k \in \mathbb{N}} \subseteq \mathcal{F}$, we have

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(A_k).$$

A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space***.*

The quantity $\mathbb{P}(A)$ is called **probability of the event A**; if $\mathbb{P}(A) = 1$ we say that the event A occurs **almost surely**, which is sometimes shortened by **a.s.**; if $\mathbb{P}(A) = 0$ we say that A is a **null set**. In general, the elements of $\mathcal{F}$ with probability zero or one will be called **trivial events** (as trivial is the information that they provide). For instance, $\mathbb{P}(\Omega) = 1$, i.e., the probability that "something happens" is one, and $\mathbb{P}(\emptyset) = \mathbb{P}(\Omega^c) = 1 - \mathbb{P}(\Omega) = 0$, i.e., the probability the "nothing happens" is zero.

Exercise 1.6 (•). *Prove the following properties:*

1. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$*;*

2. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$*;*

3. *If $A \subset B$, then $\mathbb{P}(A) < \mathbb{P}(B)$.*

Exercise 1.7 (Continuity of probability measures ($\star$)). *Let $\{A_k\}_{k\in\mathbb{N}} \subseteq \mathcal{F}$ such that $A_k \subseteq A_{k+1}$, for all $k \in \mathbb{N}$. Let $A = \cup_k A_k$. Show that*

$$\lim_{k\to\infty} \mathbb{P}(A_k) = \mathbb{P}(A).$$

Similarly, if now $\{A_k\}_{k\in\mathbb{N}} \subseteq \mathcal{F}$ such that $A_{k+1} \subseteq A_k$, for all $k \in \mathbb{N}$ and $A = \cap_k A_k$, show that

$$\lim_{k\to\infty} \mathbb{P}(A_k) = \mathbb{P}(A).$$

Let us see some examples of probability space.

- There is only one probability measure defined on the trivial σ-algebra, namely $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$.

- In this example we describe the general procedure to construct a probability space on a countable sample space $\Omega = \{\omega_k\}_{k\in\mathbb{N}}$. We pick $\mathcal{F} = 2^\Omega$ and let $0 \leq p_k \leq 1$, $k \in \mathbb{N}$, be real numbers such that

  $$\sum_{k=1}^{\infty} p_k = 1.$$

  We introduce a probability measure on $\mathcal{F}$ by first defining the probability of the atomic events $\{\omega_1\}, \{\omega_2\}, \dots$ as

  $$\mathbb{P}(\{\omega_k\}) = p_k, \quad k \in \mathbb{N}.$$

  Since every (non-empty) subset of Ω can be written as the disjoint union of atomic events, then the probability of any event can be inferred using the property (ii) in the definition of probability measure, e.g.,

  $$\begin{aligned}\mathbb{P}(\{\omega_1, \omega_3, \omega_5\}) &= \mathbb{P}(\{\omega_1\} \cup \{\omega_3\} \cup \{\omega_5\}) \\ &= \mathbb{P}(\{\omega_1\}) + \mathbb{P}(\{\omega_3\}) + \mathbb{P}(\{\omega_5\}) = p_1 + p_3 + p_5.\end{aligned}$$

  In general we define

  $$\mathbb{P}(A) = \sum_{k:\omega_k\in A} p_k, \quad A \in 2^\Omega,$$

  while $\mathbb{P}(\emptyset) = 0$.

- As a special case of the previous example we now introduce a probability measure on the sample space Ω_N of the N-coin tosses experiment. Given $0 < p < 1$ and $\omega \in \Omega_N$, we define the probability of the atomic event $\{\omega\}$ as

  $$\mathbb{P}(\{\omega\}) = p^{N_H(\omega)}(1-p)^{N_T(\omega)}, \tag{1.2}$$

  where $N_H(\omega)$ is the number of H in ω and $N_T(\omega)$ is the number of T in ω ($N_H(\omega) + N_T(\omega) = N$). We say that the coin is **fair** if $p = 1/2$. The probability of a generic event $A \in \mathcal{F} = 2^{\Omega_N}$ is obtained by adding up the probabilities of the atomic events whose disjoint union forms the event A. For instance, assume $N = 3$ and consider the event

"The first and the second toss are equal".

Denote by $A \in \mathcal{F}$ the set corresponding to this event. Then clearly A is the (disjoint) union of the atomic events

$$\{(H,H,H)\},\ \{(H,H,T)\},\ \{(T,T,T)\},\ \{(T,T,H)\}.$$

Hence,

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(\{(H,H,H)\}) + \mathbb{P}(\{(H,H,T)\}) + \mathbb{P}(\{(T,T,T)\}) + \mathbb{P}(\{(T,T,H)\}) \\ &= p^3 + p^2(1-p) + (1-p)^3 + (1-p)^2 p = 2p^2 - 2p + 1.\end{aligned}$$

- Let $f : \mathbb{R} \to [0,\infty)$ be a measurable function[1] such that

$$\int_{\mathbb{R}} f(x)\, dx = 1.$$

Then

$$\mathbb{P}(U) = \int_U f(x)\, dx, \tag{1.3}$$

defines a probability measure on $\mathcal{B}(\mathbb{R})$.

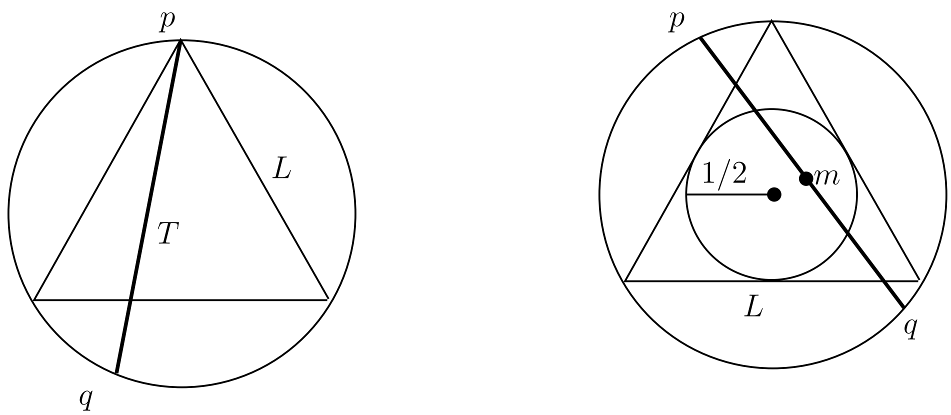
Remark 1.3 (Riemann vs. Lebesgue integral)**.** The integral in (1.3) must be understood in the Lebesgue sense, since we are integrating a general measurable function over a general Borel set. If f is a sufficiently regular (say, continuous) function, and $U = (a,b) \subset \mathbb{R}$ is an interval, then the integral in (1.3) can be understood in the Riemann sense. Although this last case is sufficient for most applications in finance, all integrals in these notes should be understood in the Lebesgue sense, unless otherwise stated. The knowledge of Lebesgue integration theory is however not required for our purposes.

Exercise 1.8 (•)**.** *Prove that* $\sum_{\omega\in\Omega_N} \mathbb{P}(\{\omega\}) = 1$*, where* $\mathbb{P}(\{\omega\})$ *is given by* (1.2).

Equivalent probability measures

A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ and if we change one element of this triple we get a different probability space. The most interesting case is when a new probability measure is introduced. Let us first show with an example (known as **Bertrand's paradox**) that there might not be just one "reasonable" definition of probability measure associated to a given experiment. We perform an experiment whose result is a pair of points p, q on the unit circle C (e.g., throw two balls in a *roulette*). The sample space for this experiment is $\Omega = \{(p,q) : p, q \in C\}$. Let T be the length of the chord joining p and q. Now let L be

[1] See Section 2.1 for the definition of measurable function.



(a) $\mathbb{P}(A) = 1/3$ (b) $\mathbb{P}(A) = 1/4$

Figure 1.1: The Bertrand paradox. The length T of the cord $\overline{pq}$ is greater then L.

the length of the side of a equilateral triangle inscribed in the circle C. Note that all such triangles are obtained one from another by a rotation around the center of the circle and all have the same sides length L. Consider the event $A = \{(p, q) \in \Omega : T > L\}$. What is a reasonable definition for $\mathbb{P}(A)$? From one hand we can suppose that one vertex of the triangle is p, and thus T will be greater than L if and only if the point q lies on the arch of the circle between the two vertexes of the triangle different from p, see Figure 1.1(a). Since the length of such arc is 1/3 the perimeter of the circle, then it is reasonable to define $\mathbb{P}(A) = 1/3$. On the other hand, it is simple to see that $T > L$ whenever the midpoint m of the chord lies within a circle of radius 1/2 concentric to C, see Figure 1.1(b). Since the area of the interior circle is 1/4 the area of C, we are led to define $\mathbb{P}(A) = 1/4$.

Whenever two probabilities are defined for the same experiment, we shall require them to be equivalent, in the following sense.

Definition 1.5. *Given two probability spaces* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$*, the probability measures* $\mathbb{P}$ *and* $\widetilde{\mathbb{P}}$ *are said to be* **equivalent** *if* $\mathbb{P}(A) = 0 \Leftrightarrow \widetilde{\mathbb{P}}(A) = 0$*.*

A complete characterization of the probability measures $\widetilde{\mathbb{P}}$ equivalent to a given $\mathbb{P}$ will be given in Theorem 3.3.

Conditional probability

It might be that the occurrence of an event B makes the occurrence of another event A more or less likely. For instance, the probability of the event $A = \{$the first two tosses of a fair coin are both head$\}$ is 1/4; however if know that the first toss is a tail, then $\mathbb{P}(A) = 0$, while $\mathbb{P}(A) = 1/2$ if we know that the first toss is a head. This leads to the important definition of conditional probability.

Definition 1.6. *Given two events A, B such that $\mathbb{P}(B) > 0$, the* **conditional probability** *of A given B is defined as*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

To justify this definition, let $\mathcal{F}_B = \{A \cap B\}_{A \in \mathcal{F}}$, and set

$$\mathbb{P}_B(\cdot) = \mathbb{P}(\cdot|B). \tag{1.4}$$

Then $(B, \mathcal{F}_B, \mathbb{P}_B)$ is a probability space in which the events that cannot occur simultaneously with B are null events. Therefore it is natural to regard $(B, \mathcal{F}_B, \mathbb{P}_B)$ as the restriction of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ when B has occurred.

If $\mathbb{P}(A|B) = \mathbb{P}(A)$, the two events are said to be independent. The interpretation is the following: if two events A, B are independent, then the occurrence of the event B does not change the probability that A occurred. By Definition 1.4 we obtain the following equivalent characterization of independent events.

Definition 1.7. *Two events A, B are said to be* **independent** *if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. In general, the events $A_1, \dots, A_N$ ($N \geq 2$) are said to be independent if, for all $1 \leq k_1 < k_2 < \dots < k_m \leq N$, we have*

$$\mathbb{P}(A_{k_1} \cap \dots \cap A_{k_m}) = \prod_{j=1}^{m} \mathbb{P}(A_{k_j}).$$

Two σ-algebras $\mathcal{F}, \mathcal{G}$ are said to be independent if A and B are independent, for all $A \in \mathcal{G}$ and $B \in \mathcal{F}$. In general the σ-algebras $\mathcal{F}_1, \dots, \mathcal{F}_N$ ($N \geq 2$) are said to be independent if $A_1, A_2, \dots, A_N$ are independent events, for all $A_1 \in \mathcal{F}_1, \dots, A_N \in \mathcal{F}_n$.

Note that if $\mathcal{F}, \mathcal{G}$ are two independent σ-algebras and $A \in \mathcal{F} \cap \mathcal{G}$, then A is trivial. In fact, if $A \in \mathcal{F} \cap \mathcal{G}$, then $\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A)^2$. Hence $\mathbb{P}(A) = 0$ or 1. The interpretation of this simple remark is that independent σ-algebras carry distinct information.

Exercise 1.9 (•). *Given a fair coin and assuming N is odd, consider the following two events $A, B \in \Omega_N$:*

$A =$ "the number of heads is greater than the number of tails",

$B =$ "The first toss is a head".

Use your intuition to guess whether the two events are independent; then verify your answer numerically (e.g., using Mathematica*).*

1.3 Filtered probability spaces

Consider again the N-coin tosses probability space. Let A_H be the event that the first toss is a head and A_T the event that it is a tail. Clearly $A_T = A_H^c$ and the σ-algebra $\mathcal{F}_1$ generated by the partition $\{A_H, A_T\}$ is $\mathcal{F}_1 = \{A_H, A_T, \Omega, \emptyset\}$. Now let A_{HH} be the event that the first 2 tosses are heads, and similarly define A_{HT}, A_{TH}, A_{TT}. These four events form a partition of Ω_N and they generate a σ-algebra $\mathcal{F}_2$ as indicated in Exercise 1.4. Clearly, $\mathcal{F}_1 \subset \mathcal{F}_2$. Going on with three tosses, four tosses, and so on, until we complete the N-toss, we construct a sequence

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_N = 2^{\Omega_N}$$

of σ-algebras. The σ-algebra $\mathcal{F}_k$ contains all the events of the experiment that depend on (i.e., which are resolved by) the first k tosses. The family $\{\mathcal{F}_k\}_{k=1,\dots,N}$ of σ-algebras is an example of filtration.

Definition 1.8. *A* **filtration** *is a one parameter family* $\{\mathcal{F}(t)\}_{t\geq 0}$ *of σ-algebras such that* $\mathcal{F}(t) \subseteq \mathcal{F}$ *for all* $t \geq 0$ *and* $\mathcal{F}(s) \subseteq \mathcal{F}(t)$ *for all* $s \leq t$*. A quadruple* $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ *is called a* **filtered probability space**.

In our applications t stands for the time variable and filtrations are associated to experiments in which "information accumulates with time". For instance, in the example given above, the more times we toss the coin, the higher is the number of events which are resolved by the experiment, i.e., the more information becomes accessible.

1.A Appendix: The ∞-coin tosses probability space

In this appendix we outline the construction of the probability space for the ∞-coin tosses experiment. The sample space is

$$\Omega_\infty = \{\omega = (\gamma_n)_{n\in\mathbb{N}},\ \gamma_n \in \{H, T\}\}.$$

Let us show first that Ω is uncountable. We use the well-known **Cantor diagonal argument**. Suppose that Ω_∞ is countable and write

$$\Omega_\infty = \{\omega_k\}_{k\in\mathbb{N}}. \tag{1.5}$$

Each $\omega_k \in \Omega_\infty$ is a sequence of infinite tosses, which we write as $\omega_k = (\gamma_j^{(k)})_{j\in\mathbb{N}}$, where $\gamma_j^{(k)}$ is either H or T, for all $j \in \mathbb{N}$ and for each fixed $k \in \mathbb{N}$. Note that $(\gamma_j^{(k)})_{j,k\in\mathbb{N}}$ is an "$\infty \times \infty$" matrix. Now consider the ∞-toss corresponding to the diagonal of this matrix, that is

$$\bar{\omega} = (\bar{\gamma}_m)_{m\in\mathbb{N}}, \quad \bar{\gamma}_m = \gamma_m^{(m)}, \text{ for all } m \in \mathbb{N}.$$

Finally consider the ∞-toss ω which is obtained by changing each single toss of $\bar{\omega}$, that is to say

$$\omega = (\gamma_m)_{m\in\mathbb{N}}, \quad \text{where } \gamma_m = H \text{ if } \bar{\gamma}_m = T, \text{ and } \gamma_m = T \text{ if } \bar{\gamma}_m = H, \text{ for all } m \in \mathbb{N}.$$

It is clear that the ∞-toss ω does not belong to the set (1.5). In fact, by construction, the first toss of ω is different from the first toss of ω_1, the second toss of ω is different from the second toss of ω_2, ..., the n^{th} toss of ω is different from the n^{th} toss of ω_n, and so on, so that each ∞-toss in (1.5) is different from ω. We conclude that the elements of Ω_∞ cannot be listed as they were comprising a countable set.

Now, let $N \in \mathbb{N}$ and recall that the sample space Ω_N for the N-tosses experiment is given by (1.1). For each $\bar{\omega} = (\bar{\gamma}_1, \dots, \bar{\gamma}_N) \in \Omega_N$ we define the event $A_{\bar{\omega}} \subset \Omega_\infty$ by

$$A_{\bar{\omega}} = \{\omega = (\gamma_n)_{n\in\mathbb{N}} : \gamma_j = \bar{\gamma}_j,\ j = 1, \dots, N\},$$

i.e., the event that the first N tosses in a ∞-toss be equal to $(\bar{\gamma}_1, \dots, \bar{\gamma}_N)$. Define the probability of this event as the probability of the N-toss $\bar{\omega}$, that is

$$\mathbb{P}_0(A_{\bar{\omega}}) = p^{N_H(\bar{\omega})}(1-p)^{N_T(\bar{\omega})},$$

where $0 < p < 1$, $N_H(\bar{\omega})$ is the number of heads in the N-toss $\bar{\omega}$ and $N_T(\bar{\omega}) = N - N_H(\bar{\omega})$ is the number of tails in $\bar{\omega}$, see (1.2). Next consider the family of events

$$\mathcal{U}_N = \{A_{\bar{\omega}}\}_{\bar{\omega}\in\Omega_N} \subset 2^{\Omega_\infty}.$$

It is clear that $\mathcal{U}_N$ is, for each fixed $N \in \mathbb{N}$, a partition of Ω_∞. Hence the σ-algebra $\mathcal{F}_N = \mathcal{F}_{\mathcal{U}_N}$ is generated according to Exercise 1.4. Note that $\mathcal{F}_N$ contains all events of Ω_∞ that are resolved by the first N tosses. Moreover $\mathcal{F}_N \subset \mathcal{F}_{N+1}$, that is to say, $\{\mathcal{F}_N\}_{N\in\mathbb{N}}$ is a filtration. Since $\mathbb{P}_0$ is defined for all $A_{\bar{\omega}} \in \mathcal{U}_N$, then it can be extended uniquely to the entire $\mathcal{F}_N$, because each element $A \in \mathcal{F}_N$ is the disjoint union of events of $\mathcal{U}_N$ (see again Exercise 1.4) and therefore the probability of A can be inferred by the property (ii) in the definition of probability measure, see Definition 1.4. But then $\mathbb{P}_0$ extends uniquely to

$$\mathcal{F}_\infty = \bigcup_{N\in\mathbb{N}} \mathcal{F}_N.$$

Hence we have constructed a triple $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P}_0)$. Is this triple a probability space? The answer is *no*, because $\mathcal{F}_\infty$ is *not* a σ-algebra. To see this, let A_k be the event that the k^{th} toss in a infinite sequence of tosses is a head. Clearly $A_k \in \mathcal{F}_k$ for all k and therefore $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{F}_\infty$. Now assume that $\mathcal{F}_\infty$ is a σ-algebra. Then the event $A = \cup_k A_k$ would belong to $\mathcal{F}_\infty$ and therefore also $A^c \in \mathcal{F}_\infty$. The latter holds if and only if there exists $N \in \mathbb{N}$ such that $A^c \in \mathcal{F}_N$. But A^c is the event that all tosses are tails, which of course cannot be resolved by the information $\mathcal{F}_N$ accumulated after just N tosses. We conclude that $\mathcal{F}_\infty$ is not a σ-algebra. In particular, we have shown that $\mathcal{F}_\infty$ is not in general closed with respect to the countable union of its elements. However it is easy to show that $\mathcal{F}_\infty$ is closed with respect to the *finite* union of its elements, and in addition satisfies the properties (i), (ii) in Definition 1.4. This set of properties makes $\mathcal{F}_\infty$ an **algebra**. To complete the construction of the probability space for the ∞-coin tosses experiment, we need the following deep result.

Theorem 1.1 (Caratheódory's theorem). *Let $\mathcal{U}$ be an algebra of subsets of Ω and $\mathbb{P}_0 : \mathcal{U} \to [0,1]$ a map satisfying $\mathbb{P}_0(\Omega) = 1$ and $\mathbb{P}_0(\cup_{i=1}^N A_i) = \sum_{i=1}^N \mathbb{P}_0(A_i)$, for every finite collection $\{A_1, \dots, A_N\} \subset \mathcal{U}$ of disjoint sets*[2]*. Then there exists a unique probability measure $\mathbb{P}$ on $\mathcal{F}_\mathcal{U}$ such that $\mathbb{P}(A) = \mathbb{P}_0(A)$, for all $A \in \mathcal{U}$.*

[2] $\mathbb{P}_0$ is called a **pre-measure**.

Hence the map $\mathbb{P}_0 : \mathcal{F}_\infty \to [0,1]$ defined above extends uniquely to a probability measure $\mathbb{P}$ defined on $\mathcal{F} = \mathcal{F}_{\mathcal{F}_\infty}$. The resulting triple $(\Omega_\infty, \mathcal{F}, \mathbb{P})$ defines the probability space for the ∞-tosses experiment.

1.B Appendix: Solutions to selected problems

Exercise 1.3. Since an event belongs to the intersection of σ-algebras if and only if it belongs to each single σ-algebra, the proof of the first statement is trivial. As an example of two σ-algebras whose union is not a σ algebra, take 1 and 3 of Exercise 1.2.

Exercise 1.6. Since A and A^c are disjoint, we have

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c) \Rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$

To prove 2 we notice that $A \cup B$ is the disjoint union of the sets $A \setminus B$, $B \setminus A$ and $A \cap B$. It follows that

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B).$$

Since A is the disjoint union of $A \cap B$ and $A \setminus B$, we also have

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \setminus B)$$

and similarly

$$\mathbb{P}(B) = \mathbb{P}(B \cap A) + \mathbb{P}(B \setminus A). \tag{1.6}$$

Combining the three identities above yields the result. Moreover, from (1.6) and assuming $A \subset B$, we obtain $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) > \mathbb{P}(A)$, which is claim 3.

Exercise 1.8. Since for all $k = 0, \dots, N$ the number of N-tosses $\omega \in \Omega_N$ having $N_H(\omega) = k$ is given by the binomial coefficient

$$\binom{N}{k} = \frac{N!}{k!(N-k)!},$$

then

$$\sum_{\omega \in \Omega_N} \mathbb{P}(\{\omega\}) = \sum_{\omega \in \Omega_N} p^{N_H(\omega)}(1-p)^{N_T(\omega)} = (1-p)^N \sum_{\omega \in \Omega_N} \left(\frac{p}{1-p}\right)^{N_H(\omega)}$$
$$= (1-p)^N \sum_{k=0}^{N} \binom{N}{k} \left(\frac{p}{1-p}\right)^k.$$

By the binomial theorem, $(1+a)^N = \sum_{k=0}^N \binom{N}{k} a^k$, hence

$$\mathbb{P}(\Omega) = \sum_{\omega \in \Omega_N} \mathbb{P}(\{\omega\}) = (1-p)^N \left(1 + \frac{p}{1-p}\right)^N = 1.$$

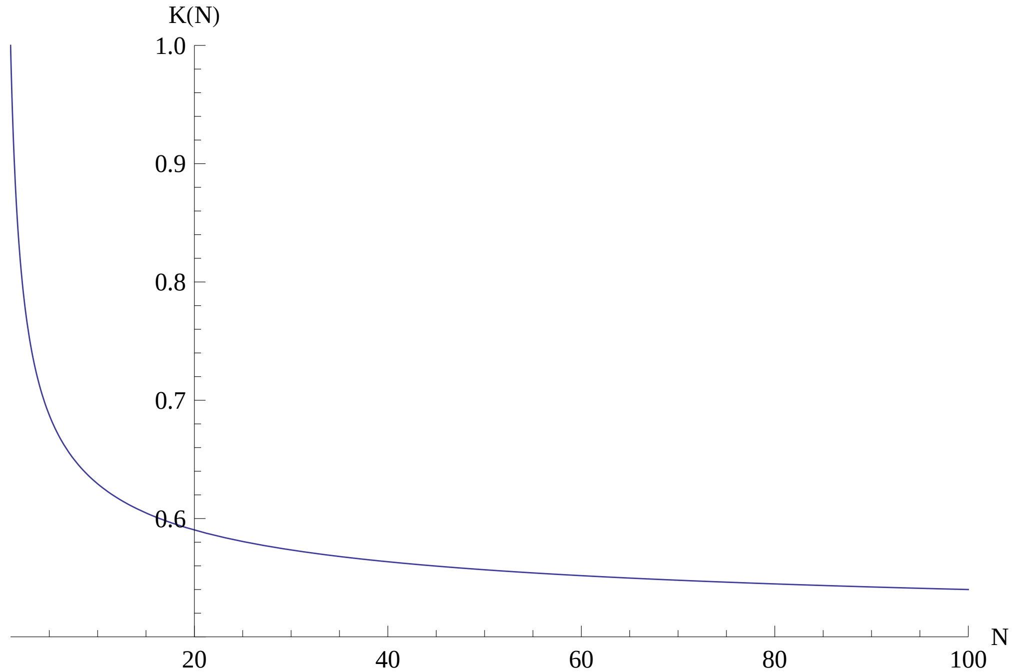


Figure 1.2: A numerical solution of Exercise 1.9 for a generic odd natural number N.

Exercise 1.9. We expect that $\mathbb{P}(A|B) > \mathbb{P}(A)$, that is to say, the first toss being a head increases the probability that the number of heads in the complete N-toss will be larger than the number of tails. To verify this, we first observe that $\mathbb{P}(A) = 1/2$, since N is odd and thus there will be either more heads or more tails in any N-toss. Moreover, $\mathbb{P}(A|B) = \mathbb{P}(C)$, where $C \in \Omega_{N-1}$ is the event that the number of heads in a $(N-1)$-toss is larger or equal to the number of tails. Letting k be the number of heads, $\mathbb{P}(C)$ is the probability that $k \in \{(N-1)/2, \dots, N-1\}$. Since there are $\binom{N-1}{k}$ possible $(N-1)$-tosses with k-heads, then

$$\mathbb{P}(C) = \sum_{k=(N-1)/2}^{N-1} \binom{N-1}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{N-1-k} = \frac{1}{2^{N-1}} \sum_{k=(N-1)/2}^{N-1} \binom{N-1}{k}.$$

Thus proving the statement for a generic odd N is equivalent to prove the inequality

$$K(N) = \frac{1}{2^{N-1}} \sum_{k=(N-1)/2}^{N-1} \binom{N-1}{k} > \frac{1}{2}.$$

A "numerical proof" of this inequality is provided in Figure 1.2. Note that the function $K(N)$ is decreasing and converges to $1/2$ as $N \to \infty$.

Chapter 2

Random variables and stochastic processes

Throughout this chapter we assume that $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ is a given filtered probability space.

2.1 Random variables

In many applications of probability theory, and in financial mathematics in particular, one is more interested in knowing the value attained by quantities that depend on the outcome of the experiment, rather than knowing which specific events have occurred. Such quantities are called random variables.

Definition 2.1. *A map $X : \Omega \to \mathbb{R}$ is called a (real-valued)* **random variable** *if $\{X \in U\} \in \mathcal{F}$, for all $U \in \mathcal{B}(\mathbb{R})$, where*

$$\{X \in U\} = \{\omega \in \Omega : X(\omega) \in U\}$$

is the pre-image of the Borel set U. If there exists $c \in \mathbb{R}$ such that $X(\omega) = c$ almost surely, we say that X is **a deterministic constant***.*

Occasionally we shall also need to consider complex-valued random variables. These are defined as the maps $Z : \Omega \to \mathbb{C}$ of the form $Z = X + iY$, where X, Y are real-valued random variables and i is the imaginary unit ($i^2 = -1$). Similarly a vector valued random variable $X = (X_1, \dots, X_N) : \Omega \to \mathbb{R}^N$ can be defined by simply requiring that each component $X_j : \Omega \to \mathbb{R}$ is a random variable in the sense of Definition 2.1.

Remark 2.1 (Notation)**.** A generic real-valued random variable will be denoted by X. If we need to consider two such random variables we will denote them by X, Y, while N real-valued random variables will be denoted by $X_1, \dots, X_N$. Note that $(X_1, \dots, X_N) : \Omega \to \mathbb{R}^N$ is a vector-valued random variable. The letter Z is used for complex-valued random variables.

Remark 2.2. Equality among random variables is always understood to hold up to a null set. That is to say, $X = Y$ always means $X = Y$ a.s., for all random variables $X, Y : \Omega \to \mathbb{R}$.

Random variables are also called **measurable functions**, but we prefer to use this terminology only when $\Omega = \mathbb{R}$ and $\mathcal{F} = \mathcal{B}(R)$. Measurable functions will be denoted by small Latin letters (e.g., $f, g, \dots$). If X is a random variable and $Y = f(X)$ for some measurable function f, then Y is also a random variable. We denote $\mathbb{P}(X \in U) = \mathbb{P}(\{X \in U\})$ the probability that X takes value in $U \in \mathcal{B}(R)$. Moreover, given two random variables $X, Y : \Omega \to \mathbb{R}$ and the Borel sets U, V, we denote

$$\mathbb{P}(X \in U, Y \in V) = \mathbb{P}(\{X \in U\} \cap \{Y \in V\}),$$

which is the probability that the random variable X takes value in U and Y takes value in V. The generalization to an arbitrary number of random variables is straightforward.

As the value attained by X depends on the result of the experiment, random variables carry information, i.e., upon knowing the value attained by X we know something about the outcome ω of the experiment. For instance, if $X(\omega) = (-1)^\omega$, where ω is the result of tossing a dice, and if we are told that X takes value 1, then we infer immediately that the dice roll is even. The information carried by a random variable X forms the σ-algebra generated by X, whose precise definition is the following.

Definition 2.2. *Let $X : \Omega \to \mathbb{R}$ be a random variable. The* **σ-algebra generated by X** *is the collection $\sigma(X) \subseteq \mathcal{F}$ of events given by*

$$\sigma(X) = \{A \in \mathcal{F} : A = \{X \in U\},\ \textit{for some } U \in \mathcal{B}(\mathbb{R})\}.$$

If $\mathcal{G} \subseteq \mathcal{F}$ is another σ-algebra of subsets of Ω and $\sigma(X) \subseteq \mathcal{G}$, we say that X is **$\mathcal{G}$-measurable**. *If $Y : \Omega \to \mathbb{R}$ is another random variable and $\sigma(Y) \subseteq \sigma(X)$, we say that Y is* **X-measurable**

Exercise 2.1 (•). *Prove that $\sigma(X)$ is a σ-algebra.*

Thus $\sigma(X)$ contains all the events that are resolved by knowing the value of X. The interpretation of X being $\mathcal{G}$-measurable is that the information contained in $\mathcal{G}$ suffices to determine the value taken by X in the experiment. Note that the σ-algebra generated by a deterministic constant consists of trivial events only.

Definition 2.3. *The σ-algebra $\sigma(X, Y)$ generated by two random variables $X, Y : \Omega \to \mathbb{R}$ is the smallest σ-algebra containing $\sigma(X) \cup \sigma(Y)$, that is to say*[1] *$\sigma(X, Y) = \mathcal{F}_{\mathcal{O}}$, where $\mathcal{O} = \sigma(X) \cup \sigma(Y)$, and similarly for any number of random variables.*

If Y is X-measurable then $\sigma(X, Y) = \sigma(X)$, i.e., the random variable Y does not add any new information to the one already contained in X. Clearly, if $Y = f(X)$ for some measurable function f, then Y is X-measurable. It can be shown that the opposite is also

[1] See Definition 1.2.

true: if $\sigma(Y) \subseteq \sigma(X)$, then there exists a measurable function f such that $Y = f(X)$ (see Prop. 3 in [18]). The other extreme is when X and Y carry distinct information, i.e., when $\sigma(X) \cap \sigma(Y)$ consists of trivial events only. This occurs in particular when the two random variables are independent.

Definition 2.4. *Let $X : \Omega \to \mathbb{R}$ be a random variable and $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. We say that X is independent of $\mathcal{G}$ if $\sigma(X)$ and $\mathcal{G}$ are independent in the sense of Definition 1.7. Two random variables $X, Y : \Omega \to \mathbb{R}$ are said to be* **independent random variables** *if the σ-algebras $\sigma(X)$ and $\sigma(Y)$ are independent. More generally, the random variables $X_1, \dots, X_N$ are independent if $\sigma(X_1), \dots, \sigma(X_N)$ are independent σ-algebras.*

In the intermediate case, i.e., when Y is neither X-measurable nor independent of X, it is expected that the knowledge on the value attained by X helps to derive information on the values attainable by Y. We shall study this case in the next chapter.

Exercise 2.2 (•). *Show that when X, Y are independent random variables, then $\sigma(X) \cap \sigma(Y)$ consists of trivial events only. Show that two deterministic constants are always independent. Finally assume $Y = g(X)$ and show that in this case the two random variables are independent if and only if Y is a deterministic constant.*

Exercise 2.3. *Which of the following pairs of random variables $X, Y : \Omega_N \to \mathbb{R}$ are independent? (Use only the intuitive interpretation of independence and not the formal definition.)*

1. *$X(\omega) = N_T(\omega)$; $Y(\omega) = 1$ if the first toss is head, $Y(\omega) = 0$ otherwise.*
2. *$X(\omega) = 1$ if there exists at least a head in ω, $X(\omega) = 0$ otherwise; $Y(\omega) = 1$ if there exists exactly a head in ω, $Y(\omega) = 0$ otherwise.*
3. *$X(\omega) =$ number of times that a head is followed by a tail; $Y(\omega) = 1$ if there exist two consecutive tail in ω, $Y(\omega) = 0$ otherwise.*

Theorem 2.1. *Let $X_1, \dots, X_N$ be independent random variables. Let us divide the set $\{X_1, \dots, X_N\}$ into m separate groups of random variables, namely, let*

$$\{X_1, \dots, X_N\} = \{X_{k_1}\}_{k_1 \in I_1} \cup \{X_{k_2}\}_{k_2 \in I_2} \cup \dots \cup \{X_{k_m}\}_{k_m \in I_m},$$

where $\{I_1, I_2, \dots I_m\}$ is a partition of $\{1, \dots, N\}$. Let n_i be the number of elements in the set I_i, so that $n_1 + n_2 + \dots + n_m = N$. Let $g_1, \dots, g_m$ be measurable functions such that $g_i : \mathbb{R}^{n_i} \to \mathbb{R}$. Then the random variables

$$Y_1 = g_1((X_{k_1})_{k_1 \in I_1}), \ Y_2 = g_2((X_{k_2})_{k_2 \in I_2}), \quad Y_m = g_m((X_{k_m})_{k_m \in I_m})$$

are independent.

For instance, in the case of $N = 2$ independent random variables X_1, X_2, Theorem 2.1 asserts that $Y_1 = g(X_1)$ and $Y_2 = f(X_2)$ are independent random variables, for all measurable functions $f, g : \mathbb{R} \to \mathbb{R}$.

Exercise 2.4 (•). *Prove Theorem 2.1 for the case $N = 2$.*

Simple and discrete Random Variables

A special role is played by simple random variables. The simplest possible one is the **indicator function** of an event: Given $A \in \mathcal{F}$, the indicator function of A is the random variable that takes value 1 if $\omega \in A$ and 0 otherwise, i.e.,

$$\mathbb{I}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \in A^c. \end{cases}$$

Obviously, $\sigma(\mathbb{I}_A) = \{A, A^c, \emptyset, \Omega\}$.

Definition 2.5. *Let $\{A_k\}_{k=1,\dots,N} \subset \mathcal{F}$ be a family of disjoint events and $a_1, \dots, a_N$ be distinct real numbers. The random variable*

$$X = \sum_{k=1}^{N} a_k \mathbb{I}_{A_k}$$

is called a **simple random variable**. *If $N = \infty$ in this definition, we call X a* **discrete random** *variable.*

Thus a simple random variable X can attain only a finite number of values, while a discrete random variable X attains countably infinite many values[2]. In both cases we have

$$\mathbb{P}(X = x) = \begin{cases} 0, & \text{if } x \notin \mathrm{Im}(X), \\ \mathbb{P}(A_k), & \text{if } x = a_k, \end{cases}$$

where $\mathrm{Im}(X) = \{x \in \mathbb{R} : X(\omega) = x, \text{ for some } \omega \in \Omega\}$ is the image of X. We remark that most references do not assume, in the definition of simple random variable, that the sets $A_1, \dots, A_N$ should be disjoint. We do so, however, because all simple random variables considered in these notes satisfy this property and because the sets $A_1, \dots, A_N$ can always be re-defined in such a way that they are disjoint, without modifying the image of the simple random variable, see Exercise 2.5. Similarly the condition that $a_1, \dots, a_N$ should be distinct can be removed from the definition of simple random variable.

Exercise 2.5 (•). *Let a random variable X have the form*

$$X = \sum_{k=1}^{M} b_k \mathbb{I}_{B_k},$$

for some non-zero $b_1, \dots, b_M \in \mathbb{R}$ and $B_1, \dots B_M \in \mathcal{F}$. Show that there exists $a_1, \dots, a_N \in \mathbb{R}$ distinct and disjoint sets $A_1, \dots A_N \in \mathcal{F}$ such that

$$X = \sum_{k=1}^{N} a_k \mathbb{I}_{A_k}.$$

[2] Not all authors distinguish between simple and discrete random variables.

Let us see two examples of simple/discrete random variables that appear in financial mathematics (and in many other applications). A simple random variable X is called a **binomial** random variable if

- $\mathrm{Range}(X) = \{0, 1, \dots, N\}$;
- There exists $p \in (0,1)$ such that $\mathbb{P}(X = k) = \binom{N}{k} p^k (1-p)^{N-k}$, $k = 1 \dots, N$.

For instance, if we let X to be the number of heads in a N-toss, then X is binomial. A widely used model for the evolution of stock prices in financial mathematics assumes that the price of the stock at any time is a binomial random variable (**binomial asset pricing model**). A discrete random variable X is called a **Poisson** variable if

- $\mathrm{Range}(X) = \mathbb{N} \cup \{0\}$;
- There exists $\mu > 0$ such that $\mathbb{P}(X = k) = \dfrac{\mu^k e^{-\mu}}{k!}$, $k = 0, 1, 2, \dots$

We denote by $\mathcal{P}(\mu)$ the set of all Poisson random variables with parameter $\mu > 0$.

The following important theorem shows that all non-negative random variables can be approximated by a sequence of simple random variables.

Theorem 2.2. *Let $X : \Omega \to [0, \infty)$ be a random variable and let $n \in \mathbb{N}$ be given. For $k = 0, 1, ...n2^n - 1$, consider the sets*

$$A_{k,n} := \Big\{X \in \Big[\frac{k}{2^n}, \frac{k+1}{2^n}\Big)\Big\}$$

and for $k = n2^n$ let

$$A_{n2^n,n} = \{X \geq n\}.$$

Note that $\{A_{k,n}\}_{k=0,\dots,n2^n}$ is a partition of Ω, for all fixed $n \in \mathbb{N}$. Define the simple random variables

$$s_n^X(\omega) = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{I}_{A_{k,n}}(\omega).$$

Then $0 \leq s_1^X(\omega) \leq s_2^X(\omega) \leq \cdots \leq s_n^X(\omega) \leq s_{n+1}^X(\omega) \leq \cdots \leq X(\omega)$, for all $\omega \in \Omega$ (i.e., the sequence $\{s_n^X\}_{n\in\mathbb{N}}$ is non-decreasing) and

$$\lim_{n\to\infty} s_n^X(\omega) = X(\omega), \quad \textit{for all } \omega \in \Omega.$$

Exercise 2.6. *Prove Theorem 2.2.*

2.2 Distribution and probability density functions

Definition 2.6. *The* **(cumulative) distribution function** *of the random variable $X : \Omega \to \mathbb{R}$ is the non-negative function $F_X : \mathbb{R} \to [0, 1]$ given by $F_X(x) = \mathbb{P}(X \leq x)$. Two random variables X, Y are said to be* **identically distributed** *if $F_X = F_Y$.*

Exercise 2.7 (•). *Show that*

$$\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a).$$

Show also that F_X is (1) right-continuous, (2) increasing and (3) $\lim_{x\to+\infty} F_X(x) = 1$.

Exercise 2.8. *Let $F : \mathbb{R} \to [0, 1]$ be a measurable function satisfying the properties (1)–(3) in Exercise 2.7. Show that there exists a probability space and a random variable X such that $F = F_X$.*

Definition 2.7. *A random variable $X : \Omega \to \mathbb{R}$ is said to admit the* **probability density function (pdf)** *$f_X : \mathbb{R} \to [0, \infty)$ if f_X is integrable on $\mathbb{R}$ and*

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy. \tag{2.1}$$

Note that if f_X is the pdf of a random variable, then necessarily

$$\int_{\mathbb{R}} f_X(x)\,dx = \lim_{x\to\infty} F_X(x) = 1.$$

All probability density functions considered in these notes are continuous, and therefore the integral in (2.1) can be understood in the Riemann sense. Moreover in this case F_X is differentiable and we have

$$f_X = \frac{dF_X}{dx}.$$

If the integral in (2.1) is understood in the Lebesgue sense, then the density f_X can be a quite irregular function. In this case, the fundamental theorem of calculus for the Lebesgue integral entails that the distribution $F_X(x)$ satisfying (2.1) is absolutely continuous, and so in particular it is continuous. Conversely, if F_X is absolutely continuous, then X admits a density function. We remark that, regardless of the notion of integral being used, a simple (or discrete) random variable X cannot admit a density in the sense of Definition 2.7, unless it is a deterministic constant. Suppose in fact that $X = \sum_{k=1}^{N} a_k \mathbb{I}_{A_k}$ is not a deterministic constant. Assume that $a_1 = \max(a_1, \dots, a_N)$. Then

$$\lim_{x\to a_1^-} F_X(x) = \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_N) < 1,$$

while

$$\lim_{x\to a_1^+} F_X(x) = 1 = F_X(a_1).$$

It follows that $F_X(x)$ is not continuous, and so in particular it cannot be written in the form (2.1). To define the pdf of simple random variables, let

$$X = \sum_{k=1}^{N} a_k \mathbb{I}_{A_k},$$

where without loss of generality we assume that the real numbers $a_1, \dots, a_N$ are distinct and the sets $A_1, \dots, A_N$ are disjoint (see Exercise 2.5). The distribution function of X is

$$F_X(x) = \mathbb{P}(X \leq x) = \sum_{a_k \leq x} \mathbb{P}(X = a_k). \tag{2.2}$$

In this case the probability density function $f_X(x)$ is defined as

$$f_X(x) = \begin{cases} \mathbb{P}(X = x), & \text{if } x = a_k \text{ for some } k \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

and thus, with a slight abuse of notation, we can rewrite (2.2) as

$$F_X(x) = \sum_{y \leq x} f_X(y), \tag{2.4}$$

which extend (2.1) to simple random variables. We remark that it is possible to unify the definition of pdf for continuum and discrete random variables by writing the sum (2.4) as an integral with respect to the Dirac measure, but we shall not do so.

We shall see that when a random variable X admits a density f_X, all the relevant statistical information on X can be deduced by f_X. We also remark that often one can prove the existence of the pdf f_X without however being able to derive an explicit formula for it. For instance, f_X is often given as the solution of a partial differential equation, or through its (inverse) Fourier transform, which is called the characteristic function of X, see Section 3.3. Some examples of density functions, which have important applications in financial mathematics, are the following.

Examples of probability density functions

- A random variable $X : \Omega \to \mathbb{R}$ is said to be a **normal** (or **normally distributed**) random variable if it admits the density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\dfrac{(x-m)^2}{2\sigma^2}},$$

  for some $m \in \mathbb{R}$ and $\sigma > 0$, which are called respectively the **expectation** (or **mean**) and the **deviation** of the normal random variable X, while σ^2 is called the **variance** of X. A typical profile of a normal density function is shown in Figure 2.1(a). We denote by $\mathcal{N}(m, \sigma^2)$ the set of all normal random variables with expectation m and variance σ^2. If $m = 0$ and $\sigma^2 = 1$, $X \in \mathcal{N}(0,1)$ is said to be a **standard** normal variable. The density function of standard normal random variables is denoted by ϕ, while their distribution is denoted by Φ, i.e.,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy.$$

- A random variable $X : \Omega \to \mathbb{R}$ is said to be an **exponential** (or **exponentially distributed**) random variable if it admits the density

$$f_X(x) = \lambda e^{-\lambda x} \mathbb{I}_{x \geq 0},$$

for some $\lambda > 0$, which is called the **intensity** of the exponential random variable X. A typical profile is shown in Figure 2.1(b) . We denote by $\mathcal{E}(\lambda)$ the set of all exponential random variables with intensity $\lambda > 0$. The distribution function of an exponential random variable X with intensity λ is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy = \lambda \int_{0}^{x} e^{-\lambda y}\, dy = 1 - e^{-\lambda x}.$$

- A random variable $X : \Omega \to \mathbb{R}$ is said to be **chi-squared distributed** if it admits the density

$$f_X(x) = \frac{x^{\delta/2-1} e^{-x/2}}{2^{\delta/2}\Gamma(\delta/2)} \mathbb{I}_{x>0},$$

for some $\delta > 0$, which is called the **degree** of the chi-squared distributed random variable. Here $\Gamma(t) = \int_0^\infty z^{t-1} e^{-z} dz$, $t > 0$ is the Gamma-function. Recall the relation

$$\Gamma(n) = (n-1)!$$

for $n \in \mathbb{N}$. We denote by $\chi^2(\delta)$ the set of all chi-squared distributed random variables with degree δ. Three typical profiles of this density are shown in Figure 2.2(a).

- A random variable $X : \Omega \to \mathbb{R}$ is said to be **non-central chi-squared distributed** with **degree** $\delta > 0$ and **non-centrality parameter** $\beta > 0$ if it admits the density

$$f_X(x) = \frac{1}{2} e^{-\frac{x+\beta}{2}} \left(\frac{x}{\beta}\right)^{\frac{\delta}{4}-\frac{1}{2}} I_{\delta/2-1}(\sqrt{\beta x}) \mathbb{I}_{x>0}, \tag{2.5}$$

where $I_\nu(y)$ denotes the modified Bessel function of the first kind. We denote by $\chi^2(\delta, \beta)$ the random variables with density (2.5). It can be shown that $\chi^2(\delta, 0) = \chi^2(\delta)$. Three typical profiles of the density (2.5) are shown in Figure 2.2(b).

- A random variable $X : \Omega \to \mathbb{R}$ is said to be **Cauchy distributed** if it admits the density

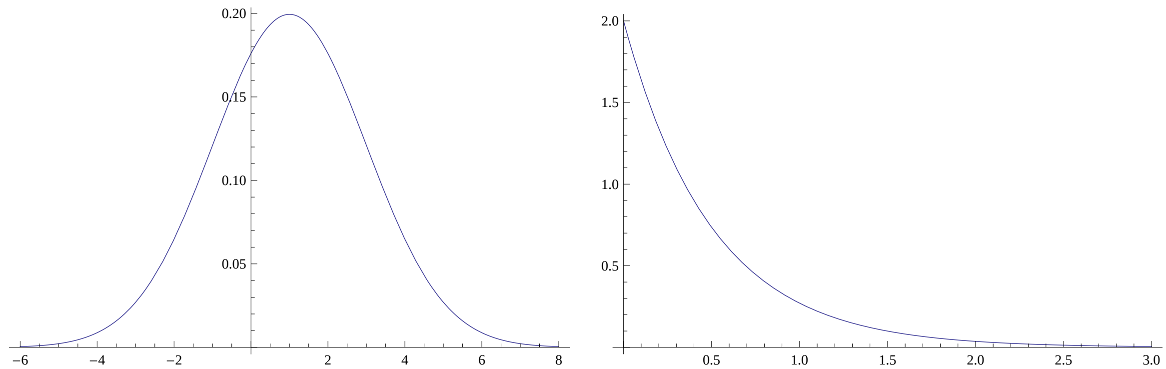
$$f_X(x) = \frac{\gamma}{\pi((x-x_0)^2 + \gamma^2)}$$

for $x_0 \in \mathbb{R}$ and $\gamma > 0$, called the **location** and the **scale** of the Cauchy pdf.

- A random variable $X : \Omega \to \mathbb{R}$ is said to be **Lévy distributed** if it admits the density

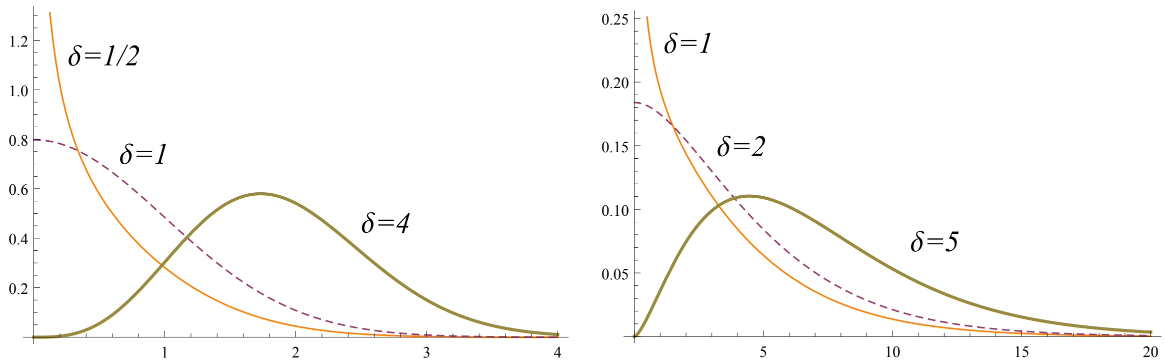
$$f_X(x) = \sqrt{\frac{c}{2\pi}} \frac{e^{-\frac{c}{2(x-x_0)}}}{(x-x_0)^{3/2}} \mathbb{I}_{x>x_0},$$

for $x_0 \in \mathbb{R}$ and $c > 0$, called the **location** and the **scale** of the Lévy pdf.



(a) $X \in \mathcal{N}(1, 2)$ (b) $X \in \mathcal{E}(2)$

Figure 2.1: Densities of a normal random variable X and of an exponential random variable Y.



(a) $X \in \chi^2(\delta)$ (b) $X \in \chi^2(\delta, 2)$

Figure 2.2: Densities of (non-central) chi-squared random variables with different degree.

If a random variable X admits a density f_X, then for all (possibly unbounded) intervals $I \subset \mathbb{R}$ the result of Exercise 2.7 entails

$$\mathbb{P}(X \in I) = \int_I f_X(y)\, dy. \tag{2.6}$$

It can be shown that (2.6) extends to

$$\mathbb{P}(g(X) \in I) = \int_{x:g(x)\in I} f_X(x)\, dx, \tag{2.7}$$

for all measurable functions $g : \mathbb{R} \to \mathbb{R}$. For example, if $X \in \mathcal{N}(0, 1)$,

$$\mathbb{P}(X^2 \leq 1) = \mathbb{P}(-1 \leq X \leq 1) = \int_{-1}^{1} \phi(x)\, dx \approx 0.683,$$

which means that a standard normal random variable has about 68.3 % chances to take value in the interval $[-1,1]$.

Exercise 2.9 (•). *Let $X \in \mathcal{N}(0,1)$ and $Y = X^2$. Show that $Y \in \chi^2(1)$.*

Exercise 2.10. *Let $X \in \mathcal{N}(0,1)$. Show that the random variable W defined by*

$$W = \begin{cases} 1/X^2 & \text{for } X \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

is Lévy distributed.

Exercise 2.11. *Let $X \in \mathcal{N}(m,\sigma^2)$ and $Y = X^2$. Show that*

$$f_Y(x) = \frac{\cosh(m\sqrt{x}/\sigma^2)}{\sqrt{2\pi x \sigma^2}} \exp\left(-\frac{x+m^2}{2\sigma^2}\right)\mathbb{I}_{x>0}.$$

2.2.1 Random variables with boundary values

Random variables in mathematical finance do not always admit a density in the classical sense described above (or in any other sense), and the purpose of this section is to present an example when one has to consider a generalized notion of density function. Suppose that X takes value on the semi-open interval $[0,\infty)$. Then clearly $F_X(x) = 0$ for $x < 0$, $F_X(0) = \mathbb{P}(X = 0)$, while for $x > 0$ we can write

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(0 \leq X \leq x) = \mathbb{P}(X = 0) + \mathbb{P}(0 < X \leq x).$$

Now assume that F_X is differentiable on the open set $x \in (0,\infty)$. Then there exists a function $f_X^+(x)$, $x > 0$, such that $F_X(x) - F_X(0) = \int_0^x f_X^+(t)\,dt$. Hence, for all $x \in \mathbb{R}$ we find

$$F_X(x) = p_0 H(x) + \int_{-\infty}^x f_X^+(t)\mathbb{I}_{t>0}\,dt,$$

where $p_0 = \mathbb{P}(X = 0)$ and $H(x)$ is the Heaviside function, i.e., $H(x) = 1$ if $x \geq 0$, $H(x) = 0$ if $x < 0$. By introducing the delta-distribution through the formal identity

$$H'(x) = \delta(x) \tag{2.8}$$

then we obtain, again formally, the following expression for the density function

$$f_X(x) = \frac{dF_X(x)}{dx} = p_0\delta(x) + f_X^+(x). \tag{2.9}$$

The formal identities (2.8)-(2.9) become rigorous mathematical expressions when they are understood in the sense of distributions. We shall refer to the term $p_0\delta(x)$ as the **discrete part** of the density. The function f_X^+ is also called the **defective density** of the random variable X. Note that

$$\int_0^\infty f_X^+(x)\,dx = 1 - p_0.$$

The defective density is the actual pdf of X if and only if $p_0 = 0$.

The typical example of financial random variable whose pdf may have a discrete part is the stock price $S(t)$ at time t. For simple models (such us the geometric Browniam motion (2.14) defined in Section 2.4 below), the stock price is strictly positive a.s. at all finite times and the density has no discrete part. However for more sophisticated models the stock price can reach zero with positive probability at any finite time and so the pdf of the stock price admits a discrete part $\mathbb{P}(S(t) = 0)\delta(x)$. Hence these models take into account the **risk of default** of the stock. We shall see an example in Section 6.5.

2.2.2 Joint distribution

If two random variables $X, Y : \Omega \to \mathbb{R}$ are given, how can we verify whether or not they are independent? This problem has a simple solution when X, Y admit a joint distribution density.

Definition 2.8. *The* **joint (cumulative) distribution** $F_{X,Y} : \mathbb{R}^2 \to [0, 1]$ *of two random variables* $X, Y : \Omega \to \mathbb{R}$ *is defined as*

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y).$$

The random variables X, Y *are said to admit the* **joint (probability) density function** $f_{X,Y} : \mathbb{R}^2 \to [0, \infty)$ *if* $f_{X,Y}$ *is integrable in* $\mathbb{R}^2$ *and*

$$F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(\eta, \xi)\, d\eta\, d\xi. \tag{2.10}$$

Note the formal identities

$$f_{X,Y} = \frac{\partial^2 F_{X,Y}}{\partial x\, \partial y}, \quad \int_{\mathbb{R}^2} f_{X,Y}(x, y)\, dx\, dy = 1.$$

Moreover, if two random variables X, Y admit a joint density $f_{X,Y}$, then each of them admits a density (called **marginal density** in this context) which is given by

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x, y)\, dy, \quad f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x, y)\, dx.$$

To see this we write

$$\mathbb{P}(X \leq x) = \mathbb{P}(X \leq x, Y \in \mathbb{R}) = \int_{-\infty}^{x} \int_{\mathbb{R}} f_{X,Y}(\eta, \xi)\, d\eta\, d\xi = \int_{-\infty}^{x} f_X(\eta)\, d\eta$$

and similarly for the random variable Y. If $W = g(X, Y)$, for some measurable function g, and $I \subset \mathbb{R}$ is an interval, the analogue of (2.7) in 2 dimensions holds, namely:

$$\mathbb{P}(g(X, Y) \in I) = \int_{x,y:g(x,y)\in I} f_{X,Y}(x, y)\, dx\, dy.$$

As an example of joint pdf, let $m = (m_1, m_2) \in \mathbb{R}^2$ and $C = (C_{ij})_{i,j=1,2}$ be a 2×2 positive definite, symmetric matrix. Two random variables $X, Y : \Omega \to \mathbb{R}$ are said to be **jointly normally distributed** with **mean** m and **covariance matrix** C if they admit the joint density

$$f_{X,Y}(x, y) = \frac{1}{\sqrt{(2\pi)^2 \det C}} \exp\left[\frac{1}{2}(z - m) \cdot C^{-1} \cdot (z - m)^T\right], \tag{2.11}$$

where $z = (x, y)$, " $\cdot$ " denotes the row by column product, C^{-1} is the inverse matrix of C and v^T is the transpose of the vector v.

Exercise 2.12 (•). *Show that two random variables X, Y are jointly normally distributed if and only if*

$$\begin{aligned} f_{X,Y}(x, y) = & \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times \\ & \times \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-m_1)^2}{\sigma_1^2} - \frac{2\rho(x-m_1)(y-m_2)}{\sigma_1\sigma_2} + \frac{(y-m_2)^2}{\sigma_2^2}\right]\right), \end{aligned} \tag{2.12}$$

where

$$\sigma_1 = C_{11}^2, \quad \sigma_2 = C_{22}^2, \quad \rho = \frac{C_{12}}{\sigma_1\sigma_2}.$$

Exercise 2.13 (⋆). *Let $X, Y \in \mathcal{N}(0, 1)$ be independent and jointly normally distributed. Show that the random variable Z defined by*

$$Z = \begin{cases} Y/X & \text{for } X \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

is Cauchy distributed.

In the next theorem we establish a simple condition for the independence of two random variables which admit a joint density[3].

Theorem 2.3. *The following holds.*

(i) If two random variables X, Y admit the densities f_X, f_Y and are independent, then they admit the joint density

$$f_{X,Y}(x, y) = f_X(x) f_Y(y).$$

(ii) If two random variables X, Y admit a joint density $f_{X,Y}$ of the form

$$f_{X,Y}(x, y) = u(x)v(y),$$

[3]A similar result holds in terms of the joint cumulative distribution, irrespective of whether a joint pdf exists.

for some functions $u, v : \mathbb{R} \to [0, \infty)$, then X, Y are independent and admit the densities f_X, f_Y given by

$$f_X(x) = cu(x), \quad f_Y(y) = \frac{1}{c}v(y),$$

where

$$c = \int_{\mathbb{R}} v(x)\, dx = \Big(\int_{\mathbb{R}} u(y)\, dy\Big)^{-1}.$$

Proof. As to (i) we have

$$\begin{aligned} F_{X,Y}(x,y) &= \mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x)\mathbb{P}(Y \le y) \\ &= \int_{-\infty}^{x} f_X(\eta)\, d\eta \int_{-\infty}^{y} f_Y(\xi)\, d\xi \\ &= \int_{-\infty}^{x}\int_{-\infty}^{y} f_X(\eta) f_Y(\xi)\, d\eta\, d\xi. \end{aligned}$$

To prove (ii), we first write

$$\{X \le x\} = \{X \le x\} \cap \Omega = \{X \le x\} \cap \{Y \le \mathbb{R}\} = \{X \le x, Y \le \mathbb{R}\}.$$

Hence,

$$\mathbb{P}(X \le x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(\eta, y)\, dy\, d\eta = \int_{-\infty}^{x} u(\eta)\, d\eta \int_{\mathbb{R}} v(y)\, dy = \int_{-\infty}^{x} cu(\eta)\, d\eta,$$

where $c = \int_{\mathbb{R}} v(y)\, dy$. Thus X admits the density $f_X(x) = c\, u(x)$. At the same fashion one proves that Y admits the density $f_Y(y) = c' v(y)$, where $c' = \int_{\mathbb{R}} u(x) dx$. Since

$$1 = \int_{\mathbb{R}}\int_{\mathbb{R}} f_{X,Y}(x,y)\, dx\, dy = \int_{\mathbb{R}} u(x)\, dx \int_{\mathbb{R}} v(y)\, dy = c'c,$$

then $c' = 1/c$. It remains to prove that X, Y are independent. This follows by

$$\begin{aligned} \mathbb{P}(X \in U, Y \in V) &= \int_U \int_V f_{X,Y}(x,y)\, dx\, dy = \int_U u(x)\, dx \int_V v(y)\, dy \\ &= \int_U cu(x)\, dx \int_V \frac{1}{c} v(y)\, dy = \int_U f_X(x)\, dx \int_V f_Y(y)\, dy \\ &= \mathbb{P}(X \in U)\mathbb{P}(Y \in V), \quad \text{for all } U, V \in \mathcal{B}(\mathbb{R}). \end{aligned}$$

□

Remark 2.3. By Theorem 2.3 and the result of Exercise 2.12, we have that two jointly normally distributed random variables are independent if and only if $\rho = 0$ in the formula (2.12).

Exercise 2.14 (•). *Let $X \in \mathcal{N}(0, 1)$ and $Y \in \mathcal{E}(1)$ be independent. Compute $\mathbb{P}(X \le Y)$.*

Exercise 2.15. *Let* $X \in \mathcal{E}(2)$*,* $Y \in \chi^2(3)$ *be independent. Compute numerically (e.g., using* Mathematica*) the following probability*

$$\mathbb{P}(\log(1 + XY) < 2).$$

Result:$\approx$ *0.893.*

In Exercise 3.19 we give another criteria to establish whether two random variables are independent, which applies also when the random variables do not admit a density.

2.3 Stochastic processes

Definition 2.9. *A* **stochastic process** *is a one-parameter family of random variables, which we denote by* $\{X(t)\}_{t\geq 0}$*, or by* $\{X(t)\}_{t\in[0,T]}$ *if the parameter* t *is restricted to the interval* $[0, T]$*,* $T > 0$*. Hence, for each* $t \geq 0$*,* $X(t) : \Omega \to \mathbb{R}$ *is a random variable. We denote by* $X(t, \omega)$ *the value of* $X(t)$ *on the sample point* $\omega \in \Omega$*, i.e.,* $X(t, \omega) = X(t)(\omega)$*. For each* $\omega \in \Omega$ *fixed, the curve* $\gamma_X^\omega : \mathbb{R} \to \mathbb{R}$*,* $\gamma_X^\omega(t) = X(t, \omega)$ *is called the* ω**-path** *of the stochastic process and is assumed to be a measurable function. If the paths of a stochastic process are all almost surely equal (i.e., independent of* ω*), we say that the stochastic process is a* **deterministic function of time***.*

The parameter t will be referred to as **time** parameter, since this is what it represents in the applications in financial mathematics. Examples of stochastic processes in financial mathematics are given in the next section.

Definition 2.10. *Two stochastic processes* $\{X(t)\}_{t\geq 0}$*,* $\{Y(t)\}_{t\geq 0}$ *are said to be independent if for all* $m, n \in \mathbb{N}$ *and* $0 \leq t_1 < t_2 < \cdots < t_n$*,* $0 \leq s_1 < s_2 < \cdots < s_m$*, the* σ*-algebras* $\sigma(X(t_1), \dots, X(t_n))$*,* $\sigma(Y(s_1), \dots, Y(s_m))$ *are independent.*

Hence two stochastic processes $\{X(t)\}_{t\geq 0}$, $\{Y(t)\}_{t\geq 0}$ are independent if the information obtained by "looking" at the process $\{X(t)\}_{t\geq 0}$ up to time T is independent of the information obtained by "looking" at the process $\{Y(t)\}_{t\geq 0}$ up to time S, for all $S, T > 0$. Similarly one defines the notion of several independent stochastic processes.

Remark 2.4 (Notation)**.** If t runs over a countable set, i.e., $t \in \{t_k\}_{k\in\mathbb{N}}$, then a stochastic process is equivalent to a sequence of random variables $X_1, X_2, \dots$, where $X_k = X(t_k)$. In this case we say that the stochastic process is **discrete** and we denote it by $\{X_k\}_{k\in\mathbb{N}}$. An example of discrete stochastic process is the random walk defined below.

A special role is played by **step** processes: given $0 = t_0 < t_1 < t_2 < \dots$, a step process is a stochastic process $\{\Delta(t)\}_{t\geq 0}$ of the form

$$\Delta(t, \omega) = \sum_{k=0}^{\infty} X_k(\omega)\mathbb{I}_{[t_k,t_{k+1})}.$$

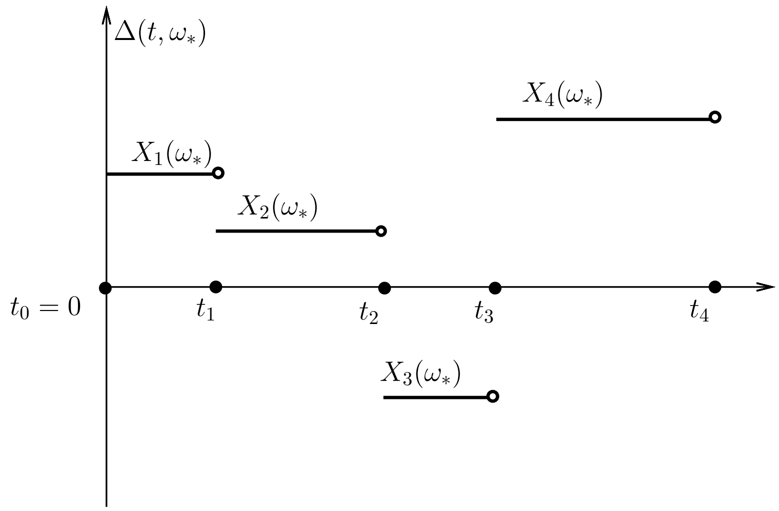


Figure 2.3: The path $\omega = \omega_*$ of a step process.

A typical path of a step process is depicted in Figure 2.3. Note that the paths of a step process are right-continuous, but not left-continuous. Moreover, since $X_k(\omega) = \Delta(t_k, \omega)$, we can rewrite $\Delta(t)$ as

$$\Delta(t) = \sum_k^\infty \Delta(t_k)\mathbb{I}_{[t_k,t_{k+1})}.$$

It will be shown in Theorem 4.2 that any sufficiently regular stochastic process can be approximated, in a suitable sense, by a sequence of step processes.

In the same way as a random variable generates a σ-algebra, a stochastic process generates a filtration. Informally, the filtration generated by a stochastic process $\{X(t)\}_{t\geq 0}$ contains the information accumulated by looking at the process for longer and longer periods of time.

Definition 2.11. *The filtration generated by the stochastic process* $\{X(t)\}_{t\geq 0}$ *is given by* $\{\mathcal{F}_X(t)\}_{t\geq 0}$, *where*

$$\mathcal{F}_X(t) = \mathcal{F}_{\mathcal{O}(t)}, \quad \mathcal{O}(t) = \cup_{0\leq s\leq t}\sigma(X(s)).$$

Hence $\mathcal{F}_X(t)$ is the smallest σ-algebra containing $\sigma(X(s))$, for all $0 \leq s \leq t$, see Definition 1.2. Similarly one defines the filtration $\{\mathcal{F}_{X,Y}(t)\}_{t\geq 0}$ generated by two stochastic processes $\{X(t)\}_{t\geq 0}$, $\{Y(t)\}_{t\geq 0}$, as well as the filtration generated by any number of stochastic processes.

Definition 2.12. *If* $\{\mathcal{F}(t)\}_{t\geq 0}$ *is a filtration and* $\mathcal{F}_X(t) \subseteq \mathcal{F}(t)$, *for all* $t \geq 0$, *we say that the stochastic process* $\{X(t)\}_{t\geq 0}$ *is* **adapted** *to the filtration* $\{\mathcal{F}(t)\}_{t\geq 0}$.

The property of $\{X(t)\}_{t\geq 0}$ being adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$ means that the information contained in $\mathcal{F}(t)$ suffices to determine the value attained by the random variable $X(s)$, for all $s \in [0, t]$. Clearly, $\{X(t)\}_{t\geq 0}$ is adapted to its own generated filtration $\{\mathcal{F}_X(t)\}_{t\geq 0}$. Moreover if $\{X(t)\}_{t\geq 0}$ is adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$ and $Y(t) = f(X(t))$, for some measurable function f, then $\{Y(t)\}_{t\geq 0}$ is also adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$.

Next we give an example of (discrete) stochastic process. Let $\{X_t\}_{t\in\mathbb{N}}$ be a sequence of independent random variables satisfying

$$X_t = 1 \quad \text{with probability } 1/2, \qquad X_t = -1 \quad \text{with probability } 1/2,$$

for all $t \in \mathbb{N}$. For a concrete realization of these random variables, we may think of X_t as being defined on the sample space Ω_∞ of the ∞-coin tosses experiment (see Appendix 1.A). In fact, letting $\omega = (\gamma_j)_{j\in\mathbb{N}} \in \Omega_\infty$, we may set

$$X_t(\omega) = \begin{cases} -1, & \text{if } \gamma_t = H, \\ 1, & \text{if } \gamma_t = T. \end{cases}$$

Hence $X_t : \Omega \to \{-1, 1\}$ is the simple random variable $X_t(\omega) = \mathbb{I}_{A_t} - \mathbb{I}_{A_t^c}$, where $A_t = \{\omega \in \Omega_\infty : \gamma_t = H\}$. Clearly, $\mathcal{F}_X(t)$ is the collection of all the events that are resolved by the first t-tosses, which is given as indicated at the beginning of Section 1.3.

Definition 2.13. *The stochastic process $\{M_t\}_{t\in\mathbb{N}}$ given by*

$$M_0 = 0, \quad M_t = \sum_{k=1}^{t} X_k,$$

is called **symmetric random walk**.

To understand the meaning of the term "random walk", consider a particle moving on the real line in the following way: if $X_t = 1$ (i.e., if the toss number t is a head), at time t the particle moves one unit of length to the right, if $X_t = -1$ (i.e., if the toss number t is a head) it moves one unit of length to the left. Then M_t gives the total amount of units of length that the particle has travelled to the right or to the left up to time t.

Exercise 2.16. *Which of the following holds?*

$$\mathcal{F}_M(t) \subset \mathcal{F}_X(t), \quad \mathcal{F}_M(t) = \mathcal{F}_X(t), \quad \mathcal{F}_X(t) \subset \mathcal{F}_M(t).$$

Justify the answer.

The **increments** of the random walk are defined as follows. If $(k_1, \dots, k_N) \in \mathbb{N}^N$, such that $1 \leq k_1 < k_2 < \dots < k_N$, we set

$$\Delta_1 = M_{k_1} - M_0 = M_{k_1}, \quad \Delta_2 = M_{k_2} - M_{k_1}, \dots, \quad \Delta_N = M_{k_N} - M_{k_{N-1}}.$$

Hence Δ_j is the total displacement of the particle from time k_{j-1} to time k_j.

Theorem 2.4. *The increments $\Delta_1, \dots, \Delta_N$ of the random walk are independent random variables.*

Proof. Since

$$\begin{aligned}
\Delta_1 &= X_1 + \cdots + X_{k_1} = g_1(X_1, \dots, X_{k_1}),\\
\Delta_2 &= X_{k_1+1} + \cdots + X_{k_2} = g_2(X_{k_1+1}, \dots, X_{k_2}),\\
&.\\
&.\\
\Delta_N &= X_{k_{N-1}+1} + \cdots + X_{k_N} = g_N(X_{k_{N-1}+1} + \cdots + X_{k_N}),
\end{aligned}$$

the result follows by Theorem 2.1. □

The interpretation of this result is that the particle has no memory of past movements: the distance travelled by the particle in a given interval of time is not affected by the motion of the particle at earlier times.

We may now define the most important of all stochastic processes.

Definition 2.14. *A* **Brownian motion** *(or* **Wiener process***) is a stochastic process* $\{W(t)\}_{t\geq 0}$ *such that*

(i) The paths are continuous and start from 0 *almost surely, i.e., the sample points* $\omega \in \Omega$ *such that* $\gamma_W^\omega(0) = 0$ *and* γ_W^ω *is a continuous function comprise a set of probability 1;*

(ii) The increments over disjoint time intervals are independent, i.e., for all $0 = t_0 < t_1 < \dots t_m \in (0, \infty)$*, the random variables*

$$W(t_1) - W(t_0),\ W(t_2) - W(t_1),\ \dots,\ W(t_m) - W(t_{m-1})$$

are independent;

(iii) For all $s < t$*, the increment* $W(t) - W(s)$ *belongs to* $\mathcal{N}(0, t - s)$*.*

Remark 2.5. Note carefully that the properties defining a Brownian motion depend on the probability measure $\mathbb{P}$. Thus a stochastic process may be a Brownian motion relative to a probability measure $\mathbb{P}$ and not a Brownian motion with respect to another (possibly equivalent) probability measure $\widetilde{\mathbb{P}}$. If we want to emphasize the probability measure $\mathbb{P}$ with respect to which a stochastic process is a Brownian motion we shall say that it is a $\mathbb{P}$-Brownian motion.

It can be shown that Brownian motions exist. In particular, it can be shown that the sequence of stochastic processes $\{W_n(t)\}_{t\geq 0}$, $n \in \mathbb{N}$, defined by

$$W_n(t) = \frac{1}{\sqrt{n}} M_{[nt]}, \tag{2.13}$$

where M_t is the symmetric random walk and $[z]$ denotes the integer part of z, converges to a Brownian motion[4]. Therefore one may think of a Brownian motion as a time-continuum

[4] The convergence holds in probability, i.e., $\lim_{n\to\infty} \mathbb{P}(|W_n(t) - W(t)| \geq \varepsilon) = 0$, for all $\varepsilon > 0$.

version of a symmetric random walk which runs for an infinite number of "infinitesimal time steps". In fact, provided the number of time steps is sufficiently large, the process $\{W_n(t)\}_{t\geq 0}$ gives a very good approximation of a Brownian motion, which is useful for numerical computations. An example of path to the stochastic process $\{W_n(t)\}_{t\geq 0}$, for $n = 1000$, is shown in Figure 2.4. Notice that there exist many Brownian motions and each of them may have some specific properties besides those listed in Definition 2.14. However, as long as we use only the properties (i)-(iii), we do not need to work with a specific example of Brownian motion.

Once a Brownian motion is introduced it is natural to require that the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ should be somehow related to it. For our future financial applications, the following class of filtrations will play a fundamental role.

Definition 2.15. *Let $\{W(t)\}_{t\geq 0}$ be a Brownian motion and denote by $\sigma^+(W(t))$ the σ-algebra generated by the increments $\{W(s) - W(t); s \geq t\}$, that is*

$$\sigma^+(W(t)) = \mathcal{F}_{O(t)}, \ \mathcal{O}(t) = \cup_{s\geq t}\sigma(W(s) - W(t)).$$

A filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ is said to be a **non-anticipating** *filtration for the Brownian motion $\{W(t)\}_{t\geq 0}$ if $\{W(t)\}_{t\geq 0}$ is adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$ and if the σ-algebras $\sigma^+(W(t))$, $\mathcal{F}(t)$ are independent for all $t \geq 0$*

The meaning is the following: the increments of the Brownian motion after time t are independent of the information available at time t in the σ-algebra $\mathcal{F}(t)$. It is clear by the previous definition that $\{\mathcal{F}_W(t)\}_{t\geq 0}$ is a non-anticipating filtration for $\{W(t)\}_{t\geq 0}$. We shall see later that many properties of Brownian motions that depend on $\{\mathcal{F}_W(t)\}_{t\geq 0}$ also holds with respect to any non-anticipating filtration (e.g., the martingale property, see Section 3.6).

Another important example of stochastic process applied in financial mathematics is the following.

Definition 2.16. *A* **Poisson process** *with* **rate** λ *is a stochastic process $\{N(t)\}_{t\geq 0}$ such that*

(i) $N(0) = 0$ *a.s.;*

(ii) The increments over disjoint time-intervals are independent;

(iii) For all $s < t$, the increment $N(t) - N(s)$ belongs to $\mathcal{P}(\lambda(t - s))$.

Note in particular that $N(t)$ is a discrete random variable, for all $t \geq 0$, and that, in contrast to the Brownian motion, the paths of a Poisson process are not continuous. The Poisson process is the building block to construct more general stochastic processes with jumps, which are very popular nowadays as models for the price of certain financial assets, see [4].

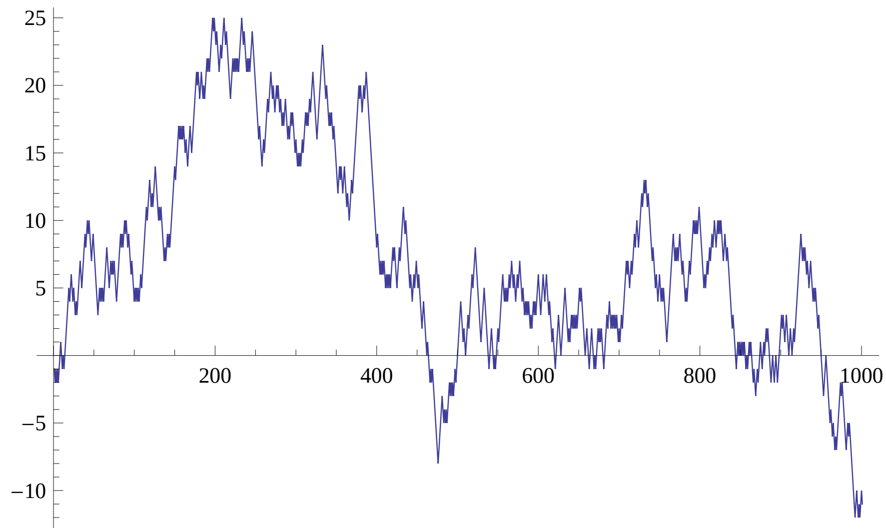


Figure 2.4: A path of the stochastic process (2.13) for $n = 1000$.

2.4 Stochastic processes in financial mathematics

Remark 2.6. For more information on the financial concepts introduced in this section, see Chapter 1 in [5].

All variables in financial mathematics are represented by stochastic processes. The most obvious example is the **price** (or **value**) of financial assets. The stochastic process representing the price per share of a generic asset at different times will be denoted by $\{\Pi(t)\}_{t\geq 0}$. Depending on the type of asset considered, we use a different specific notation for the stochastic process modeling its price.

Remark 2.7. We always assume that $t = 0$ is earlier or equal to the present time. In particular, the value of all financial variables is known at time $t = 0$. Hence, if $\{X(t)\}_{t\geq 0}$ is a stochastic process modelling a financial variable, then $X(0)$ is a deterministic constant.

Before presenting various examples of stochastic processes in financial mathematics, let us introduce an important piece of terminology. An investor is said to have a **short position** on an asset if the investor profits from a decrease of its price, and a **long position** if the investor profits from an increase of the price of the asset. The specific trading strategy that leads to a short or long position on an asset depends on the type of asset considered, as we are now ready to describe in more details.

Stock price

The price per share a time t of a stock will be denoted by $S(t)$. Typically $S(t) > 0$, for all $t \geq 0$, however, as discussed in Section 2.2.1, some models allow for the possibility that $S(t) = 0$ with positive probability at finite times $t > 0$ (risk of default). Clearly $\{S(t)\}_{t\geq 0}$

is a stochastic process. If we have several stocks, we shall denote their price by $\{S_1(t)\}_{t\geq 0}$, $\{S_2(t)\}_{t\geq 0}$, etc. Investors who own shares of a stock are those having a long position on the stock, while investors **short-selling** the stock hold a short position. We recall that short-selling a stock is the practice to sell the stock without actually owning it. Concretely, an investor is short-selling N shares of a stock if the investor borrows the shares from a third party and then sell them immediately on the market. The reason for short-selling assets is the expectation that the price of the asset will decrease. If this is the case, then upon re-purchasing the N shares in the future, and returning them to the lender, the short-seller will profit from the lower current price of the asset compared to the price at the time of short-selling.

The most popular model for the price of a stock is the **geometric Brownian motion** stochastic process, which is given by

$$S(t) = S(0)\exp(\alpha t + \sigma W(t)). \tag{2.14}$$

Here $\{W(t)\}_{t\geq 0}$ is a Brownian motion, $\alpha \in \mathbb{R}$ is the **instantaneous mean of log-return**, $\sigma > 0$ is the **instantaneous volatility**, while σ^2 is the **instantaneous variance** of the stock. Note that α and σ are constant in this model. Moreover, $S(0)$ is the price at time $t = 0$ of the stock, which, according to Remark 2.7, is a deterministic constant. In Chapter 4 we introduce a generalization of the geometric Brownian motion, in which the instantaneous mean of log-return and the instantaneous volatility of the stock are stochastic processes $\{\alpha(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$ (generalized geometric Brownian motion).

Exercise 2.17 (•). *Derive the density of the geometric Brownian motion* (2.14) *and use the result to show that* $\mathbb{P}(S(t) = 0) = 0$*, i.e., a stock whose price is described by a geometric Brownian motion cannot default.*

Risk-free assets

A **money market** is a market in which the object of trading is money. More precisely, a money market is a type of financial market where investors can borrow and lend money at a given interest rate and for a period of time $T \leq 1$ year[5]. Assets in the money market (i.e., short term loans) are assumed to be **risk-free**, which means that their value is always increasing in time. Examples of risk-free assets in the money market are repurchase agreements (repo), certificates of deposit, treasure bills, etc. The stochastic process corresponding to the price per share of a generic risk-free asset will be denoted by $\{B(t)\}_{t\in[0,T]}$. The **instantaneous interest rate** of a risk-free asset is a stochastic process $\{R(t)\}_{t\in[0,T]}$ such that $R(t) > 0$, for all $t \in [0,T]$, and such that the value of the asset at time t is given by

$$B(t) = B(0)\exp\left(\int_0^t R(s)\,ds\right), \quad t \in [0,T]. \tag{2.15}$$

This corresponds to the investor debit/credit with the money market at time t if the amount $B(0)$ is borrowed/lent by the investor at time $t = 0$. An investor lending (resp. borrowing)

[5] Loans with maturity longer than 1 year are called bonds; they will be discussed in more details in Chapter 6.

money has a long (resp. short) position on the risk-free asset (more precisely, on its interest rate). We remark that the integral in the right hand side of (2.15) is to be evaluated path by path, i.e.,

$$B(t,\omega) = B(0)\exp\left(\int_0^t R(s,\omega)\,ds\right),$$

for all *fixed* $\omega \in \Omega$. Although in the real world different risk-free assets have different interest rates, throughout these notes we make the simplifying assumption that *all* assets in the money market have the same instantaneous interest rate $\{R(t)\}_{t\in[0,T]}$, which we call the interest rate of the money market. For the applications in options pricing theory it is common to assume that the interest rate of the money market is a deterministic constant $R(t) = r$, for all $t \in [0,T]$. This assumption can be justified by the relatively short time of maturity of options, see below.

Remark 2.8. The (average) interest rate of the money market is sometimes referred to as "the cost of money", and the ratio $B(t)/B(0)$ is said to express the "time-value of money". This terminology is meant to emphasize that one reason for the "time-devaluation" of money—in the sense that the purchasing power of money decreases with time—is precisely the fact that money can grow interests by purchasing risk-free assets.

The discounting process

The stochastic process $\{D(t)\}_{t\geq 0}$ given by

$$D(t) = \exp\left(-\int_0^t R(s)\,ds\right) = \frac{B(0)}{B(t)}$$

is called the **discounting process**. In general, if an asset price is multiplied by $D(t)$, the new stochastic process is called the **discounted price** of the asset. We denote the discounted price by adding a subscript $*$ to the asset price. For instance, the discounted price of a stock with price $S(t)$ at time t is given by

$$S^*(t) = D(t)S(t).$$

Its meaning is the following: $S^*(t)$ is the amount that should be invested on the money market at time $t = 0$ in order that the value of this investment at time t replicates the value of the stock at time t. Notice that $S^*(t) < S(t)$. The discounted price of the stock measures, roughly speaking, the loss in the stock value due to the "time-devaluation" of money discussed above, see Remark 2.8.

Financial derivative

A **financial derivative** (or derivative security) is a contract whose value depends on the performance of one (or more) other asset(s), which is called the **underlying asset**. There exist various types of financial derivatives, the most common being options, futures, forwards and swaps. Financial derivatives can be traded **over the counter** (OTC), or in a regularized **market**. In the former case, the contract is stipulated between two individual investors, who agree upon the conditions and the price of the contract. In particular, the same derivative (on the same asset, with the same parameters) can have two different prices over the counter. Derivatives traded in the market, on the contrary, are standardized contracts. Anyone, after a proper authorization, can make offers to buy or sell derivatives in the market, in a way much similar to how stocks are traded. Let us see some examples of financial derivative (we shall introduce more in Chapter 6).

A **call option** is a contract between two parties, the buyer (or **owner**) of the call and the seller (or **writer**) of the call. The contract gives to the buyer the right, but not the obligation, to buy the underlying asset at some future time for a price agreed upon today, which is called **strike price** of the call. If the buyer can exercise this option only at some given time $t = T > 0$ (where $t = 0$ corresponds to the time at which the contract is stipulated) then the call option is called **European**, while if the option can be exercised at any time in the interval $(0, T]$, then the option is called **American**. The time $T > 0$ is called **maturity time**, or **expiration date** of the call. The seller of the call is obliged to sell the asset to the buyer if the latter decides to exercise the option. If the option to buy in the definition of a call is replaced by the option to sell, then the option is called a **put option**.

In exchange for the option, the buyer must pay a **premium** to the seller. Suppose that the option is a European option with strike price K, maturity time T and premium Π_0 on a stock with price $S(t)$ at time t. In which case is it then convenient for the buyer to exercise the call? Let us define the **payoff** of a European call as

$$Y = (S(T) - K)_+ := \max(0, S(T) - K),$$

i.e., $Y > 0$ if the stock price at the expiration date is higher than the strike price of the call and it is zero otherwise; similarly for a European put we set

$$Y = (K - S(T))_+.$$

Note that Y is a random variable, because it depends on the random variable $S(T)$. Clearly, if $Y > 0$ it is more convenient for the buyer to exercise the option rather than buying/selling the asset on the market. Note however that the real **profit** for the buyer is given by $N(Y - \Pi_0)$, where N is the number of option contracts owned by the buyer. Typically, options are sold in stocks of 100 shares, that is to say, the minimum amount of options that one can buy is 100, which cover 100 shares of the underlying asset.

One reason why investors buy calls in the market is to protect a short position on the underlying asset. In fact, suppose that an investor short-sells 100 shares of a stock at time $t = 0$ with the agreement to return them to the original owner at time $t_0 > 0$. The investor believes that the price of the stock will go down in the future, but of course the price may go up instead. To avoid possible large losses, at time $t = 0$ the investor buys 100 shares of an American call option on the stock expiring at $T \geq t_0$, and with strike price $K = S(0)$. If the price of the stock at time t_0 is not lower than the price $S(0)$ as the investor expected, then the investor will exercise the call, i.e., will buy 100 shares of the stock at the price $K = S(0)$. In this way the investor can return the shares to the lender with minimal losses. At the same fashion, investors buy put options to protect a long position on the underlying asset. The reason why investors write options is mostly to get liquidity (cash) to invest in other assets[6].

Let us introduce some further terminology. A European call (resp. put) is said to be **in the money** at time t if $S(t) > K$ (resp. $S(t) < K$). The call (resp. put) is said to be **out of the money** if $S(t) < K$ (resp. $S(t) > K$). If $S(t) = K$, the (call or put) option is said to be **at the money** at time t. The meaning of this terminology is self-explanatory.

The premium that the buyer has to pay to the seller for the option is the price (or value) of the option. It depends on time (in particular, on the time left to expiration). Clearly, the deeper in the money is the option, the higher will be its price. Therefore the holder of the long position on the option is the buyer, while the seller holds the short position on the option.

European call and put options are examples of more general contracts called **European derivatives**. Given a function $g : (0, \infty) \to \mathbb{R}$, a **standard European derivative** with pay-off $Y = g(S(T))$ and maturity time $T > 0$ is a contract that pays to its owner the amount Y at time $T > 0$. Here $S(T)$ is the price of the underlying asset (which we take to be a stock) at time T. The function g is called **pay-off function** of the derivative. The term "European" refers to the fact that the contract cannot be exercised before time T, while the term "standard" refers to the fact that the pay-off depends only on the price of the underlying at time T. The pay-off of a non-standard European derivative depends on the path of the asset price during the interval $[0, T]$. For example, the pay-off of an **Asian call** is given by $Y = (\int_0^T S(t)\,dt - K)_+$.

The price at time t of a European derivative (standard or not) with pay-off Y and expiration date T will be denoted by $\Pi_Y(t)$. Hence $\{\Pi_Y(t)\}_{t\in[0,T]}$ is a stochastic process. In addition, we now show that $\Pi_Y(T) = Y$ holds, i.e., there exist no offers to buy (sell) a derivative for less (more) than Y at the time of maturity. In fact, suppose that a derivative is sold for $\Pi_Y(T) < Y$ "just before" it expires at time T. In this way the buyer would make the sure profit $Y - \Pi_Y(T)$ at time T, which means that the seller would loose the same amount. On the contrary, upon buying a derivative "just before" maturity for more than Y, the buyer would loose $Y - \Pi_Y(T)$. Thus in a **rational market**, $\Pi_Y(T) = Y$ (or, more precisely, $\Pi_Y(t) \to Y$, as $t \to T$).

A **standard American derivative** with pay-off function g is a contract which can be exercised at any time $t \in (0, T]$ prior or equal to its maturity and that, upon exercise, pays the amount $g(S(t))$ to the holder of the derivative. In these notes we are mostly concerned with (standard) European derivatives, but in Chapter 6.9 we also discuss briefly some properties of American call/put options.

[6]Of course, speculation is also an important motivation to buy/sell options. However the standard theory of options pricing is firmly based on the interpretation of options as derivative securities and does not take speculation into account.

Portfolio

The portfolio of an investor is the set of all assets in which the investor is trading. Mathematically it is described by a collection of N stochastic processes

$$\{h_1(t)\}_{t\geq 0}, \{h_2(t)\}_{t\geq 0}, \dots, \{h_N(t)\}_{t\geq 0},$$

where $h_k(t)$ represents the number of shares of the asset k at time t in the investor portfolio. If $h_k(t)$ is positive, resp. negative, the investor has a long, resp. short, position on the asset k at time t. If $\Pi_k(t)$ denotes the value of the asset k at time t, then $\{\Pi_k(t)\}_{t\geq 0}$ is a stochastic process; the **portfolio value** is the stochastic process $\{V(t)\}_{t\geq 0}$ given by

$$V(t) = \sum_{k=1}^{N} h_k(t)\Pi_k(t).$$

Remark 2.9. For modeling purposes, it is convenient to assume that an investor can trade any fraction of shares of an asset, i.e., $h_k(t) : \Omega \to \mathbb{R}$, rather than $h_k(t) : \Omega \to \mathbb{Z}$.

The investor makes a profit in the time interval $[t_0, t_1]$ if $V(t_1) > V(t_0)$; the investor incurs in a loss in the interval $[t_0, t_1]$ if $V(t_1) < V(t_0)$. We now introduce the important definition of arbitrage portfolio.

Definition 2.17. *An* **arbitrage portfolio** *is a portfolio whose value* $\{V(t)\}_{t\geq 0}$ *satisfies the following properties, for some* $T > 0$*:*

(i) $V(0) = 0$ *almost surely;*

(ii) $V(T) \geq 0$ *almost surely;*

(iii) $\mathbb{P}(V(T) > 0) > 0$.

Hence an arbitrage portfolio is a risk-free investment in the interval $[0, T]$ which requires no initial wealth and with a positive probability to give profit. We remark that the arbitrage property depends on the probability measure $\mathbb{P}$. However, it is clear that if two measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, then the arbitrage property is satisfied with respect to $\mathbb{P}$ if and only if it is satisfied with respect to $\widetilde{\mathbb{P}}$. The guiding principle to devise theoretical models for asset prices in financial mathematics is to ensure that one cannot set-up an arbitrage portfolio by investing on these assets (**arbitrage-free principle**).

Markets

A market in which the objects of trading are N risky assets (e.g., stocks) and M risk-free assets in the money market is said to be "N+M dimensional". Most of these notes focus on the case of 1+1 dimensional markets in which we assume that the risky asset is a stock. A portfolio invested in this market is a pair $\{h_S(t), h_B(t)\}_{t\geq 0}$ of stochastic processes, where

$h_S(t)$ is the number of shares of the stock and $h_B(t)$ the number of shares of the risk-free asset in the portfolio at time t. The value of such portfolio is given by

$$V(t) = h_S(t)S(t) + h_B(t)B(t),$$

where $S(t)$ is the price of the stock (given for instance by (2.14)), while $B(t)$ is the value at time t of the risk-free asset, which is given by (2.15).

2.A Appendix: Solutions to selected problems

Exercise 2.1. Since $\Omega = \{X \in \mathbb{R}\}$, then $\Omega \in \sigma(X)$. $\{X \in U\}^c$ is the set of sample points $\omega \in \Omega$ such that $X(\omega) \notin U$. The latter is equivalent to $X(\omega) \in U^c$, hence $\{X \in U\}^c = \{X \in U^c\}$. Since $U^c \in \mathcal{B}(\mathbb{R})$, it follows that $\{X \in U\} \in \sigma(X)$. Finally we have to prove that $\sigma(X)$ is closed with respect to the countable union of sets. Let $\{A_k\}_{k\in\mathbb{N}} \subset \sigma(X)$. By definition of $\sigma(X)$, there exist $\{U_k\}_{k\in\mathbb{N}} \subset \mathcal{B}(\mathbb{R})$ such that $A_k = \{X \in U_k\}$. Thus we have the following chain of equivalent statements

$$\omega \in \cup_{k\in\mathbb{N}} A_k \Leftrightarrow \exists \bar{k} \in \mathbb{N} : X(\omega) \in U_{\bar{k}} \Leftrightarrow X(\omega) \in \cup_{k\in\mathbb{N}} U_k \Leftrightarrow \omega \in \{X \in \cup_{k\in\mathbb{N}} U_k\}.$$

Hence $\cup_{k\in\mathbb{N}} A_k = \{X \in \cup_{k\in\mathbb{N}} U_k\}$. Since $\cup_{k\in\mathbb{N}} U_k \in \mathcal{B}(\mathbb{R})$, then $\cup_{k\in\mathbb{N}} A_k \in \sigma(X)$.

Exercise 2.2. Let A be an event that is resolved by both variables X, Y. This means that there exist $I, J \subseteq \mathbb{R}$ such that $A = \{X \in I\} = \{Y \in J\}$. Hence, using the independence of X, Y,

$$\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(X \in I, Y \in J) = \mathbb{P}(X \in I)\mathbb{P}(Y \in J) = \mathbb{P}(A)\mathbb{P}(A) = \mathbb{P}(A)^2.$$

Therefore $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

Now let a, b two deterministic constants. Note that, for all $I \subset \mathbb{R}$,

$$\mathbb{P}(a \in I) = \begin{cases} 1 & \text{if } a \in I \\ 0 & \text{otherwise} \end{cases}$$

and similarly for b. Hence

$$\mathbb{P}(a \in I, b \in J) = \begin{cases} 1 & \text{if } a \in I \text{ and } b \in J \\ 0 & \text{otherwise} \end{cases} = \mathbb{P}(a \in I)\mathbb{P}(b \in J).$$

Finally we show that X and $Y = g(X)$ are independent if and only if Y is a deterministic constant. For the "if" part we use that

$$\mathbb{P}(a \in I, X \in J) = \begin{cases} \mathbb{P}(X \in J) & \text{if } a \in I \\ 0 & \text{otherwise} \end{cases} = \mathbb{P}(a \in I)\mathbb{P}(X \in J).$$

For the "only if" part, let $z \in \mathbb{R}$ and $I = \{g(X) \le z\} = \{X \in g^{-1}(-\infty, z]\}$. Then, using the independence of X and $Y = g(X)$,

$$\begin{aligned}\mathbb{P}(g(X) \le z) &= \mathbb{P}(g(X) \le z, g(X) \le z) = \mathbb{P}(X \in g^{-1}(-\infty, z], g(X) \le z) \\ &= \mathbb{P}(X \in g^{-1}(-\infty, z])\mathbb{P}(g(X) \le z) = \mathbb{P}(g(X) \le z)\mathbb{P}(g(X) \le z).\end{aligned}$$

Hence $\mathbb{P}(Y \leq z)$ is either 0 or 1, which implies that Y is a deterministic constant.

Exercise 2.4. $A \in \sigma(f(X))$ if and only if $A = \{f(X) \in U\}$, for some $U \in \mathcal{B}(\mathbb{R})$. The latter is equivalent to $X(\omega) \in \{f \in U\}$, hence $A = \{X \in \{f \in U\}\}$. Similarly, $B = \{Y \in \{g \in V\}\}$, for some $V \in \mathcal{B}(\mathbb{R})$. Hence

$$\mathbb{P}(A \cap B) = \mathbb{P}(\{X \in \{f \in U\}\} \cap \{Y \in \{g \in V\}\}).$$

As X and Y are independent, the right hand side is equal to $\mathbb{P}(\{X \in \{f \in U\}\})\mathbb{P}(\{Y \in \{g \in V\}\})$, hence $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, as claimed.

Exercise 2.5. Let's assume for simplicity that $M = 2$, i.e., $X = b_1\mathbb{I}_{B_1} + b_2\mathbb{I}_{B_2}$ with $B_1 \cap B_2 \neq \emptyset$ (the generalization to $M > 2$ is straightforward). Hence

$$X(\omega) = \begin{cases} b_1 & \text{for } \omega \in B_1 \setminus (B_1 \cap B_2) \\ b_2 & \text{for } \omega \in B_2 \setminus (B_1 \cap B_2) \\ b_1 + b_2 & \text{for } \omega \in B_1 \cap B_2 \end{cases}$$

Assume $b_1 \neq b_2$. Then upon defining the disjoint sets $A_1 = B_1 \setminus (B_1 \cap B_2)$, $A_2 = B_2 \setminus (B_1 \cap B_2)$, $A_3 = B_1 \cap B_2$, and the real numbers $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_1 + b_2$, we can rewrite X as the simple random variable

$$X = a_1\mathbb{I}_{A_1} + a_2\mathbb{I}_{A_2} + a_3\mathbb{I}_{A_3}.$$

If $b_1 = b_2 \neq 0$ we define $A_1 = B_1 \setminus (B_1 \cap B_2) \cup B_2 \setminus (B_1 \cap B_2)$, $A_2 = B_1 \cap B_2$, $a_1 = b_1 = b_2$, $a_2 = b_1 + b_2 = 2b_1$ and write X in the form

$$X = a_1\mathbb{I}_{A_1} + a_2\mathbb{I}_{A_2}.$$

Exercise 2.7. Write $(-\infty, b]$ as the disjoint union of the sets $(-\infty, a]$ and $(a, b]$. Hence

also $\{X \in (-\infty, a]\}$, $\{X \in (a, b]\}$ are disjoint. It follows that

$$\mathbb{P}(-\infty < X \leq b) = \mathbb{P}(-\infty < X \leq a) + \mathbb{P}(a < X \leq b),$$

that is, $F(b) = F(a) + \mathbb{P}(a < X \leq b)$, by which the claim follows. To establish that F_X is right-continuous we now show that

$$\mathbb{P}(X \leq x_0 + \frac{1}{n}) \to \mathbb{P}(X \leq x_0) \quad \text{as } n \to \infty, \text{ for all } x_0 \in \mathbb{R}.$$

By the first part of the exercise it suffices to show that $\mathbb{P}(x_0 < x \leq x_0 + \frac{1}{n}) \to 0$ as $n \to \infty$. The intervals $A_n = (x_0, x_0 + n^{-1}]$ satisfy $A_{n+1} \subset A_n$ and $\cap_n A_n = \emptyset$. Hence by Exercise 1.7 we have $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\emptyset) = 0$. The facts that F_X is increasing and converges to 1 as $x \to \infty$ are obvious.

Exercise 2.9. We first compute the distribution function F_Y of $Y = X^2$. Clearly, $F_Y(y) = \mathbb{P}(X^2 \leq y) = 0$, if $y \leq 0$. For $y > 0$ we have

$$F_Y(y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} < X < \sqrt{y}) = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{y}}^{\sqrt{y}} e^{-x^2/2}\, dx.$$

Hence, for $y > 0$,

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{\sqrt{2\pi}}\left(e^{-y/2}\frac{d}{dy}(\sqrt{y}) - e^{-y/2}\frac{d}{dy}(-\sqrt{y})\right) = \frac{1}{\sqrt{2\pi}}\frac{e^{-y/2}}{\sqrt{y}}.$$

Since $\Gamma(1/2) = \sqrt{\pi}$, this is the claim.

Exercise 2.12. A 2×2 symmetric matrix

$$C = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is positive definite if and only if $\operatorname{Tr} C = a + c > 0$ and $\det C = ac - b^2 > 0$. In particular, $a, c > 0$. Let us denote

$$a = \sigma_1^2, \quad c = \sigma_2^2, \quad \rho = \frac{b}{\sigma_1\sigma_2}.$$

Note that $\rho^2 = \frac{b^2}{ac} < 1$. Thus

$$C = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix},$$

and so

$$C^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2} \\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix}.$$

Substituting into (2.11) proves (2.12).

Exercise 2.14. $X \in \mathcal{N}(0,1)$ means that X has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},$$

while $Y \in \mathcal{E}(1)$ means that Y has density

$$f_Y(y) = e^{-y}\mathbb{I}_{y\geq 0}.$$

Since X, Y are independent, they have the joint density $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. Hence

$$\mathbb{P}(X \leq Y) = \iint\limits_{x\leq y} f_{X,Y}(x,y)\,dx\,dy = \int_{\mathbb{R}} dx\, e^{-\frac{x^2}{2}} \int_x^\infty dy\, \frac{1}{\sqrt{2\pi}}e^{-y}\mathbb{I}_{y\geq 0}.$$

To compute this integral, we first divide the domain of integration on the variable x in $x \leq 0$ and $x \geq 0$. So doing we have

$$\mathbb{P}(X \leq Y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 dx\, e^{-\frac{x^2}{2}} \int_0^\infty dy\, e^{-y} + \frac{1}{\sqrt{2\pi}}\int_0^\infty dx\, e^{-\frac{x^2}{2}} \int_x^\infty dy\, e^{-y}.$$

Computing the integrals we find

$$\mathbb{P}(X \leq Y) = \frac{1}{2} + e^{1/2}(1 - \Phi(1)),$$

where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$ is the standard normal distribution.

Exercise 2.17. The density of $S(t)$ is given by

$$f_{S(t)}(x) = \frac{d}{dx} F_{S(t)}(x),$$

provided the distribution $F_{S(t)}$, i.e.,

$$F_{S(t)}(x) = \mathbb{P}(S(t) \leq x)$$

is differentiable. Clearly, $f_{S(t)}(x) = F_{S(t)}(x) = 0$, for $x < 0$. For $x > 0$ we use that

$$S(t) \leq x \quad \text{if and only if} \quad W(t) \leq \frac{1}{\sigma}\left(\log \frac{x}{S(0)} - \alpha t\right) := A(x).$$

Thus,

$$\mathbb{P}(S(t) \leq x) = \mathbb{P}(-\infty < W(t) \leq A(x)) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{A(x)} e^{-\frac{y^2}{2t}}\, dy,$$

where for the second equality we used that $W(t) \in N(0, t)$. Hence

$$f_{S(t)}(x) = \frac{d}{dx}\left(\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{A(x)} e^{-\frac{y^2}{2t}}\, dy\right) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{A(x)^2}{2t}} \frac{dA(x)}{dx},$$

for $x > 0$, that is

$$f_{S(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \frac{1}{x} \exp\left\{-\frac{(\log x - \log S(0) - \alpha t)^2}{2\sigma^2 t}\right\} \mathbb{I}_{x>0}.$$

Since

$$\int_0^\infty f_{S(t)}(y) dy = 1,$$

then $p_0 = \mathbb{P}(S(t) = 0) = 0$.

Chapter 3

Expectation

Throughout this chapter we assume that $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ is a given filtered probability space.

3.1 Expectation and variance of random variables

Suppose that we want to estimate the value of a random variable X before the experiment has been performed. What is a reasonable definition for our "estimate" of X? Let us first assume that X is a simple random variable of the form

$$X = \sum_{k=1}^{N} a_k \mathbb{I}_{A_k},$$

for some finite partition $\{A_k\}_{k=1,\dots,N}$ of Ω and real distinct numbers $a_1, \dots, a_N$. In this case, it is natural to define the **expected value** (or **expectation**) of X as

$$\mathbb{E}[X] = \sum_{k=1}^{N} a_k \mathbb{P}(A_k) = \sum_{k=1}^{N} a_k \mathbb{P}(X = a_k).$$

That is to say, $\mathbb{E}[X]$ is a weighted average of all the possible values attainable by X, in which each value is weighted by its probability of occurrence. This definition applies also for $N = \infty$ (i.e., for discrete random variables) provided of course the infinite series converges. For instance, if $X \in \mathcal{P}(\mu)$ we have

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=0}^{\infty} k\mathbb{P}(X = k) = \sum_{k=0}^{\infty} k \frac{\mu^k e^{-\mu}}{k!} \\ &= e^{-\mu} \sum_{k=1}^{\infty} \frac{\mu^k}{(k-1)!} = e^{-\mu} \sum_{r=0}^{\infty} \frac{\mu^{r+1}}{r!} = e^{-\mu}\mu \sum_{r=0}^{\infty} \frac{\mu^r}{r!} = \mu.\end{aligned}$$

Exercise 3.1 (•). *Compute the expectation of binomial variables.*

Now let X be a non-negative random variable and consider the sequence $\{s_n^X\}_{n\in\mathbb{N}}$ of simple functions defined in Theorem 2.2. Recall that s_n^X converges pointwise to X as $n\to\infty$, i.e., $s_n^X(\omega)\to X(\omega)$, for all $\omega\in\Omega$ (see Exercise 2.6). Since

$$\mathbb{E}[s_n^X]=\sum_{k=1}^{n2^n-1}\frac{k}{2^n}\mathbb{P}(\frac{k}{2^n}\leq X<\frac{k+1}{2^n})+n\mathbb{P}(X\geq n), \tag{3.1}$$

it is natural to introduce the following definition.

Definition 3.1. *Let $X:\Omega\to[0,\infty)$ be a non-negative random variable. We define the expectation of X as*

$$\mathbb{E}[X]=\lim_{n\to\infty}\sum_{k=1}^{n2^n-1}\frac{k}{2^n}\mathbb{P}(\frac{k}{2^n}\leq X<\frac{k+1}{2^n})+n\mathbb{P}(X\geq n), \tag{3.2}$$

i.e., $\mathbb{E}[X]=\lim_{n\to\infty}\mathbb{E}[s_n^X]$, where $s_1^X,s_2^X,\dots$ is the sequence of simple functions converging pointwise to X and defined in Theorem 2.2.

We remark that the limit in (3.2) exists, because (3.1) is an increasing sequence (see next exercise), although this limit could be infinity. When the limit is finite we say that X has finite expectation. This happens for instance when X is bounded, i.e., $0\leq X\leq C$ a.s., for some positive constant C.

Exercise 3.2. *Show that $\mathbb{E}[s_n^X]$ is increasing in $n\in\mathbb{N}$. Show that the limit* (3.2) *is finite when the non-negative random variable X is bounded.*

Remark 3.1 (Monotone convergence theorem)**.** It can be shown that that the limit (3.2) is the same along any non-decreasing sequence of non-negative random variables that converge pointwise to X, hence we can use any such sequence to define the expectation of a non-negative random variable. This follows by the **monotone convergence theorem**, whose precise statement is the following: If $X_1,X_2,,\dots$ is a non-decreasing sequence of non-negative random variables such that $X_n\to X$ pointwise a.s., then $\mathbb{E}[Z_n]\to\mathbb{E}[Z]$.

Remark 3.2 (Dominated convergence theorem)**.** The sequence of simple random variables used to define the expectation of a non-negative random variable need not be non-decreasing either. This follows by the **dominated convergence theorem**, whose precise statement is the following: if $X_1,X_2,\dots$ is a sequence of non-negative random variables such that $X_n\to X$, as $n\to\infty$, pointwise a.s., and $\sup_n X_n\leq Y$ for some non-negative random variable Y with finite expectation, then $\lim_{n\to\infty}\mathbb{E}[X_n]=\mathbb{E}[X]$.

Next we extend the definition of expectation to general random variables. For this purpose we use that every random variable $X:\Omega\to\mathbb{R}$ can be written as

$$X=X_+-X_-,$$

where

$$X_+=\max(0,X),\quad X_-=-\min(X,0)$$

are respectively the positive and negative part of X. Since $X_\pm$ are non-negative random variables, then their expectation is given as in Definition 3.1.

Definition 3.2. *Let $X : \Omega \to \mathbb{R}$ be a random variable and assume that at least one of the random variables X_+, X_- has finite expectation. Then we define the expectation of X as*

$$\mathbb{E}[X] = \mathbb{E}[X_+] - \mathbb{E}[X_-].$$

If $X_\pm$ have both finite expectation, we say that X has finite expectation or that it is an **integrable** *random variable. The set of all integrable random variables on Ω will be denoted by $L^1(\Omega)$, or by $L^1(\Omega, \mathbb{P})$ if we want to specify the probability measure.*

Remark 3.3 (Notation)**.** Of course the expectation of a random variable depends on the probability measure. If another probability measure $\widetilde{\mathbb{P}}$ is defined on the σ-algebra of events (not necessarily equivalent to $\mathbb{P}$), we denote the expectation of X in $\widetilde{\mathbb{P}}$ by $\widetilde{\mathbb{E}}[X]$.

Remark 3.4 (Expectation=Lebesgue integral)**.** The expectation of a random variable X with respect to the probability measure $\mathbb{P}$ is also called the Lebesgue integral of X over Ω in the measure $\mathbb{P}$ and it is also denoted by

$$\mathbb{E}[X] = \int_\Omega X(\omega) d\mathbb{P}(\omega).$$

We shall not use this notation.

The following theorem collects some useful properties of the expectation:

Theorem 3.1. *Let $X, Y : \Omega \to \mathbb{R}$ be integrable random variables. Then the following holds:*

(i) Linearity: For all $\alpha, \beta \in \mathbb{R}$, $\mathbb{E}[\alpha X + \beta Y] = \alpha\mathbb{E}[X] + \beta\mathbb{E}[Y]$;

(ii) If $X \leq Y$ a.s. then $\mathbb{E}[X] \leq \mathbb{E}[Y]$;

(iii) If $X \geq 0$ a.s., then $\mathbb{E}[X] = 0$ if and only if $X = 0$ a.s.;

(iv) If X, Y are independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

Sketch of the proof. For all claims, the argument of the proof is divided in three steps: STEP 1: Show that it suffices to prove the claim for non-negative random variables. STEP 2: Prove the claim for simple functions. STEP 3: Take the limit along the sequences $\{s_n^X\}_{n\in\mathbb{N}}$, $\{s_n^Y\}_{n\in\mathbb{N}}$ of simple functions converging to X, Y. Carrying out these three steps for (i), (ii) and (iii) is simpler, so let us focus on (iv). Let $X_+ = f(X)$, $X_- = g(X)$, and similarly for Y, where $f(s) = \max(0, s)$, $g(s) = -\min(0, s)$. By Exercise 2.4, each of (X_+, Y_+), (X_-, Y_+), (X_+, Y_-) and (X_-, Y_-) is a pair of independent (non-negative) random variables. Assume that the claim is true for non-negative random variables. Then, using $X = X_+ - X_-$, $Y = Y_+ - Y_-$

and the linearity of the expectation, we find

$$\begin{aligned}\mathbb{E}[XY] &= \mathbb{E}[(X_+ - X_-)(Y_+ - Y_-)] \\ &= \mathbb{E}[X_+Y_+] - \mathbb{E}[X_-Y_+] - \mathbb{E}[X_+Y_-] + \mathbb{E}[X_-Y_-] \\ &= \mathbb{E}[X_+]\mathbb{E}[Y_+] - \mathbb{E}[X_-]\mathbb{E}[Y_+] - \mathbb{E}[X_+]\mathbb{E}[Y_-] + \mathbb{E}[X_-]\mathbb{E}[Y_-] \\ &= (\mathbb{E}[X_+] - \mathbb{E}[X_-])(\mathbb{E}[Y_+] - \mathbb{E}[Y_-]) = \mathbb{E}[X]\mathbb{E}[Y].\end{aligned}$$

Hence it suffices to prove the claim for non-negative random variables. Next assume that X, Y are independent simple functions and write

$$X = \sum_{j=1}^{N} a_j \mathbb{I}_{A_j}, \quad Y = \sum_{k=1}^{M} b_k \mathbb{I}_{B_k}.$$

We have

$$XY = \sum_{j=1}^{N}\sum_{k=1}^{M} a_j b_k \mathbb{I}_{A_j}\mathbb{I}_{B_k} = \sum_{j=1}^{N}\sum_{k=1}^{M} a_j b_k \mathbb{I}_{A_j \cap B_k}.$$

Thus by linearity of the expectation, and since the events A_j, B_k are independent, for all j, k, we have

$$\begin{aligned}\mathbb{E}[XY] &= \sum_{j=1}^{N}\sum_{k=1}^{M} a_j b_k \mathbb{E}[\mathbb{I}_{A_j \cap B_k}] = \sum_{j=1}^{N}\sum_{k=1}^{M} a_j b_k \mathbb{P}(A_j \cap B_k) \\ &= \sum_{j=1}^{N}\sum_{k=1}^{M} a_j b_k \mathbb{P}(A_j)\mathbb{P}(B_k) = \sum_{j=1}^{N} a_j \mathbb{P}(A_j) \sum_{k=1}^{M} b_k \mathbb{P}(B_k) = \mathbb{E}[X]\mathbb{E}[Y].\end{aligned}$$

Hence the claim holds for simple functions. It follows that

$$\mathbb{E}[s_n^X s_n^Y] = \mathbb{E}[s_n^X]\mathbb{E}[s_n^Y].$$

Letting $n \to \infty$, the right hand side converges to $\mathbb{E}[X]\mathbb{E}[Y]$. To complete the proof we have to show that the left hand side converges to $\mathbb{E}[XY]$. This follows by applying the monotone convergence theorem (see Remark 3.1) to the sequence $Z_n = s_n^X s_n^Y$. □

As $|X| = X_+ + X_-$, a random variable X is integrable if and only if $\mathbb{E}[|X|] < \infty$. Hence we have

$$X \in L^1(\Omega) \Leftrightarrow \mathbb{E}[X] < \infty \Leftrightarrow \mathbb{E}[|X|] < \infty.$$

The set of random variables $X : \Omega \to \mathbb{R}$ such that $|X|^2$ is integrable, i.e., $\mathbb{E}[|X|^2] < \infty$, will be denoted by $L^2(\Omega)$ or $L^2(\Omega, \mathbb{P})$.

Exercise 3.3 (•). *Prove the Schwarz inequality,*

$$\mathbb{E}[XY] \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}, \tag{3.3}$$

for all random variables $X, Y \in L^2(\Omega)$.

Letting $Y = 1$ in (3.3), we find

$$L^1(\Omega) \subset L^2(\Omega).$$

The **covariance** $\mathrm{Cov}(X, Y)$ of two random variables $X, Y \in L^2(\Omega)$ is defined as

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Two random variables are said to be **uncorrelated** if $\mathrm{Cov}(X, Y) = 0$. By Theorem 3.1(iv), if X, Y are independent then they are uncorrelated, but the opposite is not true in general. Consider for example the simple random variables

$$X = \begin{cases} -1 & \text{with probability } 1/3 \\ 0 & \text{with probability } 1/3 \\ 1 & \text{with probability } 1/3 \end{cases}$$

and

$$Y = X^2 = \begin{cases} 0 & \text{with probability } 1/3 \\ 1 & \text{with probability } 2/3 \end{cases}$$

Then X and Y are clearly not independent, but

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[X^3] - 0 = 0,$$

since $\mathbb{E}[X^3] = \mathbb{E}[X] = 0$.

Definition 3.3. *The* **variance** *of a random variable* $X \in L^2(\Omega)$ *is given by*

$$\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

Using the linearity of the expectation we can rewrite the definition of variance as

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - 2\mathbb{E}[\mathbb{E}[X]X] + \mathbb{E}[X]^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathrm{Cov}(X, X).$$

Note that a random variable has zero variance if and only if $X = \mathbb{E}[X]$ a.s., hence we may view $\mathrm{Var}[X]$ as a measure of the "randomness of X". As a way of example, let us compute the variance of $X \in \mathcal{P}(\mu)$. We have

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{k=0}^{\infty} k^2 \mathbb{P}(X = k) = \sum_{k=0}^{\infty} k^2 \frac{\mu^k e^{-\mu}}{k!} = e^{-\mu} \sum_{k=1}^{\infty} \frac{k}{(k-1)!}\mu^k \\
&= e^{-\mu} \sum_{r=0}^{\infty} \frac{r+1}{r!}\mu^{r+1} = e^{-\mu}\mu \sum_{r=0}^{\infty} \frac{\mu^r}{r!} + \mu \sum_{r=0}^{\infty} r\mathbb{P}(X = r) = \mu + \mu\mathbb{E}[X] = \mu + \mu^2.
\end{aligned}$$

Hence

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mu + \mu^2 - \mu^2 = \mu.$$

Exercise 3.4. *Compute the variance of binomial random variables.*

Exercise 3.5 (•). *Prove the following:*

1. $\text{Var}[\alpha X] = \alpha^2 \text{Var}[X]$, *for all constants* $\alpha \in \mathbb{R}$*;*
2. $\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}(X,Y)$*;*
3. $-\sqrt{\text{Var}[X]\text{Var}[Y]} \leq \text{Cov}(X,Y) \leq \sqrt{\text{Var}[X]\text{Var}[Y]}$. *The left (resp. right) inequality becomes an equality if and only if there exists a negative (resp. positive) constant* a_0 *and a real constant* b_0 *such that* $Y = a_0 X + b_0$ *almost surely.*

By the previous exercise, $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$ holds if and only if X, Y are uncorrelated. Moreover, if we define the **correlation** of X, Y as

$$\text{Cor}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}},$$

then $\text{Cor}(X,Y) \in [-1,1]$ and $|\text{Cor}(X,Y)| = 1$ if and only if Y is a linear function of X. The interpretation is the following: the closer is $\text{Cor}(X,Y)$ to 1 (resp. -1), the more the variable X and Y have tendency to move in the same (resp. opposite) direction (for instance, ($\text{Cor}(X,-2X) = -1$, $\text{Cor}(X,2X) = 1$). An important problem in quantitative finance is to find correlations between the price of different assets.

Exercise 3.6. *Let* $\{M_k\}_{k\in\mathbb{N}}$ *be a symmetric random walk. Show that* $\mathbb{E}[M_k] = 0$*, and that* $\text{Var}[M_k] = k$*, for all* $k \in \mathbb{N}$*.*

Exercise 3.7. *Show that the function* $\|\cdot\|_2$ *which maps a random variable* Z *to* $\|Z\|_2 = \sqrt{\mathbb{E}[Z^2]}$ *is a norm in* $L^2(\Omega)$*.*

Remark 3.5 (L^2-norm)**.** The norm defined in the previous exercise is called L^2 **norm.** It can be shown that it is a complete norm, i.e., if $\{X_N\}_{n\in\mathbb{N}} \subset L^2(\Omega)$ is a Cauchy sequence of random variables in the norm L^2, then there exists a random variable $X \in L^2(\Omega)$ such that $\|X_N - X\|_2 \to 0$ as $n \to \infty$.

Exercise 3.8 (•). *Let* $\{W_n(t)\}_{t\geq 0}$*,* $n \in \mathbb{N}$*, be the sequence of stochastic processes defined in* (2.13)*. Compute* $\mathbb{E}[W_n(t)]$, $\text{Var}[W_n(t)]$, $\text{Cov}[W_n(t), W_n(s)]$*. Show that*

$$\text{Var}(W_n(t)) \to t, \quad \text{Cov}(W_n(t), W_n(s)) \to \min(s,t), \quad \textit{as } n \to +\infty.$$

Next we want to present a first application in finance of the theory outlined above. In particular we establish a sufficient condition which ensures that a portfolio is not an arbitrage.

Theorem 3.2. *Let a portfolio be given with value* $\{V(t)\}_{t\geq 0}$*. Let* $V^*(t) = D(t)V(t)$ *be the discounted portfolio value. If there exists a measure* $\widetilde{\mathbb{P}}$ *equivalent to* $\mathbb{P}$ *such that* $\widetilde{\mathbb{E}}[V^*(t)]$ *is constant (independent of* t*), then the portfolio is not an arbitrage.*

Proof. Assume that the portfolio is an arbitrage. Then $V(0) = 0$ almost surely; as $V^*(0) =$

$V(0)$, the assumption of constant expectation in the probability measure $\widetilde{\mathbb{P}}$ gives

$$\widetilde{\mathbb{E}}[V^*(t)] = 0, \quad \text{for all } t \geq 0. \tag{3.4}$$

Let $T > 0$ be such that $\mathbb{P}(V(T) \geq 0) = 1$ and $\mathbb{P}(V(T) > 0) > 0$. Since $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, we also have $\widetilde{\mathbb{P}}(V(T) \geq 0) = 1$ and $\widetilde{\mathbb{P}}(V(T) > 0) > 0$. Since the discounting process is positive, we also have $\widetilde{\mathbb{P}}(V^*(T) \geq 0) = 1$ and $\widetilde{\mathbb{P}}(V^*(T) > 0) > 0$. However this contradicts (3.4), due to Theorem 3.1(iii). Hence our original hypothesis that the portfolio is an arbitrage portfolio is false. □

Theorem 3.2 will be applied in Chapter 6. To this purpose we shall need the following characterization of equivalent probability measures.

Theorem 3.3. *The following statements are equivalents:*

(i) $\mathbb{P}$ *and* $\widetilde{\mathbb{P}}$ *are equivalent probability measures;*

(ii) There exists a unique (up to null sets) random variable $Z : \Omega \to \mathbb{R}$ *such that* $Z > 0$ *almost surely,* $\mathbb{E}[Z] = 1$ *and* $\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z\mathbb{I}_A]$*, for all* $A \in \mathcal{F}$*.*

Moreover, assuming any of these two equivalent conditions, for all random variables X *such that* $XZ \in L^1(\Omega, \mathbb{P})$*, we have* $X \in L^1(\Omega, \widetilde{\mathbb{P}})$ *and*

$$\widetilde{\mathbb{E}}[X] = \mathbb{E}[ZX]. \tag{3.5}$$

Proof. The implication $(i) \Rightarrow (ii)$ follows by the Radon-Nikodým theorem, whose proof can be found for instance in [6]. As to the implication $(ii) \Rightarrow (i)$, we first observe that $\widetilde{\mathbb{P}}(\Omega) = \mathbb{E}[Z\mathbb{I}_\Omega] = \mathbb{E}[Z] = 1$. Hence, to prove that $\widetilde{\mathbb{P}}$ is a probability measure, it remains to show that it satisfies the countable additivity property: for all families $\{A_k\}_{k\in\mathbb{N}}$ of disjoint events, $\widetilde{\mathbb{P}}(\cup_k A_k) = \sum_k \widetilde{\mathbb{P}}(A_k)$. To prove this let

$$B_n = \cup_{k=1}^n A_k.$$

Clearly, $Z\mathbb{I}_{B_n}$ is an increasing sequence of random variables. Hence, by the monotone convergence theorem (see Remark 3.1) we have

$$\lim_{n\to\infty} \mathbb{E}[Z\mathbb{I}_{B_n}] = \mathbb{E}[Z\mathbb{I}_{B_\infty}], \quad B_\infty = \cup_{k=1}^\infty A_k,$$

i.e.,

$$\lim_{n\to\infty} \widetilde{\mathbb{P}}(B_n) = \widetilde{\mathbb{P}}(B_\infty). \tag{3.6}$$

On the other hand, by linearity of the expectation,

$$\widetilde{\mathbb{P}}(B_n) = \mathbb{E}[ZB_n] = \mathbb{E}[Z\mathbb{I}_{\cup_{k=1}^n A_k}] = \mathbb{E}[Z(\mathbb{I}_{A_1} + \cdots + \mathbb{I}_{A_n})] = \sum_{k=1}^n \mathbb{E}[ZA_k] = \sum_{k=1}^n \widetilde{\mathbb{P}}(A_k).$$

Hence (3.6) becomes

$$\sum_{k=1}^\infty \widetilde{\mathbb{P}}(A_k) = \widetilde{\mathbb{P}}(\cup_{k=1}^\infty A_k).$$

This proves that $\widetilde{\mathbb{P}}$ is a probability measure. To show that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, let A be such that $\widetilde{\mathbb{P}}(A) = 0$. Since $Z\mathbb{I}_A \geq 0$ almost surely, then $\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z\mathbb{I}_A] = 0$ is equivalent, by Theorem 3.1(iii), to $Z\mathbb{I}_A = 0$ almost surely. Since $Z > 0$ almost surely, then this is equivalent to $\mathbb{I}_A = 0$ a.s., i.e., $\mathbb{P}(A) = 0$. Thus $\widetilde{\mathbb{P}}(A) = 0$ if and only if $\mathbb{P}(A) = 0$, i.e., the probability measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent. It remains to prove the identity (3.5). If X is the simple random variable $X = \sum_k a_k \mathbb{I}_{A_k}$, then the proof is straightforward:

$$\widetilde{\mathbb{E}}[X] = \sum_k a_k \widetilde{\mathbb{P}}(A_k) = \sum_k a_k \mathbb{E}[Z\mathbb{I}_{A_k}] = \mathbb{E}[Z \sum_k a_k \mathbb{I}_{A_k}] = \mathbb{E}[ZX].$$

For a general non-negative random variable X the result follows by applying (3.5) to an increasing sequence of simple random variables converging to X and then passing to the limit (using the monotone convergence theorem). The result for a general random variable $X : \Omega \to \mathbb{R}$ follows by applying (3.5) to the positive and negative part of X and using the linearity of the expectation. □

Remark 3.6 (Radon-Nikodým derivative)**.** Using the Lebesgue integral notation (see Remark 3.4) we can write (3.5) as

$$\int_\Omega X(\omega) d\widetilde{\mathbb{P}}(\omega) = \int_\Omega X(\omega) Z(\omega) d\mathbb{P}(\omega).$$

This leads to the formal identity $d\widetilde{\mathbb{P}}(\omega) = Z(\omega)d\mathbb{P}(\omega)$, or $Z(\omega) = \frac{d\widetilde{\mathbb{P}}(\omega)}{d\mathbb{P}(\omega)}$, which explains why Z is also called the Radon-Nikodým derivative of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$.

An application of Theorem (3.3) is given in Exercise 3.11 below.

3.2 Computing the expectation of a random variable

Next we discuss how to compute the expectation of a random variable X. Definition 3.1 is clearly not very useful to this purpose, unless X is a simple random variable. There exist several methods to compute the value for $\mathbb{E}[X]$, some of which will be presented later in these notes. In this section we show that the expectation and the variance of a random variable can be computed easily when the random variable admits a density.

Theorem 3.4. *Let $X : \Omega \to \mathbb{R}$ be a random variable and $g : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $g(X) \in L^1(\Omega)$. Assume that X admits the density f_X. Then*

$$\mathbb{E}[g(X)] = \int_\mathbb{R} g(x) f_X(x)\, dx.$$

In particular, the expectation and the variance of X are given by

$$\mathbb{E}[X] = \int_\mathbb{R} x f_X(x)\, dx, \quad \mathrm{Var}[X] = \int_\mathbb{R} x^2 f_X(x)\, dx - \left(\int_\mathbb{R} x f_X(x)\, dx\right)^2.$$

Proof. We prove the theorem under the assumption that g is a simple measurable function, the proof for general functions g follows by a limit argument similar to the one used in the proof of Theorem 3.1, see Theorem 1.5.2 in [21] for the details. Hence we assume

$$g(x) = \sum_{k=1}^{N} \alpha_k \mathbb{I}_{U_k}(x),$$

for some disjoint Borel sets $U_1, \dots, U_N \subset \mathbb{R}$. Thus

$$\mathbb{E}[g(X)] = \mathbb{E}\big[\sum_k \alpha_k \mathbb{I}_{U_k}(X)\big] = \sum_k \alpha_k \mathbb{E}[\mathbb{I}_{U_k}(X)].$$

Let $Y_k = \mathbb{I}_{U_k}(X) : \Omega \to \mathbb{R}$. Then Y_k is the simple random variable that takes value 1 if $\omega \in A_k$ and 0 if $\omega \in A_k^c$, where $A_k = \{X \in U_k\}$. Thus the expectation of Y_k is given by $\mathbb{E}[Y_k] = \mathbb{P}(A_k)$ and so

$$\mathbb{E}[g(X)] = \sum_k \alpha_k \mathbb{P}(X \in U_k) = \sum_k \alpha_k \int_{U_k} f(x)\,dx = \int_{\mathbb{R}} \sum_k \alpha_k \mathbb{I}_{U_k}(x) f(x)\,dx = \int_{\mathbb{R}} g(x) f(x)\,dx,$$

as claimed. □

For instance, if $X \in \mathcal{N}(m, \sigma^2)$, we have

$$\mathbb{E}[X] = \int_{\mathbb{R}} x e^{-\frac{(x-m)^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi\sigma^2}} = m,$$
$$\mathrm{Var}[X] = \int_{\mathbb{R}} x^2 e^{-\frac{(x-m)^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi\sigma^2}} - m^2 = \sigma^2,$$

which explains why we called m the expectation and σ^2 the variance of the normal random variable X. Note in particular that, for a Brownian motion $\{W(t)\}_{t\geq 0}$, there holds

$$\mathbb{E}[W(t) - W(s)] = 0, \quad \mathrm{Var}[W(t) - W(s)] = |t - s|, \quad \text{for all } s, t \geq 0. \tag{3.7}$$

Let us show that[1]

$$\mathrm{Cov}(W(t), W(s)) = \min(s, t). \tag{3.8}$$

For $s = t$, the claim is equivalent to $\mathrm{Var}[W(t)] = t$, which holds by definition of Brownian motion (see (3.7)). For $t > s$ we have

$$\begin{aligned}\mathrm{Cov}(W(t), W(s)) &= \mathbb{E}[W(t)W(s)] - \mathbb{E}[W(t)]\mathbb{E}[W(s)] \\ &= \mathbb{E}[W(t)W(s)] \\ &= \mathbb{E}[(W(t) - W(s))W(s)] + \mathbb{E}[W(s)^2].\end{aligned}$$

Since $W(t)-W(s)$ and $W(s)$ are independent random variables, then $\mathbb{E}[(W(t)-W(s))W(s)] = \mathbb{E}(W(t) - W(s)]\mathbb{E}[W(s)] = 0$, and so

$$\mathrm{Cov}(W(t), W(s)) = \mathbb{E}[W(s)^2] = \mathrm{Var}[W(s)] = s = \min(s, t), \quad \text{for } t > s.$$

A similar argument applies for $t < s$.

[1] Compare (3.8) with the result of Exercise 3.8.

Exercise 3.9. *The moment of order n of a random variable X is the quantity* $\mu_n = \mathbb{E}[X^n]$, $n = 1, 2, \dots$. *Let* $X \in \mathcal{N}(0, \sigma^2)$. *Prove that*

$$\mu_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 \cdot 3 \cdot 5 \cdot \dots \cdot \dots (n-1)\sigma^n & \text{if } n \text{ is even.} \end{cases}$$

Exercise 3.10 (•). *Compute the expectation and the variance of exponential random variables.*

Exercise 3.11 (•). *Let* $X \in \mathcal{E}(\lambda)$ *an exponential random variable with intensity* λ. *Given* $\widetilde{\lambda} > 0$, *let*

$$Z = \frac{\widetilde{\lambda}}{\lambda} e^{-(\widetilde{\lambda}-\lambda)X}.$$

Define $\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z\mathbb{I}_A]$, $A \in \mathcal{F}$. *Show that* $\widetilde{\mathbb{P}}$ *is a probability measure equivalent to* $\mathbb{P}$. *Prove that* $X \in \mathcal{E}(\widetilde{\lambda})$ *in the probability measure* $\widetilde{\mathbb{P}}$.

Exercise 3.12. *Compute the expectation and the variance of Cauchy distributed random variables. Compute the expectation and the variance of Lévy distributed random variables.*

Exercise 3.13. *Compute the expectation and the variance of the geometric Brownian motion* (2.14).

Exercise 3.14. *Show that the paths of the Brownian motion have unbounded linear variation. Namely, given* $0 = t_0 < t_1 < \dots < t_n = t$ *with* $t_k - t_{k-1} = h$, *for all* $k = 1, \dots, n$, *show that*

$$\mathbb{E}[\sum_{k=1}^{n} |W(t_k) - W(t_{k-1})|] \to \infty, \quad \text{as } n \to \infty.$$

(However, Brownian motions have finite quadratic variation, see Section 3.4).

A result similar to Theorem 3.4 can be used to compute the correlation between two random variables that admit a joint density.

Theorem 3.5. *Let* $X, Y : \Omega \to \mathbb{R}$ *be two random variables with joint density* $f_{X,Y} : \mathbb{R}^2 \to [0, \infty)$ *and let* $g : \mathbb{R}^2 \to \mathbb{R}$ *be a measurable function such that* $g(X, Y) \in L^1(\Omega)$. *Then*

$$\mathbb{E}[g(X, Y)] = \int_{\mathbb{R}^2} g(x, y) f_{X,Y}(x, y)\, dx\, dy.$$

In particular, for $X, Y \in L^2(\Omega)$,

$$\mathrm{Cov}(X, Y) = \int_{\mathbb{R}^2} xy f_{X,Y}(x, y)\, dx\, dy - \int_{\mathbb{R}} x f_X(x)\, dx \int_{\mathbb{R}} y f_Y(y)\, dy,$$

where

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x, y)\, dy, \quad f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x, y)\, dx$$

are the (marginal) densities of X and Y.

Exercise 3.15. *Show that if* $X_1, X_2 : \Omega \to \mathbb{R}$ *are jointly normally distributed with covariant matrix* $C = (C_{ij})_{i,j=1,2}$*, then* $C_{ij} = \mathrm{Cov}(X_i, X_j)$*.*

Combining the results of Exercises 2.12 and 3.15, we see that the parameter ρ in Equation (2.12) is precisely the correlation of the two jointly normally distributed random variables X, Y. It follows by Remark 2.3 that two jointly normally distributed random variables are independent if and only if they are uncorrelated. Recall that for general random variables, independence implies uncorrelation, but the opposite is in general not true.

3.3 Characteristic function

In this section, and occasionally in the rest of the notes, we shall need to take the expectation of a complex-valued random variable $Z : \Omega \to \mathbb{C}$. Letting $Z = \mathrm{Re}(Z) + i\mathrm{Im}(Z)$, the expectation of Z is the complex number defined by

$$\mathbb{E}[Z] = \mathbb{E}[\mathrm{Re}(Z)] + i\mathbb{E}[\mathrm{Im}(Z)].$$

Definition 3.4. *Let* $X \in L^1(\Omega)$*. The function* $\theta_X : \mathbb{R} \to \mathbb{C}$ *given by*

$$\theta_X(u) = \mathbb{E}[e^{iuX}]$$

is called the **characteristic function of** X**.** *The positive, real-valued function* $M_X(u) = \mathbb{E}[e^{uX}]$*, when it exists in some neighborhood of* $u = 0$*, is called the* **moment-generating function of** X**.**

Note that if the random variable X admits the density f_X, then

$$\theta_X(u) = \int_{\mathbb{R}} e^{iux} f_X(x)\, dx,$$

i.e., the characteristic function is the inverse Fourier transform of the density. Table 3.1 contains some examples of characteristic functions.

Note carefully that, while θ_X is defined for all $u \in \mathbb{R}$, the moment-generating function of a random variable may be defined only in a subset of the real line, or not defined at all (see Exercise 3.16). For instance, when $X \in \mathcal{E}(\lambda)$ we have

$$M_X(u) = \mathbb{E}[e^{uX}] = \lambda \int_0^\infty e^{(u-\lambda)x}\, dx = \begin{cases} +\infty & \text{if } u \geq \lambda \\ (1 - u/\lambda)^{-1} & \text{if } u < \lambda \end{cases}$$

Hence $M_X(u)$ which is defined (as a positive function) only for $u < \lambda$.

Exercise 3.16. *Show that Cauchy random variables do not have a well-defined moment-generating function.*

The characteristic function of a random variable provides a lot of information. In particular, it determines completely the distribution function of the random variable, as shown in the following theorem (for the proof, see [6, Sec. 9.5]).

Density	Characteristic function
$\mathcal{N}(m,\sigma^2)$	$\exp(ium - \frac{1}{2}\sigma^2u^2)$
$\mathcal{E}(\lambda)$	$(1-iu/\lambda)^{-1}$
$\chi^2(\delta)$	$(1-2iu)^{-\delta/2}$
$\chi^2(\delta,\beta)$	$(1-2iu)^{-\delta/2}\exp\left(-\frac{\beta u}{2u+i}\right)$

Table 3.1: Examples of characteristic functions

Theorem 3.6. *Let $X, Y \in L^1(\Omega)$. Then $\theta_X = \theta_Y$ if and only if $F_X = F_Y$. In particular, if $\theta_X = \theta_Y$ and one of the two variables admits a density, the other does too and the densities are equal.*

According to the previous theorem, if we want for instance to prove that a random variable X is normally distributed, we may try to show that its characteristic function θ_X is given by $\theta_X(u) = \exp(ium - \frac{1}{2}\sigma^2u^2)$, see Table 3.1. Another useful property of characteristic functions is proved in the following theorem.

Theorem 3.7. *Let $X_1, \dots, X_N \in L^1(\Omega)$ be independent random variables. Then*

$$\theta_{X_1+\dots+X_N} = \theta_{X_1}\dots\theta_{X_N}.$$

Proof. We have

$$\theta_{X_1+\dots+X_N}(u) = \mathbb{E}[e^{iu(X_1+\dots+X_N)}] = \mathbb{E}[e^{iuX_1}e^{iuX_2}\dots e^{iuX_N}].$$

Using that the variables $Y_1 = e^{iuX_1}, \dots, Y_N = e^{iuX_N}$ are independent (see Theorem 2.1) and that the expectation of the product of independent random variables is equal to the product of their expectations (see Theorem 3.1(iv)) we obtain

$$\mathbb{E}[e^{iuX_1}e^{iuX_2}\dots e^{iuX_N}] = \mathbb{E}[e^{iuX_1}]\dots\mathbb{E}[e^{uX_N}] = \theta_{X_1}(u)\dots\theta_{X_N}(u),$$

which concludes the proof. □

As an example of application of the previous theorem, we now show that if $X_1, \dots X_N$ are independent normally distributed random variables with expectations $m_1, \dots, m_N$ and variances $\sigma_1^2, \dots, \sigma_N^2$, then the random variable

$$Y = X_1 + \dots + X_N$$

is normally distributed with mean m and variance σ^2 given by

$$m = m_1 + \dots + m_N, \quad \sigma^2 = \sigma_1^2 + \dots + \sigma_N^2. \tag{3.9}$$

In fact,

$$\theta_{X_1+\dots+X_N}(u) = \theta_{X_1}(u)\dots\theta_{X_N}(u) = e^{ium_1-\frac{1}{2}\sigma_1^2u^2}\dots e^{ium_N-\frac{1}{2}\sigma_N^2u^2} = e^{ium-\frac{1}{2}\sigma^2u^2}.$$

The right hand side of the previous equation is the characteristic function of a normal variable with expectation m and variance σ^2 given by (3.9). Thus Theorem 3.6 implies that $X_1 + \dots + X_N \in \mathcal{N}(m, \sigma^2)$.

Exercise 3.17. *Let $X_1 \in \mathcal{N}(m_1, \sigma_1^2), \dots, X_N \in \mathcal{N}(m_N, \sigma_N^2)$, $N \geq 2$, be independent. Show that $Y = \sum_{k=1}^N (X_k/\sigma_k)^2 \in \chi^2(N, \beta)$ where $\beta = (m_1/\sigma_1)^2 + \dots + (m_N/\sigma_N)^2$ (compare with Exercise 2.9).*

Exercise 3.18. *Let $X, Y \in L^1(\Omega)$ be independent random variables with densities f_X, f_Y. Show that $X + Y$ has the density*

$$f_{X+Y}(x) = \int_{\mathbb{R}} f_X(x-y)f_Y(y)\,dy.$$

Remark: The right hand side of the previous identity defines the **convolution product** *of the functions f_X, f_Y.*

The characteristic function is also very useful to establish whether two random variables are independent, as shown in the following exercise.

Exercise 3.19. *Let $X, Y \in L^1(\Omega)$ and define their joint characteristic function as*

$$\theta_{X,Y}(u, v) = \mathbb{E}[e^{iuX+ivY}], \quad u, v \in \mathbb{R}.$$

Show that X, Y are independent if and only if $\theta_{X,Y}(u, v) = \theta_X(u)\theta_Y(v)$.

3.4 Quadratic variation of stochastic processes

We continue this chapter by discussing the important concept of quadratic variation. We introduce this concept to measure how "wild" a stochastic process oscillates in time, which in financial mathematics is a measure of the volatility of an asset price.

Let $\{X(t)\}_{t\geq 0}$ be a stochastic process. A partition of the interval $[0, T]$ is a set of points $\Pi = \{t_0, t_1, \dots t_m\}$ such that

$$0 = t_0 < t_1 < t_2 < \dots < t_m = T.$$

The size of the partition is given by

$$\|\Pi\| = \max_{j=0,\dots,m-1}(t_{j+1} - t_j).$$

To measure the amount of oscillations of $\{X(t)\}_{t\geq 0}$ in the interval $[0, T]$ along the partition Π, we compute

$$Q_\Pi(\omega) = \sum_{j=0}^{m-1}(X(t_{j+1}, \omega) - X(t_j, \omega))^2.$$

Note carefully that Q_Π is a random variable and that it depends on the partition. For example, let $\{\Delta(s)\}_{s\geq 0}$ be the step process

$$\Delta(s,\omega) = \sum_{k=0}^{\infty} X_k(\omega)\mathbb{I}_{[s_k,s_{k+1})}.$$

Then if the partition $\Pi = \{0, t_1, \dots, t_m = T\}$ is such that $s_{k-1} < t_k < s_k$, for all $k = 1, 2, \dots m$, we have

$$Q_\Pi(\omega) = (X_2(\omega) - X_1(\omega))^2 + (X_3(\omega) - X_2(\omega))^2 + \cdots + (X_{m+1}(\omega) - X_m(\omega))^2.$$

However if two points in the partition belong to the same interval $[s_k, s_{k+1})$, the variation within these two instants of time clearly gives no contribution to the total variation Q_Π.

To define the quadratic variation of the stochastic process $\{X(t)\}_{t\geq 0}$, we compute Q_{Π_n} along a sequence $\{\Pi_n\}_{n\in\mathbb{N}}$ of partitions to the interval $[0,T]$ such that $\|\Pi_n\| \to 0$ as $n \to \infty$ and then we take the limit of Q_{Π_n} as $n \to \infty$. Since $\{Q_{\Pi_n}\}_{n\in\mathbb{N}}$ is a sequence of random variables, there are several ways to define its limit as $n \to \infty$. The precise definition that we adopt is that of L^2-quadratic variation, in which the limit is taken in the norm $\|\cdot\|_2$ defined in Exercise 3.5.

Definition 3.5. *The* L^2**-quadratic variation** *of the stochastic process* $\{X(t)\}_{t\geq 0}$ *in the interval* $[0,T]$ *along the sequence of partitions* $\{\Pi_n\}_{n\in\mathbb{N}}$ *is a random variable denoted by* $[X,X](T)$ *such that*

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{j=0}^{m(n)-1} (X(t_{j+1}^{(n)}) - X(t_j^{(n)}))^2 - [X,X](T)\right)^2\right] = 0,$$

where $m(n)+1$ *is the number of points in the partition* $\Pi_n = \{t_0, t_1^{(n)}, t_2^{(n)}, \dots, t_{m(n)-1}^{(n)}, T\}$.

If the limit in the previous definition does not exist, the quadratic variation cannot be defined as we did (an alternative definition is possible, but we shall not need it)

Note that the quadratic variation depends in general on the sequence of partitions of the interval $[0,T]$ along which it is computed, although this is not reflected in our notation $[X,X](T)$. However for several important examples of stochastic processes—and in particular for all applications considered in these notes—the quadratic variation is independent of the sequence of partitions. To distinguish this important special case, we shall use the following (standard) notation:

$$dX(t)dX(t) = dY(t),$$

to indicate that the quadratic variation of the stochastic process $\{X(t)\}_{t\geq 0}$ in any interval $[0,t]$ is given by the random variable $Y(t)$, independently from the sequence of partitions of the interval $[0,t]$ along which it is computed. Note that $\{Y(t)\}_{t\geq 0}$ is a stochastic process.

Now we show that if the paths of the stochastic process $\{X(t)\}_{t\geq 0}$ are sufficiently regular, then its quadratic variation is zero along any sequence of partitions.

Theorem 3.8. *Assume that the paths of the stochastic process* $\{X(t)\}_{t\geq 0}$ *satisfy*

$$\mathbb{P}(|X(t)-X(s)|\leq C|t-s|^{\gamma})=1, \tag{3.10}$$

for some positive constant $C>0$ *and* $\gamma>1/2$. *Then*

$$dX(t)dX(t)=0.$$

Proof. We have

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{j=0}^{m(n)-1}(X(t_{j+1}^{(n)})-X(t_j^{(n)}))^2\right)^2\right] &\leq C^4\mathbb{E}\left[\left(\sum_{j=0}^{m(n)-1}(t_{j+1}^{(n)}-t_j^{(n)})^{2\gamma}\right)^2\right]\\ &= C^4\mathbb{E}\left[\left(\sum_{j=0}^{m(n)-1}(t_{j+1}^{(n)}-t_j^{(n)})^{2\gamma-1}(t_{j+1}^{(n)}-t_j^{(n)})\right)^2\right].\end{aligned}$$

Now we use that $t_{j+1}^{(n)}-t_j^{(n)}\leq\|\Pi_n\|$ and $\sum_j(t_{j+1}^{(n)}-t_j^{(n)})=T$, so that

$$\mathbb{E}\left[\left(\sum_{j=0}^{m(n)-1}(X(t_{j+1}^{(n)})-X(t_j^{(n)}))^2\right)^2\right]\leq\left(C^2\|\Pi_n\|^{2\gamma-1}T\right)^2\to 0,\quad\text{as }\|\Pi_n\|\to 0.$$

□

As a special important case we have that

$$dtdt=0. \tag{3.11}$$

Next we compute the quadratic variation of Brownian motions.

Theorem 3.9. *For a Brownian motion* $\{W(t)\}_{t\geq 0}$ *there holds*

$$dW(t)dW(t)=dt. \tag{3.12}$$

Proof. Let

$$Q_{\Pi_n}(\omega)=\sum_{j=0}^{m(n)-1}(W(t_{j+1}^{(n)},\omega)-W(t_j^{(n)},\omega))^2,$$

where we recall that $m(n)+1$ is the number of points in the partition Π_n of $[0,T]$. We compute

$$\mathbb{E}[(Q_{\Pi_n}-T)^2]=\mathbb{E}[Q_{\Pi_n}^2]+T^2-2T\mathbb{E}[Q_{\Pi_n}].$$

But

$$\begin{aligned}\mathbb{E}[Q_{\Pi_n}]&=\sum_{j=0}^{m(n)-1}\mathbb{E}[(W(t_{j+1}^{(n)})-W(t_j^{(n)}))^2]=\sum_{j=0}^{m(n)-1}\mathrm{Var}[W(t_{j+1}^{(n)})-W(t_j^{(n)})]\\ &=\sum_{j=0}^{m(n)-1}(t_{j+1}^{(n)}-t_j^{(n)})=T.\end{aligned}$$

Hence we have to prove that

$$\lim_{\|\Pi_n\|\to 0} \mathbb{E}[Q_{\Pi_n}^2] - T^2 = 0,$$

or equivalently (as we have just proved that $\mathbb{E}[Q_{\Pi_n}] = T$),

$$\lim_{\|\Pi_n\|\to 0} \mathrm{Var}(Q_{\Pi_n}) = 0. \tag{3.13}$$

Since the increments of a Brownian motion are independent, we have

$$\mathrm{Var}(Q_{\Pi_n}) = \sum_{j=0}^{m(n)-1} \mathrm{Var}[(W(t_{j+1}^{(n)}) - W(t_j^{(n)}))^2] = \sum_{j=0}^{m(n)-1} \mathbb{E}[(W(t_{j+1}^{(n)}) - W(t_j^{(n)}))^4]$$
$$- \sum_{j=0}^{m(n)-1} \mathbb{E}[(W(t_{j+1}^{(n)}) - W(t_j^{(n)}))^2]^2$$

Now we use that

$$\mathbb{E}[(W(t_{j+1}^{(n)}) - W(t_j^{(n)}))^2] = \mathrm{Var}[W(t_{j+1}^{(n)}) - W(t_j^{(n)})] = t_{j+1}^{(n)} - t_j^{(n)},$$

and, as it follows by Exercise 3.9,

$$\mathbb{E}[(W(t_{j+1}^{(n)}) - W(t_j^{(n)}))^4] = 3(t_{j+1}^{(n)} - t_j^{(n)})^2.$$

We conclude that

$$\mathrm{Var}[Q_{\Pi_n}] = 2\sum_{j=0}^{m(n)-1} (t_{j+1}^{(n)} - t_j^{(n)})^2 \leq 2\|\Pi_n\|T \to 0, \quad \text{as } \|\Pi_n\| \to 0,$$

which proves (3.13) and thus the theorem. □

Remark 3.7 (No-where differentiability of Brownian motions)**.** Combining Theorem 3.9 and Theorem 3.8, we conclude that the paths of a Brownian motion $\{W(t)\}_{t\geq 0}$ cannot satisfy the regularity condition (3.10). In fact, while the paths of a Brownian motion are a.s. continuous by definition, they turn out to be **no-where differentiable**, in the sense that the event $\{\omega \in \Omega : \gamma_W^\omega \in C^1\}$ is a null set. A proof of this can be found for instance in [7].

Finally we need to consider a slight generalization of the concept of quadratic variation.

Definition 3.6. *We say that two stochastic processes $\{X_1(t)\}_{t\geq 0}$ and $\{X_2(t)\}_{t\geq 0}$ have L^2-* **cross variation** *$[X_1, X_2](T) \in \mathbb{R}$ in the interval $[0,T]$ along the sequence of partitions $\{\Pi_n\}_{n\in\mathbb{N}}$, if*

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{j=0}^{m(n)-1} (X_1(t_{j+1}^{(n)}) - X_1(t_j^{(n)}))(X_2(t_{j+1}^{(n)}) - X_2(t_j^{(n)})) - [X_1, X_2](T)\right)^2\right] = 0,$$

where $m(n)+1$ is the number of points in the partition Π_n.

As for the quadratic variation of a stochastic process, we use a special notation to express that the cross variation of two stochastic processes is independent of the sequence of partitions along which it is computed. Namely, we write

$$dX_1(t)dX_2(t) = dY(t),$$

to indicate that the cross variation $[X_1, X_2](t)$ equals $Y(t)$ along any sequence of partitions $\{\Pi_n\}_{n\in\mathbb{N}}$. The following generalization of Theorem 3.8 is easily established.

Theorem 3.10. *Assume that the paths of the stochastic processes* $\{X_1(t)\}_{t\geq 0}$, $\{X_2(t)\}_{t\geq 0}$ *satisfy*

$$\mathbb{P}(|X_1(t) - X_1(s)| \leq C|t-s|^{\gamma}) = 1, \quad \mathbb{P}(|X_2(t) - X_2(s)| \leq C|t-s|^{\lambda}) = 1,$$

for some positive constants C, γ, λ *such that* $\gamma + \lambda > 1/2$. *Then* $dX_1(t)dX_2(t) = 0$.

Exercise 3.20. *Prove the theorem.*

As a special case we find that

$$dW(t)dt = 0. \tag{3.14}$$

It is important to memorize the identities (3.11), (3.12) and (3.14), as they will be used several times in the following chapters.

Exercise 3.21 ($\star$)**.** *Let* $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ *be two independent Brownian motions. Show that* $dW_1(t)dW_2(t) = 0$.

3.5 Conditional expectation

Recall that the expectation value $\mathbb{E}[X]$ is an estimate on the average value of the random variable X. This estimate does not depend on the σ-algebra $\mathcal{F}$, nor on any sub-σ-algebra thereof. However, if some information is known in the form of a σ-algebra $\mathcal{G}$, then one expects to be able to improve the estimate on the value of X. To quantify this we introduce the definition of "expected value of X given $\mathcal{G}$", or conditional expectation, which we denote by $\mathbb{E}[X|\mathcal{G}]$. We want the conditional expectation to verify the following properties:

(i) If X is $\mathcal{G}$-measurable, then it should hold that $\mathbb{E}[X|\mathcal{G}] = X$, because the information provided by $\mathcal{G}$ is sufficient to determine X;

(ii) If X is independent of $\mathcal{G}$, then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$, because the occurrence of the events in $\mathcal{G}$ does not effect the probability distribution of X;

Note that (i) already indicates that $\mathbb{E}[X|\mathcal{G}]$ is a random variable. To begin with we define the conditional expectation of a random variable X with respect to an event $A \in \mathcal{F}$. Let's assume first that X is the simple random variable

$$X = \sum_{k=1}^{N} a_k \mathbb{I}_{A_k},$$

where $a_1, \dots, a_N \in \mathbb{R}$ and $\{A_k\}_{k=1,\dots,N}$ is a family of disjoints subsets of Ω. Let $B \in \mathcal{F} : \mathbb{P}(B) > 0$ and $\mathbb{P}_B(A_k) = \frac{\mathbb{P}(A_k \cap B)}{\mathbb{P}(B)}$ be the conditional probability of A_k given B, see Definition 1.4. It is natural to define the conditional expectation of X given the event B as

$$\mathbb{E}[X|B] = \sum_{k=1}^{N} a_k \mathbb{P}_B(A_k).$$

Moreover, since

$$X\mathbb{I}_B = \sum_{k=1}^{N} a_k \mathbb{I}_{A_k} \mathbb{I}_B = \sum_{k=1}^{N} a_k \mathbb{I}_{A_k \cap B},$$

we also have the identity $\mathbb{E}[X|B] = \frac{\mathbb{E}[X\mathbb{I}_B]}{\mathbb{P}(B)}$. We use the latter identity to define the conditional expectation given B of general random variables.

Definition 3.7. *Let $X \in L^1(\Omega)$ and $B \in \mathcal{F}$. When $\mathbb{P}(B) > 0$ we define the* **conditional expectation of X given the event B** *as*

$$\mathbb{E}[X|B] = \frac{\mathbb{E}[X\mathbb{I}_B]}{\mathbb{P}(B)}.$$

When $\mathbb{P}(B) = 0$ we define $\mathbb{E}[X|B] = \mathbb{E}[X]$.

Note that $\mathbb{E}[X|B]$ is a deterministic constant. Next we discuss the concept of conditional expectation given a σ-algebra $\mathcal{G}$. We first assume that $\mathcal{G}$ is generated by a (say, finite) partition $\{A_k\}_{k=1,\dots,M}$ of Ω, see Exercise 1.4. Then it is natural to define

$$\mathbb{E}[X|\mathcal{G}] = \sum_{k=1}^{M} \mathbb{E}[X|A_k]\mathbb{I}_{A_k}. \tag{3.15}$$

Note that $\mathbb{E}[X|\mathcal{G}]$ is a $\mathcal{G}$-measurable simple function. It will now be shown that (3.15) satisfies the identity

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|B] = \mathbb{E}[X|B], \quad \text{for all } B \in \mathcal{G} : \mathbb{P}(B) > 0. \tag{3.16}$$

In fact,

$$\begin{aligned}
\mathbb{P}(B)\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|B] &= \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbb{I}_B] \\
&= \mathbb{E}[\sum_{k=1}^{M} \mathbb{E}[X|A_k]\mathbb{I}_{A_k}\mathbb{I}_B] \\
&= \sum_{k=1}^{M} \mathbb{E}[\mathbb{E}[X|A_k]\mathbb{I}_{A_k \cap B}] \\
&= \sum_{k=1}^{M} \mathbb{E}[\frac{\mathbb{E}[X\mathbb{I}_{A_k}]}{\mathbb{P}(A_k)}\mathbb{I}_{A_k \cap B}] \\
&= \sum_{k=1}^{M} \frac{1}{\mathbb{P}(A_k)}\mathbb{E}[X\mathbb{I}_{A_k}]\mathbb{E}[\mathbb{I}_{A_k \cap B}].
\end{aligned}$$

Since $\{A_1, \dots, A_M\}$ is a partition of Ω, there exists $I \subset \{1, \dots, M\}$ such that $B = \cup_{k \in I} A_k$; hence the above sum may be restricted to $k \in I$. Since $\mathbb{E}[\mathbb{I}_{A_k \cap B}] = \mathbb{E}[\mathbb{I}_{A_k}] = \mathbb{P}(A_k)$, for $k \in I$, we obtain

$$\mathbb{P}(B)\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|B] = \sum_{k \in I} \mathbb{E}[X\mathbb{I}_{A_k}] = \mathbb{E}[X\mathbb{I}_{\cup_{k \in I} A_k}] = \mathbb{E}[X\mathbb{I}_B],$$

by which (3.16) follows.

Exercise 3.22. *What is the interpretation of* (3.16)*?*

The conditional expectation of a random variable with respect to a general σ-algebra can be constructed explicitly only in some special cases (see Section 3.7). However an abstract definition is still possible, which we give after the following theorem.

Theorem 3.11. *Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and $X \in L^1(\Omega)$. If $Y_1, Y_2 \in L^1(\Omega)$ are $\mathcal{G}$-measurable and satisfy*

$$\mathbb{E}[Y_i|A] = \mathbb{E}[X|A], \quad \textit{for } i = 1, 2 \textit{ and all } A \in \mathcal{G} : \mathbb{P}(A) > 0, \tag{3.17}$$

then $Y_1 = Y_2$ a. s.

Proof. We want to prove that $\mathbb{P}(B) = 0$, where

$$B = \{\omega \in \Omega : Y_1(\omega) \neq Y_2(\omega)\}.$$

Let $B_+ = \{Y_1 > Y_2\}$ and assume $\mathbb{P}(B_+) > 0$. Then, by (3.17) and Definition 3.7,

$$\mathbb{E}[(Y_1 - Y_2)\mathbb{I}_{B_+}] = \mathbb{E}[Y_1\mathbb{I}_{B_+}] - \mathbb{E}[Y_2\mathbb{I}_{B_+}] = \mathbb{P}(B_+)(\mathbb{E}[Y_1|B_+] - \mathbb{E}[Y_2|B_+]) = 0.$$

By Theorem 3.1(iii), this is possible if and only if $(Y_1 - Y_2)\mathbb{I}_{B_+} = 0$ a.s., which entails $\mathbb{P}(B_+) = 0$. At the same fashion one proves that $\mathbb{P}(B_-) = 0$, where $\mathbb{P}(B_-) = \{Y_1 < Y_2\}$. Hence $\mathbb{P}(B) = \mathbb{P}(B_+) + \mathbb{P}(B_-) = 0$, as claimed. □

Theorem 3.12 (and Definition). *Let $X \in L^1(\Omega)$ and $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. There exists a $\mathcal{G}$-measurable random variable $\mathbb{E}[X|\mathcal{G}] \in L^1(\Omega)$ such that*

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|A] = \mathbb{E}[X|A], \quad \textit{for all } A \in \mathcal{G} : \mathbb{P}(A) > 0. \tag{3.18}$$

The random variable $\mathbb{E}[X|\mathcal{G}]$, which by Theorem 3.11 is uniquely defined up to a null set, is called the **conditional expectation of X given the σ-algebra $\mathcal{G}$**. *If $\mathcal{G}$ is the σ-algebra generated by a random variable Y, i.e., $\mathcal{G} = \sigma(Y)$, we write $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X|Y]$.*

Proof. See [21, Appendix B]. □

Remark 3.8. Following Remark 3.3, we denote by $\widetilde{\mathbb{E}}[X|\mathcal{G}]$ the conditional expectation of X in a new probability measure $\widetilde{\mathbb{P}}$, not necessarily equivalent to $\mathbb{P}$.

We continue this section with a list of properties of the conditional expectation, which we divide in three theorems.

Theorem 3.13. *The conditional expectation of $X \in L^1(\Omega)$ satisfies the following identities almost surely:*

(i) $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X]$*;*

(ii) If X is $\mathcal{G}$-measurable, then $\mathbb{E}[X|\mathcal{G}] = X$;

(iii) If $\mathcal{H} \subset \mathcal{G}$ is a sub-σ-algebra, then

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}];$$

(iv) Linearity: $\mathbb{E}[\alpha X + \beta Y|\mathcal{G}] = \alpha\mathbb{E}[X|\mathcal{G}] + \beta\mathbb{E}[Y|\mathcal{G}]$, for all $\alpha, \beta \in \mathbb{R}$ and $Y \in L^1(\Omega)$.

(v) If $\mathcal{G}$ consists of trivial events only, then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$.

Proof. (i) Replace $A = \Omega$ into (3.18). (ii) Note that (3.17) is satisfied by $Y_1 = \mathbb{E}[X|\mathcal{G}]$ and $Y_2 = X$; hence when X is $\mathcal{G}$-measurable, we have $\mathbb{E}[X|\mathcal{G}] = X$ a.s., by uniqueness. (iii) Using (3.18), and since $\mathcal{H} \subset \mathcal{G}$, the random variables $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}]$, $\mathbb{E}[X|\mathcal{G}]$ and $\mathbb{E}[X|\mathcal{H}]$ satisfy

$$\begin{aligned}
&\mathbb{E}[\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}]|A] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]|A],\\
&\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|A] = \mathbb{E}[X|A],\\
&\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|A] = \mathbb{E}[X|A],
\end{aligned}$$

for all $A \in \mathcal{H} : \mathbb{P}(A) > 0$. It follows that

$$\mathbb{E}[\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}]|A] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]|A]$$

and thus by uniqueness the claim follows. (iv) The variables $Y_1 = \mathbb{E}[\alpha X + \beta Y|\mathcal{G}]$ and $Y_2 = \alpha\mathbb{E}[X|\mathcal{G}] + \beta\mathbb{E}[Y|\mathcal{G}]$ satisfy (3.17), and so they are equal almost surely. (v) See next exercise. □

Exercise 3.23. *Prove the property (v) in Theorem 3.13.*

The following theorem collects some properties of the conditional expectation in the presence of two variables and is given without proof.

Theorem 3.14. *Let $X, Y \in L^1(\Omega)$. Then the following identities holds almost surely:*

(i) If X is independent of $\mathcal{G}$, then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$;

(ii) If X is $\mathcal{G}$-measurable, then $\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]$;

(iii) If $X \leq Y$ then $\mathbb{E}[X|\mathcal{G}] \leq \mathbb{E}[Y|\mathcal{G}]$.

Exercise 3.24 (•). *Prove the property (i) in Theorem 3.14 when $\mathcal{G}$ is generated by a partition $\{A_k\}_{k=1\dots,M}$ of Ω, i.e., using Definition 3.15. Prove the property (ii) when Y is a simple random variable.*

Exercise 3.25. *Prove the property (iii) in Theorem 3.14.*

Exercise 3.26 (•). *The purpose of this exercise is to show that the conditional expectation is the best estimator of a random variable when some information is given in the form of a sub-σ-algebra. Let $X \in L^1(\Omega)$ and $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. Define* $\mathrm{Err} = X - \mathbb{E}[X|\mathcal{G}]$. *Show that* $\mathbb{E}[\mathrm{Err}] = 0$ *and*

$$\mathrm{Var}[\mathrm{Err}] = \min_{Y} \mathrm{Var}[Y - X],$$

where the minimum is taken with respect to all $\mathcal{G}$-measurable random variables Y.

Exercise 3.27 (•). *Let $X, Y \in L^1(\Omega)$ and consider the decomposition*

$$X = \mathbb{E}[X|Y] + (X - \mathbb{E}[X|Y]) = X_1 + X_2.$$

Show that X_2 and Y are uncorrelated. Hence any random variable X can be written as the sum of a Y-measurable random variable and a remainder which is uncorrelated to X.

Theorem 3.15. *Let $X, Y : \Omega \to \mathbb{R}$ be two random variables and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$ such that X is $\mathcal{G}$-measurable and $\mathcal{Y}$ is independent of $\mathcal{G}$. Then for any measurable function $g : \mathbb{R}^2 \to [0, \infty)$, the function $f : \mathbb{R} \to [0, \infty)$ defined by*

$$f(x) = \mathbb{E}[g(x, Y)]$$

is measurable and moreover

$$\mathbb{E}[g(X, Y)|\mathcal{G}] = f(X).$$

The previous theorem tells us that, under the stated assumptions, we can compute the random variable $\mathbb{E}[g(X, Y)|\mathcal{G}]$ as if X were a constant.

3.6 Martingales

A martingale is a stochastic process which has no tendency to rise or fall. The precise definition is the following:

Definition 3.8. *A stochastic process $\{M(t)\}_{t\geq 0}$ is called a* **martingale** *relative to the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ if it is adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$, $M(t) \in L^1(\Omega)$ for all $t \geq 0$, and*

$$\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s), \quad \textit{for all } 0 \leq s \leq t, \tag{3.19}$$

for all $t \geq 0$.

Hence a stochastic process is martingale if the information available up to time s does not help to predict whether the stochastic process will raise or fall after time s.

Remark 3.9. If the condition (3.19) is replaced by $\mathbb{E}[M(t)|\mathcal{F}(s)] \geq M(s)$, for all $0 \leq$

$s \le t$, the stochastic process $\{M(t)\}_{t\ge0}$ is called a **sub-martingale**. The interpretation is that $M(t)$ has no tendency to fall, but our expectation is that it will increase. If the condition (3.19) is replaced by $\mathbb{E}[M(t)|\mathcal{F}(s)] \le M(s)$, for all $0 \le s \le t$, the stochastic process $\{M(t)\}_{t\ge0}$ is called a **super-martingale**. The interpretation is that $M(t)$ has not tendency to rise, but our expectation is that it will decrease.

Remark 3.10. If we want to emphasize that the martingale property is satisfied with respect to the probability measure $\mathbb{P}$, we shall say that $\{M(t)\}_{t\ge0}$ is a $\mathbb{P}$-martingale.

Since the conditional expectation of a random variable X is uniquely determined by (3.18), then the property (3.19) is satisfied if and only if

$$\mathbb{E}[M(s)\mathbb{I}_A] = \mathbb{E}[M(t)\mathbb{I}_A], \quad \text{for all } 0 \le s \le t \text{ and for all } A \in \mathcal{F}(s). \tag{3.20}$$

In particular, letting $A = \Omega$, we obtain that the expectation of a martingale is constant, i.e.,

$$\mathbb{E}[M(t)] = \mathbb{E}[M(0)], \quad \text{for all } t \ge 0. \tag{3.21}$$

Combining the latter result with Theorem 3.2, we obtain the following sufficient condition for no arbitrage.

Theorem 3.16. *Let a portfolio be given with value $\{V(t)\}_{t\ge0}$. If there exists a measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ and a filtration $\{\mathcal{F}(t)\}_{t\ge0}$ such that the discounted value of the portfolio $\{V^*(t)\}_{t\ge0}$ is a martingale, then the portfolio is not an arbitrage.*

Proof. The assumption is that

$$\widetilde{\mathbb{E}}[D(t)V(t)|\mathcal{F}(s)] = D(s)V(s), \quad \text{for all } 0 \le s \le t.$$

Hence, by (3.21), $\widetilde{\mathbb{E}}[D(t)V(t)] = \widetilde{\mathbb{E}}[D(0)V(0)] = \widetilde{\mathbb{E}}[V(0)]$. The result follows by Theorem 3.2. □

Theorem 3.17. *Let $\{\mathcal{F}(t)\}_{t\ge0}$ be a non-anticipating filtration for the Brownian motion $\{W(t)\}_{t\ge0}$. Then $\{W(t)\}_{t\ge0}$ is a martingale relative to $\{\mathcal{F}(t)\}_{t\ge0}$.*

Proof. The martingale property for $s = t$, i.e., $\mathbb{E}[W(t)|\mathcal{F}(t)] = W(t)$, follows by the fact $W(t)$ is $\mathcal{F}(t)$-measurable, and thus Theorem 3.13(ii) applies. For $0 \le s < t$ we have

$$\begin{aligned}\mathbb{E}[W(t)|\mathcal{F}(s)] &= \mathbb{E}[(W(t) - W(s))|\mathcal{F}(s)] + \mathbb{E}[W(s)|\mathcal{F}(s)],\\ &= \mathbb{E}[W(t) - W(s)] + W(s) = W(s),\end{aligned}$$

where we used that $W(t) - W(s)$ is independent of $\mathcal{F}(s)$ (and so $\mathbb{E}[(W(t) - W(s))|\mathcal{F}(s)] = \mathbb{E}[(W(t) - W(s))]$ by Theorem 3.14(i)), and the fact that $W(s)$ is $\mathcal{F}(s)$-measurable (and so $\mathbb{E}[W(s)|\mathcal{F}(s)] = W(s)$). □

Exercise 3.28. *In the ∞-coin tosses experiment, let $\mathcal{F}_N$ be the σ-algebra of the events resolved by the first N tosses. Show that the random walk is a martingale with respect to the filtration $\{\mathcal{F}_N\}_{N\in\mathbb{N}}$. The proof can be found in*

Thus Brownian motions are martingales, they have a.s. continuous paths and have quadratic variation t in the interval $[0, t]$, see Theorem 3.9. The following theorem, which is a special case of the so called **Lévy characterization** of Brownian motion, show that these three properties characterize Brownian motions and is often used to prove that a given stochastic process is a Brownian motion. The proof can be found in [14].

Theorem 3.18. *Let $\{M(t)\}_{t\geq 0}$ be a martingale relative to a filtration $\{\mathcal{F}(t)\}_{t\geq 0}$. Assume that (i) $M(0) = 0$ a.s., (ii) the paths $t \to M(t, \omega)$ are a.s. continuous and (iii) $dM(t)dM(t) = dt$. Then $\{M(t)\}_{t\geq 0}$ is a Brownian motion and $\{\mathcal{F}(t)\}_{t\geq 0}$ a non-anticipating filtration thereof.*

Exercise 3.29. *Consider the stochastic process $\{Z(t)\}_{t\geq 0}$ given by*

$$Z(t) = \exp\big(\sigma W(t) - \frac{1}{2}\sigma^2 t\big),$$

where $\{W(t)\}_{t\geq 0}$ is a Brownian motion and $\sigma \in \mathbb{R}$ is a constant. Let $\{\mathcal{F}(t)\}_{t\geq 0}$ be a non-anticipating filtration for $\{W(t)\}_{t\geq 0}$. Show that $\{Z(t)\}_{t\geq 0}$ is a martingale relative to $\{\mathcal{F}(t)\}_{t\geq 0}$.

Exercise 3.30 (•). *Let $\{N(t)\}_{t\geq 0}$ be a Poisson process generating the filtration $\{\mathcal{F}_N(t)\}_{t\geq 0}$. Show that (i) $\{N(t)\}_{t\geq 0}$ is a sub-martingale relative to $\{\mathcal{F}_N(t)\}_{t\geq 0}$ and (ii) the so-called* **compound Poisson process** *$\{N(t) - \lambda t\}_{t\geq 0}$ is a martingale relative to$\{\mathcal{F}_N(t)\}_{t\geq 0}$, where λ is the rate of the Poisson process (see Definition 2.16).*

Exercise 3.31 (•). *Let $\{\mathcal{F}(t)\}_{t\in[0,T]}$ be a filtration and $\{M(t)\}_{t\in[0,T]}$ a stochastic process adapted to $\{\mathcal{F}(t)\}_{t\in[0,T]}$. Show that $\{M(t)\}_{t\in[0,T]}$ is a martingale if and only if there exists a $\mathcal{F}(T)$-measurable random variable $H \in L^1(\Omega)$ such that*

$$M(t) = \mathbb{E}[H|\mathcal{F}(t)].$$

Now assume that $\{Z(t)\}_{t\geq 0}$ is a martingale such that $Z(t) > 0$ a.s. for all $t \geq 0$ and $Z(0) = 1$. In particular, $\mathbb{E}[Z(t)] = \mathbb{E}[Z(0)] = 1$ and therefore, by Theorem 3.3, the map $\widetilde{\mathbb{P}} : \mathcal{F} \to [0, 1]$ given by

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z(T)\mathbb{I}_A], \quad A \in \mathcal{F} \tag{3.22}$$

is a probability measure equivalent to $\mathbb{P}$, for all $T > 0$. Note that $\widetilde{\mathbb{P}}$ depends on $T > 0$ and $\widetilde{\mathbb{P}} = \mathbb{P}$, for $T = 0$. The dependence on T is however not reflected in our notation. As usual, the (conditional) expectation in the probability measure $\widetilde{\mathbb{P}}$ will be denoted $\widetilde{\mathbb{E}}$. The relation between $\mathbb{E}$ and $\widetilde{\mathbb{E}}$ is revealed in the following theorem.

Theorem 3.19. *Let $t \in [0, T]$ and let X be a $\mathcal{F}_W(t)$-measurable random variable such that $Z(t)X \in L^1(\Omega, \mathbb{P})$. Then $X \in L^1(\Omega, \widetilde{\mathbb{P}})$ and*

$$\widetilde{\mathbb{E}}[X] = \mathbb{E}[Z(t)X]. \tag{3.23}$$

Moreover, for all $0 \leq s \leq t$ and for all random variables Y such that $Z(t)Y \in L^1(\Omega, \mathbb{P})$, there holds

$$\widetilde{\mathbb{E}}[Y|\mathcal{F}_W(s)] = \frac{1}{Z(s)}\mathbb{E}[Z(t)Y|\mathcal{F}_W(s)] \quad \textit{(almost surely)}. \tag{3.24}$$

Proof. As shown in Theorem 3.3, $\widetilde{\mathbb{E}}[X] = \mathbb{E}[Z(T)X]$. By Theorem 3.13(i), Theorem 3.14(ii), and the martingale property of $\{Z(t)\}_{t\geq 0}$, we have

$$\mathbb{E}[Z(T)X] = \mathbb{E}[\mathbb{E}[Z(T)X|\mathcal{F}_W(t)]] = \mathbb{E}[X\mathbb{E}[Z(T)|\mathcal{F}_W(t)]] = \mathbb{E}[Z(t)X].$$

To prove (3.24), recall that the conditional expectation is uniquely defined (up to null sets) by (3.18). Hence the identity (3.24) follows if we show that

$$\widetilde{\mathbb{E}}[Z(s)^{-1}\mathbb{E}[Z(t)Y|\mathcal{F}_W(s)]\mathbb{I}_A] = \widetilde{\mathbb{E}}[Y\mathbb{I}_A],$$

for all $A \in \mathcal{F}_W(s)$. Since $\mathbb{I}_A$ is $\mathcal{F}_W(s)$-measurable, and using (3.23) with $X = Z(s)^{-1}\mathbb{E}[Z(t)Y\mathbb{I}_A|\mathcal{F}_W(s)]$ and $t = s$, we have

$$\begin{aligned}\widetilde{\mathbb{E}}[Z(s)^{-1}\mathbb{E}[Z(t)Y|\mathcal{F}_W(s)]\mathbb{I}_A] &= \widetilde{\mathbb{E}}[Z(s)^{-1}\mathbb{E}[Z(t)Y\mathbb{I}_A|\mathcal{F}_W(s)]] = \mathbb{E}[\mathbb{E}[Z(t)Y\mathbb{I}_A|\mathcal{F}_W(s)]] \\ &= \mathbb{E}[Z(t)Y\mathbb{I}_A] = \widetilde{\mathbb{E}}[Y\mathbb{I}_A],\end{aligned}$$

where in the last step we used again (3.23). The proof is complete. □

3.7 Markov processes

In this section we introduce another class of stochastic processes, which will play a fundamental role in the following chapters.

Definition 3.9. *A stochastic process $\{X(t)\}_{t\geq 0}$ is called a* **Markov process** *with respect to the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ if it is adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$ and if for every measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $g(X(t)) \in L^1(\Omega)$ for all $t \geq 0$, there exists a measurable function $f_g : [0, \infty) \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that*

$$\mathbb{E}[g(X(t))|\mathcal{F}(s)] = f_g(t, s, X(s)), \quad \textit{for all } 0 \leq s \leq t. \tag{3.25}$$

If there exists a measurable function $\widetilde{f}_g : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that $f_g(t, s, x) = \widetilde{f}_g(t - s, x)$, the Markov process is said to be **homogeneous***. If there exists a measurable function $p : [0, \infty) \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that*

$$f_g(t, s, x) = \int_{\mathbb{R}} g(y)p(t, s, x, y)\, dy, \quad \textit{for } 0 \leq s < t, \tag{3.26}$$

then we call p the **transition density** *of the Markov process.*

Thus for a Markov process, the conditional expectation of $g(X(t))$ at the future time t depends only on the random variable $X(s)$ at time s, and not on the behavior of the process before or after time s. Note that in the case of a homogeneous Markov process, the transition density, if it exists, has the form $p(t, s, x, y) = p_*(t - s, x, y)$, for some measurable function $p_* : [0, \infty) \times \mathbb{R} \to \mathbb{R}$.

Remark 3.11. We will say that a stochastic process is a $\mathbb{P}$-Markov process if we want to emphasize that the Markov property holds in the probability measure $\mathbb{P}$.

Exercise 3.32 ($\bullet$). *Show that the function $f_g(t,s,x)$ in the right hand side of* (3.25) *is given by*

$$f_g(t,s,x) = \mathbb{E}[g(X(t))|X(s)=x] \quad \text{for all } 0 \leq s \leq t. \tag{3.27}$$

Theorem 3.20. *Let $\{X(t)\}_{t\geq 0}$ be a Markov process with transition density $p(t,s,x,y)$ relative to the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$. Assume $X(s) = x \in \mathbb{R}$ is a deterministic constant. Then for all $t \geq s$, $X(t)$ admits the density $f_{X(t)}$ given by*

$$f_{X(t)}(y) = p(t,s,x,y).$$

Proof. By definition of density

$$\mathbb{P}(X(t) \leq x) = \int_{-\infty}^{x} f_{X(t)}(y)\, dy,$$

see Definition 2.7. Letting $X(s) = x$ into (3.25)-(3.26) we obtain

$$\mathbb{E}[g(X(t))] = \int_{\mathbb{R}} g(y) p(t,s,x,y)\, dy.$$

Choosing $g = \mathbb{I}_{(-\infty,x]}$, we obtain

$$\mathbb{P}(X(t) \leq x) = \int_{-\infty}^{x} p(t,s,x,y)\, dy,$$

hence $f_X(t)(y) = p(t,s,x,y)$. □

Theorem 3.21. *Let $\{\mathcal{F}(t)\}_{t\geq 0}$ be a non-anticipating filtration for the Brownian motion $\{W(t)\}_{t\geq 0}$. Then $\{W(t)\}_{t\geq 0}$ is a homogeneous Markov process relative to $\{\mathcal{F}(t)\}_{t\geq 0}$ with transition density $p(t,s,x,y) = p_*(t-s,x,y)$, where*

$$p_*(\tau,x,y) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y-x)^2}{2\tau}}. \tag{3.28}$$

Proof. The statement holds for $s = t$, with $f_g(t,t,x) = g(x)$. For $s < t$ we write

$$\mathbb{E}[g(W(t))|\mathcal{F}(s)] = \mathbb{E}[g(W(t) - W(s) + W(s))|\mathcal{F}(s)] = \mathbb{E}[\widetilde{g}(W(s), W(t) - W(s))|\mathcal{F}(s)],$$

where $\widetilde{g}(x,y) = g(x+y)$. Since $W(t) - W(s)$ is independent of $\mathcal{F}(s)$ and $W(s)$ is $\mathcal{F}(s)$ measurable, then we can apply Theorem 3.15. Precisely, letting

$$f_g(t,s,x) = \mathbb{E}[\widetilde{g}(x, W(t) - W(s))],$$

we have

$$\mathbb{E}[g(W(t))|\mathcal{F}(s)] = f_g(t,s,W(s)),$$

which proves that the Brownian motion is a Markov process relative to $\{\mathcal{F}(t)\}_{t\geq 0}$. To derive the transition density we use that $Y = W(t) - W(s) \in \mathcal{N}(0, t-s)$, so that

$$\mathbb{E}[g(x+Y)] = \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}} g(x+y) e^{-\frac{y^2}{2(t-s)}}\, dy = \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}} g(y) e^{-\frac{(y-x)^2}{2(t-s)}}\, dy,$$

hence

$$\mathbb{E}[g(W(t))|\mathcal{F}(s)] = \left[\int_{\mathbb{R}} g(y) p_*(t-s, x, y)\, dy\right]_{x=W(s)},$$

where p_* is given by (3.28). This concludes the proof of the theorem. □

Exercise 3.33. *Show that, when p is given by* (3.28)*, the function*

$$u(t,x) = \int_{\mathbb{R}} g(y) p_*(t-s, x, y)\, dy \tag{3.29}$$

solves the **heat equation** *with initial datum g at time t = s, namely*

$$\partial_t u = \frac{1}{2}\partial_x^2 u, \quad u(s,x) = g(x), \quad t > s,\ x \in \mathbb{R}. \tag{3.30}$$

Exercise 3.34. *Let $\{\mathcal{F}(t)\}_{t\geq 0}$ be a non-anticipating filtration for the Brownian motion $\{W(t)\}_{t\geq 0}$. Show that the geometric Brownian motion*

$$S(t) = S(0) e^{\sigma W(t) + \alpha t}$$

is a homogeneous Markov process in the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ with transition density $p(t,s,x,y) = p_*(t-s,x,y)$, where

$$p_*(\tau, x, y) = \frac{1}{\sigma y \sqrt{2\pi\tau}} \exp\left\{-\frac{(\log(y/x) - \alpha\tau)^2}{2\sigma^2\tau}\right\} \mathbb{I}_{y>0}. \tag{3.31}$$

Show also that, when p is given by (3.31)*, the function $v : (s,\infty) \times (0,\infty) \to \mathbb{R}$ given by*

$$v(t,x) = \int_{\mathbb{R}} g(y) p_*(t-s, x, y)\, dy \tag{3.32}$$

satisfies

$$\partial_t v_s - (\alpha + \sigma^2/2) x \partial_x v_s - \frac{1}{2}\sigma^2 x^2 \partial_x^2 v_s = 0, \quad \text{for } x > 0,\ t > s, \tag{3.33a}$$

$$v(s,x) = g(x), \quad \text{for } x > 0. \tag{3.33b}$$

The correspondence between Markov processes and PDEs alluded to in the last two exercises is a general property which will be further discussed later in the notes.

3.A Appendix: Solutions to selected problems

Exercise 3.1. Let X be a binomial random variable. Then

$$\mathbb{E}[X] = \sum_{k=1}^{N} k\mathbb{P}(X = k) = \sum_{k=1}^{N} k\binom{N}{k} p^k(1-p)^{N-k} = (1-p)^{N-1}p \sum_{k=0}^{N} k\binom{N}{k}\Big(\frac{p}{1-p}\Big)^{k-1}.$$

Now, by the binomial theorem

$$\sum_{k=0}^{N} \binom{N}{k} x^k = (1+x)^N,$$

for all $x \in \mathbb{R}$. Differentiating with respect to x we get

$$\sum_{k=0}^{N} k\binom{N}{k} x^{k-1} = N(1+x)^{N-1},$$

Letting $x = p/(1-p)$ in the last identity we find $\mathbb{E}[X] = Np$.

Exercise 3.3. If $Y = 0$ almost surely, the claim is obvious. Hence we may assume that $\mathbb{E}[Y^2] > 0$. Let

$$Z = X - \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}Y.$$

Then

$$0 \leq \mathbb{E}[Z^2] = \mathbb{E}[X^2] - \frac{\mathbb{E}[XY]^2}{\mathbb{E}[Y^2]},$$

by which (3.3) follows.

Exercise 3.5. The first and second properties follow by the linearity of the expectation. In fact

$$\mathrm{Var}[\alpha X] = \mathbb{E}[\alpha^2 X^2] - \mathbb{E}[\alpha X]^2 = \alpha^2\mathbb{E}[X^2] - \alpha^2\mathbb{E}[X]^2 = \alpha^2\mathrm{Var}[X],$$

and

$$\begin{aligned}\mathrm{Var}[X+Y] &= \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2 = \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY] \\ &\quad - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 - 2\mathbb{E}[X]\mathbb{E}[Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}(X,Y).\end{aligned}$$

For the third property, let $a \in \mathbb{R}$ and compute, using 1 and 2,

$$\mathrm{Var}[Y - aX] = a^2\mathrm{Var}[X] + \mathrm{Var}[Y] - 2a\mathrm{Cov}(X,Y).$$

Since the variance of a random variable is always non-negative, the parabola $y(a) = a^2\mathrm{Var}[X] + \mathrm{Var}[Y] - 2a\mathrm{Cov}(X,Y)$ must always lie above the a-axis, or touch it at one single point $a = a_0$. Hence

$$\mathrm{Cov}(X,Y)^2 - \mathrm{Var}[X]\mathrm{Var}[Y] \leq 0,$$

which proves the first part of the claim 3. Moreover $\mathrm{Cov}(X,Y)^2 = \mathrm{Var}[X]\mathrm{Var}[Y]$ if and only if there exists a_0 such that $\mathrm{Var}[-a_0X+Y]=0$, i.e., $Y = a_0X+b_0$ almost surely, for some constant b_0. Substituting in the definition of covariance, we see that $\mathrm{Cov}(X, a_0X+b_0) = a_0\mathrm{Var}[X]$, by which the second claim of property 3 follows immediately.

Exercise 3.8. By linearity of the expectation,

$$\mathbb{E}[W_n(t)] = \frac{1}{\sqrt{n}}\mathbb{E}[M_{[nt]}] = 0,$$

where we used the fact that $\mathbb{E}[X_k] = \mathbb{E}[M_k] = 0$. Since $\mathrm{Var}[M_k] = k$, we obtain

$$\mathrm{Var}[W_n(t)] = \frac{[nt]}{n}.$$

Since $nt \sim [nt]$, as $n \to \infty$, then $\lim_{n\to\infty} \mathrm{Var}[W_n(t)] = t$. As to the covariance of $W_n(t)$ and $W_n(s)$ for $s \neq t$, we compute

$$\begin{aligned}\mathrm{Cov}[W_n(t), W_n(s)] &= \mathbb{E}[W_n(t)W_n(s)] - \mathbb{E}[W_n(t)]\mathbb{E}[W_n(s)] = \mathbb{E}[W_n(t)W_n(s)] \\ &= \mathbb{E}\left[\frac{1}{\sqrt{n}}M_{[nt]}\frac{1}{\sqrt{n}}M_{[ns]}\right] = \frac{1}{n}\mathbb{E}[M_{[nt]}M_{[ns]}]. \end{aligned} \tag{3.34}$$

Assume $t > s$ (a similar argument applies to the case $t < s$). If $[nt] = [ns]$ we have $\mathbb{E}[M_{[nt]}M_{[ns]}] = \mathrm{Var}[M_{[ns]}] = [ns]$. If $[nt] \geq 1 + [ns]$ we have

$$\mathbb{E}[M_{[nt]}M_{[ns]}] = \mathbb{E}[(M_{[nt]}-M_{[ns]})M_{[ns]}]+\mathbb{E}[M_{[ns]}^2] = \mathbb{E}[M_{[nt]}-M_{[ns]}]\mathbb{E}[M_{[ns]}]+\mathrm{Var}[M_{[ns]}] = [ns],$$

where we used that the increment $M_{[nt]} - M_{[ns]}$ is independent of $M_{[ns]}$. Replacing into (3.34) we obtain

$$\mathrm{Cov}[W_n(t), W_n(s)] = \frac{[ns]}{n}.$$

It follows that $\lim_{n\to\infty} \mathrm{Cov}[W_n(t), W_n(s)] = s$.

Exercise 3.10. We have

$$\mathbb{E}[X] = \int_{\mathbb{R}} \lambda x e^{-\lambda x}\mathbb{I}_{x\geq 0}\, dx = \lambda \int_0^\infty x e^{-\lambda x}\, dx = \frac{1}{\lambda},$$

and

$$\begin{aligned}\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 &= \int_{\mathbb{R}} \lambda x^2 e^{-\lambda x}\mathbb{I}_{x\geq 0}\, dx - \frac{1}{\lambda^2} \\ &= \lambda \int_0^\infty x^2 e^{-\lambda x}\, dx - \frac{1}{\lambda^2} = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.\end{aligned}$$

Exercise 3.11. According to Theorem 3.3, to prove that $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$ we have to show that $\mathbb{E}[Z] = 1$. Using the density of exponential random variables we have

$$\mathbb{E}[Z] = \frac{\widetilde{\lambda}}{\lambda}\mathbb{E}[e^{-(\widetilde{\lambda}-\lambda)X}] = \frac{\widetilde{\lambda}}{\lambda}\int_0^\infty e^{-(\widetilde{\lambda}-\lambda)x}\lambda e^{-\lambda x}\, dx = 1.$$

To show that $X \in \mathcal{E}(\widetilde{\lambda})$ in the probability measure $\widetilde{\mathbb{P}}$ we compute

$$\widetilde{\mathbb{P}}(X \leq x) = \mathbb{E}[Z\mathbb{I}_{X\leq x}] = \frac{\widetilde{\lambda}}{\lambda}\mathbb{E}[e^{-(\widetilde{\lambda}-\lambda)X}\mathbb{I}_{X\leq x}] = \frac{\widetilde{\lambda}}{\lambda}\int_0^x e^{-(\widetilde{\lambda}-\lambda)y}\lambda e^{-\lambda y}\,dy = 1 - e^{-\widetilde{\lambda}x}.$$

Exercise 3.24. Let $\mathcal{G}$ be generated by the partition $\{A_k\}_{k=1,\dots M}$ of Ω. Since X is independent of $\mathcal{G}$, then X and $\mathbb{I}_{A_k}$ are independent random variables, for all $k = 1, \dots, M$. It follows that

$$\mathbb{E}[X|\mathcal{G}] = \sum_{k=1}^M \frac{\mathbb{E}[X\mathbb{I}_{A_k}]}{\mathbb{P}(A_k)}\mathbb{I}_{A_k} = \sum_{k=1}^M \frac{\mathbb{E}[X]\mathbb{E}[\mathbb{I}_{A_k}]}{\mathbb{P}(A_k)}\mathbb{I}_{A_k} = \sum_{k=1}^M \frac{\mathbb{E}[X]\mathbb{P}(A_k)}{\mathbb{P}(A_k)}\mathbb{I}_{A_k} = \mathbb{E}[X]\sum_{k=1}^M \mathbb{I}_{A_k} = \mathbb{E}[X].$$

The proof of (ii) when Y is a simple random variable is a straightforward application of Theorem 3.13(ii).

Exercise 3.26. We have $\mathbb{E}[\mathrm{Err}] = \mathbb{E}[X] - \mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = 0$, by Theorem 3.13(i). Let Y be $\mathcal{G}$-measurable and set $\mu = \mathbb{E}[Y - X]$. Then

$$\begin{aligned}\mathrm{Var}[Y - X] &= \mathbb{E}[(Y - X - \mu)^2] = \mathbb{E}[(Y - X - \mu + \mathbb{E}[X|\mathcal{G}] - \mathbb{E}[X|\mathcal{G}])^2]\\ &= \mathbb{E}\Big[(\mathbb{E}[X|\mathcal{G}] - X)^2 + (Y - \mu - \mathbb{E}[X|\mathcal{G}])^2 + 2(\mathbb{E}[X|\mathcal{G}] - X)(Y - \mu - \mathbb{E}[X|\mathcal{G}])\Big]\\ &= \mathrm{Var}[\mathrm{Err}] + \mathbb{E}[\alpha] + 2\mathbb{E}[\beta],\end{aligned}$$

where $\alpha = (Y - \mu - \mathbb{E}[X|\mathcal{G}])^2$ and $\beta = (\mathbb{E}[X|\mathcal{G}] - X)(Y - \mu - \mathbb{E}[X|\mathcal{G}])$. As $\mathbb{E}[\alpha] \geq 0$ we have $\mathrm{Var}[Y - X] \geq \mathrm{Var}[\mathrm{Err}] + 2\mathbb{E}[\beta]$. Furthermore, as $Y - \mu - \mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable, then

$$\mathbb{E}[\beta] = \mathbb{E}[\mathbb{E}[\beta|\mathcal{G}]] = \mathbb{E}[(Y - \mu - \mathbb{E}[X|\mathcal{G}])\mathbb{E}[(\mathbb{E}[X|\mathcal{G}] - X)|\mathcal{G}] = 0.$$

Hence $\mathrm{Var}[Y - X] \geq \mathrm{Var}[\mathrm{Err}]$, for all $\mathcal{G}$-measurable random variables Y.

Exercise 3.27. We have, since $\mathbb{E}[X_2] = 0$,

$$\begin{aligned}\mathrm{Cov}(X_2, Y) &= \mathbb{E}[(X_2 - \mathbb{E}[X_2])(Y - \mathbb{E}[Y])] = \mathbb{E}[(X - \mathbb{E}[X|Y])(Y - \mathbb{E}[Y])]\\ &= \mathbb{E}[XY - \mathbb{E}[Y]X - Y\mathbb{E}[X|Y] + \mathbb{E}[Y]|\mathbb{E}[X|Y]]\\ &= \mathbb{E}\Big[\mathbb{E}[XY - \mathbb{E}[Y]X - Y\mathbb{E}[X|Y] + \mathbb{E}[Y]|\mathbb{E}[X|Y]]\Big|Y\Big]\\ &= \mathbb{E}[Y\mathbb{E}[X|Y] - \mathbb{E}[Y]\mathbb{E}[X|Y] - Y\mathbb{E}[X|Y] + \mathbb{E}[Y]\mathbb{E}[X|Y]] = 0.\end{aligned}$$

Exercise 3.30. First we observe that claim (i) follows by claim (ii). In fact, if the compound Poisson process is a martingale, then

$$\mathbb{E}[N(t) - \lambda t|\mathcal{F}_N(s)] = N(s) - \lambda s, \quad \text{for all } 0 \leq s \leq t,$$

by which it follows that

$$\mathbb{E}[N(t)|\mathcal{F}_N(s)] = N(s) + \lambda(t - s) \geq N(s), \quad \text{for all } 0 \leq s \leq t.$$

Hence it remains to prove (ii). We have

$$\begin{aligned}\mathbb{E}[N(t)-\lambda t|\mathcal{F}_N(s)] &= \mathbb{E}[N(t)-N(s)+N(s)-\lambda t|\mathcal{F}_N(s)]\\ &= \mathbb{E}[N(t)-N(s)|\mathcal{F}_N(s)]+\mathbb{E}[N(s)|\mathcal{F}_N(s)]-\lambda t\\ &= \mathbb{E}[N(t)-N(s)]+N(s)-\lambda t = \lambda(t-s)+N(s)-\lambda t = N(s)-\lambda s.\end{aligned}$$

Exercise 3.31. If $M(t)$ is a martingale, then $\mathbb{E}[M(T)|\mathcal{F}(t)] = M(t)$, hence we can pick $H = M(T)$. Viceversa, by the iterative property of the conditional expectation, the process $\mathbb{E}[H|\mathcal{F}(t)]$ satisfies, for all $s > t$,

$$\mathbb{E}[\mathbb{E}[H|\mathcal{F}(t)]|\mathcal{F}(s)] = \mathbb{E}[H|\mathcal{F}(s)],$$

hence it is a martingale.

Exercise 3.32. Taking the conditional expectation of both sides of (3.25) with respect to the event $\{X(s) = x\}$ gives (3.27).

Chapter 4

Stochastic calculus

Throughout this chapter we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the Brownian motion $\{W(t)\}_{t\geq 0}$ are given. Moreover we denote by $\{\mathcal{F}(t)\}_{t\geq 0}$ a non-anticipating filtration for the Brownian motion, e.g., $\mathcal{F}(t) = \mathcal{F}_W(t)$ (see Definition 2.15).

4.1 Introduction

So far we have studied in detail only one example of stochastic process, namely the Brownian motion $\{W(t)\}_{t\geq 0}$. In this chapter we define several other processes which are naturally derived from $\{W(t)\}_{t\geq 0}$ and which in particular are adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$. To begin with, if $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a measurable function, then we can introduce the stochastic processes

$$\{f(t, W(t))\}_{t\geq 0}, \quad \{\int_0^t f(s, W(s))\, ds\}_{t\geq 0}.$$

Note that the integral in the second stochastic process is the standard Lebesgue integral on the s-variable. It is well-defined for instance when f is a continuous function.

The next class of stochastic processes that we want to consider are those obtained by integrating along the paths of a Brownian motion, i.e., we want to give sense to the integral

$$I(t) = \int_0^t X(s) dW(s), \tag{4.1}$$

where $\{X(t)\}_{t\geq 0}$ is a stochastic process adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$. For our purposes we need to give a meaning to $I(t)$ when $\{X(t)\}_{t\geq 0}$ has continuous paths a.s. (e.g., $X(t) = W(t)$). The problem now is that the integral $\int X(t)dg(t)$ is well-defined for continuous functions X (in the Riemann-Stieltjes sense) only when g is of bounded variation. As shown in Exercise 3.14, the paths of the Brownian motion are not of bounded variation, hence we have to find another way to define (4.1). We begin in the next section by assuming that $\{X(t)\}_{t\geq 0}$ is a step process. Then we shall extend the definition to stochastic processes $\{X(t)\}_{t\geq 0}$ such that

$$\{X(t)\}_{t\geq 0} \text{ is } \{\mathcal{F}(t)\}_{t\geq 0}\text{-adapted and } \mathbb{E}[\int_0^T X(t)^2\, dt] < \infty, \text{ for all } T > 0. \tag{4.2}$$

We denote by $\mathbb{L}^2$ the family of stochastic processes satisfying (4.2). The integral (4.1) can be defined for more general processes than those in the class $\mathbb{L}^2$, as we briefly discuss in Theorem 4.4.

4.2 The Itô integral of step processes

Given $0 = t_0 < t_1 < \dots < t_k < t_{k+1} < \dots$ and a sequence $X_1, X_2, \dots$ of random variables such that, for all $j \in \mathbb{N}$, $X_j \in L^2(\Omega)$ and X_j is $\mathcal{F}(t_j)$-measurable, consider the $\{\mathcal{F}(t)\}_{t\geq 0}$-adapted step process

$$\Delta(t) = \sum_{j=0}^{\infty} \Delta(t_j)\mathbb{I}_{[t_j,t_{j+1})}, \quad \Delta(t_j) = X_j. \tag{4.3}$$

Note that $\{\Delta(t)\}_{t\geq 0} \in \mathbb{L}^2$, by the assumption $X_j \in L^2(\Omega)$, for all $j \in \mathbb{N}$. If we had to integrate $\Delta(t)$ along a stochastic process $\{Y(t)\}_{t\geq 0}$ with differentiable paths, we would have, assuming $t \in (t_k, t_{k+1})$,

$$\begin{aligned}\int_0^t \Delta(s)dY(s) &= \int_0^t \sum_{j=0}^{\infty} \Delta(t_j)\mathbb{I}_{[t_j,t_{j+1})}dY(t) = \sum_{j=0}^{k-1} \Delta(t_j) \int_{t_j}^{t_{j+1}} dY(t) + \Delta(t_k)\int_{t_k}^t dY(t) \\ &= \sum_{j=0}^{k-1} \Delta(t_j)(Y(t_{j+1}) - Y(t_j)) + \Delta(t_k)(Y(t) - Y(t_k)).\end{aligned}$$

The second line makes sense also for stochastic processes $\{Y(t)\}_{t\geq 0}$ whose paths are nowhere differentiable, and thus in particular for the Brownian motion. We then introduce the following definition.

Definition 4.1. *The* **Itô integral** *over the interval* $[0, t]$ *of a step process* $\{\Delta(t)\}_{t\geq 0} \in \mathbb{L}^2$ *is given by*

$$I(t) = \int_0^t \Delta(s)dW(s) = \sum_{j=0}^{k-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)) + \Delta(t_k)(W(t) - W(t_k)),$$

where t_k *is such that* $t_k \leq t < t_{k+1}$.

Note that $\{I(t)\}_{t\geq 0}$ is $\{F(t)\}_{t\geq 0}$-adapted stochastic process adapted with a.s. continuous paths (in fact, the only dependence on the t variable is through $W(t)$). The following theorem collects some other important properties of the Itô integral of a step process.

Theorem 4.1. *The Itô integral of a step process* $\{\Delta(t)\}_{t\geq 0} \in \mathbb{L}^2$ *satisfies the following properties.*

(i) **Linearity:** *for every pair of* $\{\mathcal{F}(t)\}_{t\geq 0}$*-adapted step processes* $\{\Delta_1(t)\}_{t\geq 0}, \{\Delta_2(t)\}_{t\geq 0}$ *and real constants* $c_1, c_2 \in \mathbb{R}$ *there holds*

$$\int_0^t (c_1\Delta_1(s) + c_2\Delta_2(s))dW(s) = c_1 \int_0^t \Delta_1(s)dW(s) + c_2 \int_0^t \Delta_2(s)dW(s).$$

(ii) **Martingale property:** *the stochastic process* $\{I(t)\}_{t\geq 0}$ *is a martingale in the filtration* $\{\mathcal{F}(t)\}_{t\geq 0}$. *In particular,* $\mathbb{E}[I(t)] = \mathbb{E}[I(0)] = \mathbb{E}[0] = 0$.

(iii) **Quadratic variation:** *the quadratic variation of the stochastic process* $\{I(t)\}_{t\geq 0}$ *on the interval* $[0, T]$ *is independent of the sequence of partitions along which it is computed and it is given by*

$$[I, I](T) = \int_0^T \Delta^2(s)\, ds.$$

(iv) **Itô's isometry:** $\mathbb{E}[I^2(t)] = \mathbb{E}[\int_0^t \Delta^2(s)\, ds]$, *for all* $t \geq 0$.

Proof. The proof of (i) is straightforward. For the remaining claims, see the following theorems in [21]: Theorem 4.2.1 (martingale property), Theorem 4.2.2 (Itô's isometry), Theorem 4.2.3 (quadratic variation). Here we present the proof of (ii). First we remark that the condition $I(t) \in L^1(\Omega)$, for all $t \geq 0$, follows easily by the assumption that $\Delta(t_j) = X_j \in L^2(\Omega)$, for all $j \in \mathbb{N}$ and the Schwartz inequality. Hence we have to prove that

$$\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s), \quad \text{for all } 0 \leq s \leq t.$$

There are two possibilities: (1) either $s, t \in [t_k, t_{k+1})$, for some $k \in \mathbb{N}$, or (2) there exists $l < k$ such that $s \in [t_l, t_{l+1})$ and $t \in [t_k, t_{k+1})$. We assume that (2) holds, the proof in the case (1) being simpler. We write

$$\begin{aligned} I(t) &= \sum_{j=0}^{l-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)) + \Delta(t_l)(W(t_{l+1}) - W(t_l)) \\ &\quad + \sum_{j=l+1}^{k-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)) + \Delta(t_k)(W(t) - W(t_k)) \\ &= I(t_{l+1}) + \int_{t_{l+1}}^t \Delta(u) dW(u). \end{aligned}$$

Taking the conditional expectation of $I(t_{l+1})$ we obtain

$$\mathbb{E}[I(t_{l+1})|\mathcal{F}(s)] = \sum_{j=0}^{l-1} \mathbb{E}[\Delta(t_j)(W(t_{j+1}) - W(t_j))|\mathcal{F}(s)] + \mathbb{E}[\Delta(t_l)(W(t_{l+1}) - W(t_l))|\mathcal{F}(s)].$$

As $t_{l-1} < s$, all random variables in the sum in the right hand side of the latter identity are $\mathcal{F}(s)$-measurable. Hence, by Theorem 3.13(ii),

$$\sum_{j=0}^{l-1} \mathbb{E}[\Delta(t_j)(W(t_{j+1}) - W(t_j))|\mathcal{F}(s)] = \sum_{j=0}^{l-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)).$$

Similarly,

$$\begin{aligned} \mathbb{E}[\Delta(t_l)(W(t_{l+1}) - W(t_l))|\mathcal{F}(s)] &= \mathbb{E}[\Delta(t_l)W(t_{l+1})|\mathcal{F}(s)] - \mathbb{E}[\Delta(t_l)W(t_l)|\mathcal{F}(s)] \\ &= \Delta(t_l)\mathbb{E}[W(t_{l+1})|\mathcal{F}(s)] - \Delta(t_l)W(t_l) \\ &= \Delta(t_l)W(s) - \Delta(t_l)W(t_l), \end{aligned}$$

where for the last equality we used that $\{W(t)\}_{t\geq 0}$ is a martingale in the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$. Hence

$$\mathbb{E}[I(t_{l+1})|\mathcal{F}(s)] = \sum_{j=0}^{l-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)) + \Delta(t_l)(W(s) - W(t_l)) = I(s).$$

To conclude the proof we have to show that

$$\mathbb{E}[\int_{t_{l+1}}^{t} \Delta(u)dW(u)] = \sum_{j=l+1}^{k-1} \mathbb{E}[\Delta(t_j)(W(t_{j+1}) - W(t_j))|\mathcal{F}(s)] + \mathbb{E}[\Delta(t_k)(W(t) - W(t_k))|\mathcal{F}(s)] = 0.$$

To prove this, we first note that, as before,

$$\mathbb{E}[\Delta(t_j)(W(t_{j+1}) - W(t_j))|\mathcal{F}(t_j)] = \Delta(t_j)W(t_j) - \Delta(t_j)W(t_j) = 0.$$

Moreover, since $\mathcal{F}(s) \subset \mathcal{F}(t_j)$, for $j = l+1, \dots, k-1$, as using Theorem 3.13(iii),

$$\mathbb{E}[\Delta(t_j)(W(t_{j+1}) - W(t_j))] = \mathbb{E}[\mathbb{E}[\Delta(t_j)(W(t_{j+1}) - W(t_j))|\mathcal{F}(t_j)]|\mathcal{F}(s)] = \mathbb{E}[0|\mathcal{F}(s)] = 0.$$

At the same fashion, since $\mathcal{F}(s) \subset \mathcal{F}(t_k)$, we have

$$\mathbb{E}[\Delta(t_k)(W(t) - W(t_k))|\mathcal{F}(s)] = \mathbb{E}[\mathbb{E}[\Delta(t_k)(W(t) - W(t_k))|\mathcal{F}(t_k)]|\mathcal{F}(s)] = 0.$$

□

Next we show that any stochastic process can be approximated, in a suitable sense, by step processes.

Theorem 4.2. *Let $\{X(t)\}_{t\geq 0} \in \mathbb{L}^2$. Then for all $T > 0$ there exists a sequence of step processes $\{\{\Delta_n^T(t)\}_{t\geq 0}\}_{n\in\mathbb{N}}$ such that $\Delta_n^T(t) \in \mathbb{L}^2$ for all $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \mathbb{E}[\int_0^T |\Delta_n^T(t) - X(t)|^2 \, dt] = 0. \tag{4.4}$$

Proof. For simplicity we argue under the stronger assumption that the stochastic process $\{X(t)\}_{t\geq 0}$ is bounded and with continuous paths, namely

$$\omega \to X(t,\omega) \text{ is bounded in } \Omega, \text{ for all } t \geq 0,$$
$$t \to X(t,\omega) \text{ is continuous for all } \omega \in \Omega \text{ and } t \geq 0.$$

Now consider the partition of $[0,T]$ given by

$$0 = t_0^{(n)} < t_1^{(n)} < \dots < t_n^{(n)} = T, \quad t_j^{(n)} = \frac{jT}{n}$$

and define

$$\Delta_n^T(t) = \sum_{j=0}^{n-1} X(t_k^{(n)}) \mathbb{I}_{[t_k^{(n)}, t_{k+1}^{(n)})}, \quad t \geq 0,$$

see Figure 4.1. Let us show that $\{\Delta_n^T(t)\}_{t\geq 0}$ is adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$. This is obvious for $t \geq T$

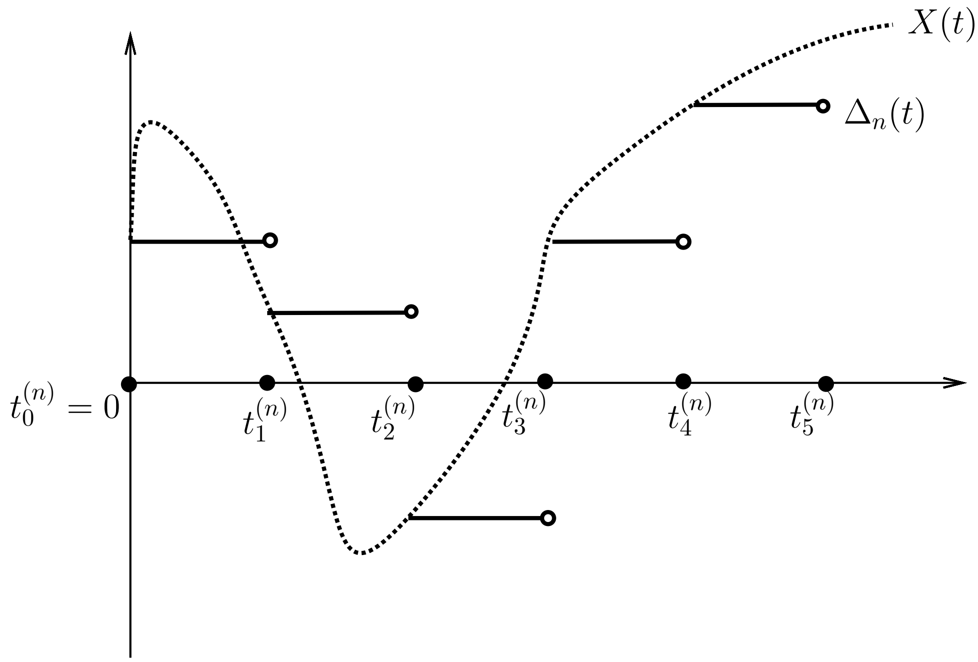


Figure 4.1: A step process approximating a general stochastic process

(since the step process is identically zero for $t \geq T$). For $t \in [0, T)$ we have $\Delta_n^T(t) = X(t_k^{(n)})$, for $t \in [t_k^{(n)}, t_{k+1}^{(n)})$, hence

$$\mathcal{F}_{\Delta_n^T(t)} = \mathcal{F}_{X(t_k^{(n)})} \underset{(*)}{\subset} \mathcal{F}(t_k^{(n)}) \underset{(**)}{\subset} \mathcal{F}(t),$$

where in (*) we used that $\{X(t)\}_{t\geq 0}$ is adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$ and in (**) the fact that $t_k^{(n)} < t$. Moreover,

$$\lim_{n\to\infty} \Delta_n^T(t) = X(t), \quad \text{for all } \omega \in \Omega,$$

by the assumed continuity of the paths of $\{X(t)\}_{t\geq 0}$. For the next step we use the dominated convergence theorem, see Remark 3.2. Since $\Delta_n^T(t)$ and $X(t)$ are bounded on $[0, T] \times \Omega$, there exists a constant C_T such that $|\Delta_n(t) - X(t)|^2 \leq C_T$. Hence we may move the limit on the left hand side of (4.4) across the expectation and integral operators and conclude that

$$\lim_{n\to\infty} \mathbb{E}[\int_0^T |\Delta_n^T(t) - X(t)|^2 \, dt] = \mathbb{E}[\int_0^T \lim_{n\to\infty} |\Delta_n^T(t) - X(t)|^2 \, dt] = 0,$$

as claimed. □

4.3 Itô's integral of general stochastic processes

The Itô integral of a general stochastic process is defined as the limit of the Itô integral along a sequence of approximating step processes (in the sense of Theorem 4.2). The precise definition is the following.

Theorem 4.3 (and Definition). *Let* $\{X(t)\}_{t\geq 0} \in \mathbb{L}^2$, $T > 0$ *and* $\{\{\Delta_n^T(t)\}_{t\geq 0}\}_{n\in\mathbb{N}}$ *be a sequence of* $\mathbb{L}^2$*-step processes converging to* $\{X(t)\}_{t\geq 0}$ *in the sense of* (4.4)*. Let*

$$I_n(T) = \int_0^T \Delta_n^T(s)dW(s).$$

Then there exists a random variable $I(T)$ *such that*

$$\|I_n(T) - I(T)\|_2 := \sqrt{\mathbb{E}[|I_n(T) - I(T)|^2]} \to 0, \quad \text{as } n \to \infty.$$

The random variable $I(T)$ *is independent of the sequence of* $\mathbb{L}^2$*-step processes converging to* $\{X(t)\}_{t\geq 0}$*.* $I(T)$ *is called* **Itô's integral** *of* $\{X(t)\}_{t\geq 0}$ *on the interval* $[0, T]$ *and denoted by*

$$I(T) = \int_0^T X(s)dW(s).$$

Proof. By Itô's isometry,

$$\mathbb{E}[|I_n(T) - I_m(T)|^2] = \mathbb{E}[\int_0^T |\Delta_n^T(s) - \Delta_m^T(s)|^2 ds].$$

We have

$$\mathbb{E}[\int_0^T |\Delta_n^T(s) - \Delta_m^T(s)|^2 ds] \leq 2\mathbb{E}[\int_0^T |\Delta_n^T(s) - X(s)|^2 ds] + 2\mathbb{E}[\int_0^T |\Delta_m^T(s) - X(s)|^2 ds] \to 0 \text{ as } n, m \to \infty.$$

It follows that $\{I_n(T)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the norm $\|\cdot\|_2$. As mentioned in Remark 3.5, the norm $\|\cdot\|_2$ is complete, i.e., Cauchy sequences converge. This proves the existence of $I(T)$ such that $\|I_n(T) - I(T)\|_2 \to 0$. To prove that the limit is the same along any sequence of $\mathbb{L}^2$-step processes converging to $\{X(t)\}_{t\geq 0}$, assume that $\{\{\Delta_n(t)\}_{t\geq 0}\}_{n\in\mathbb{N}}$, $\{\{\widetilde{\Delta}_n(t)\}_{t\geq 0}\}_{n\in\mathbb{N}}$ are two such sequences and denote

$$I_n(T) = \int_0^T \Delta_n(s)dW(s), \quad \widetilde{I}_n(t) = \int_0^T \widetilde{\Delta}_n(s)dW(s).$$

Then, using (i), (iv) in Theorem 4.1, we compute

$$\begin{aligned}\mathbb{E}[(I_n(T) - \widetilde{I}_n(T))^2] &= \mathbb{E}[(\int_0^T (\Delta_n(s) - \widetilde{\Delta}_n(s))dW(s))^2] = \mathbb{E}[\int_0^T |\Delta_n(s) - \widetilde{\Delta}_n(s)|^2 ds] \\ &\leq 2\mathbb{E}[\int_0^T |\Delta_n(s) - X(s)|^2 ds] + 2\mathbb{E}[\int_0^T |\widetilde{\Delta}_n(s) - X(s)|^2 ds] \to 0, \\ &\text{as } n \to \infty,\end{aligned}$$

which proves that $I_n(T)$ and $\widetilde{I}_n(T)$ have the same limit. This completes the proof of the theorem. □

As a way of example, we compute the Itô integral of the Brownian motion. We claim that, for all $T > 0$,

$$\int_0^T W(t)dW(t) = \frac{W^2(T)}{2} - \frac{T}{2}. \tag{4.5}$$

To prove the claim, we approximate the Brownian motion by the sequence of step processes introduced in the proof of Theorem 4.2. Hence we define

$$\Delta_n^T(t) = \sum_{j=0}^{n-1} W(\frac{jT}{n})\mathbb{I}_{[\frac{jT}{n},\frac{j+1}{n}T)}.$$

By definition

$$I_n(T) = \int_0^T \Delta_n^T(t)dW(t) = \sum_{j=0}^{n-1} W(\frac{jT}{n})[W(\frac{(j+1)T}{n}) - W(\frac{jT}{n})].$$

To simplify the notation we let $W_j = W(jT/n)$. Hence our goal is to prove

$$\mathbb{E}[\left|\sum_{j=0}^{n-1} W_j(W_{j+1} - W_j) - \frac{W^2(T)}{2} + \frac{T}{2}\right|^2] \to 0, \quad \text{as } n \to \infty. \tag{4.6}$$

We prove below that the sum within the expectation can be rewritten as

$$\sum_{j=0}^{n-1} W_j(W_{j+1} - W_j) = \frac{1}{2}W(T)^2 - \frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1} - W_j)^2. \tag{4.7}$$

Hence (4.6) is equivalent to

$$\frac{1}{4}\mathbb{E}[\left|\sum_{j=0}^{n-1}(W_{j+1} - W_j)^2 - T\right|^2] \to 0, \quad \text{as } n \to \infty,$$

which holds true by the already proven fact that $[W, W](T) = T$, see Theorem 3.9. It remains to establish (4.7). Since $W(T) = W_n$, we have

$$\begin{aligned}
\frac{W(T)}{2} - \frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1} - W_j)^2 &= \frac{1}{2}W_n^2 - \frac{1}{2}\sum_{j=0}^{n-1} W_{j+1}^2 - \frac{1}{2}\sum_{j=0}^{n-1} W_j^2 + \sum_{j=0}^{n-1} W_j W_{j+1} \\
&= -\frac{1}{2}\sum_{j=0}^{n-2} W_{j+1}^2 - \frac{1}{2}\sum_{j=1}^{n-1} W_j^2 + \sum_{j=1}^{n-1} W_j W_{j+1} \\
&= -\sum_{j=1}^{n-1} W_j^2 + \sum_{j=1}^{n-1} W_j W_{j+1} = \sum_{j=1}^{n-1} W_j(W_{j+1} - W_j) \\
&= \sum_{j=0}^{n-1} W_j(W_{j+1} - W_j).
\end{aligned}$$

The proof of (4.5) is complete.

Exercise 4.1. *Use the definition of Itô's integral to prove that*

$$TW(T) = \int_0^T W(t)dt + \int_0^T t dW(t). \tag{4.8}$$

The Itô integral can be defined under weaker assumptions on the integrand stochastic process than those considered so far. As this fact will be important in the following sections, it is worth to briefly discuss it. Let $\mathcal{M}^2$ denote the set of $\{\mathcal{F}(t)\}_{t\geq 0}$-adapted stochastic processes $\{X(t)\}_{t\geq 0}$ such that $\int_0^T X(t)^2\, dt$ is bounded a.s. for all $T > 0$ (of course, $\mathbb{L}^2 \subset \mathcal{M}^2$).

Theorem 4.4 (and Definition). *For every process $\{X(t)\}_{t\geq 0} \in \mathcal{M}^2$ and $T > 0$ there exists a sequence of step processes $\{\{\Delta_n^T(t)\}_{t\geq 0}\}_{n\in\mathbb{N}} \subset \mathcal{M}^2$ such that*

$$\lim_{n\to\infty} \int_0^T |X(s) - \Delta_n^T(s)|^2\, ds \to 0 \quad a.s.$$

and

$$\int_0^T \Delta_n(t) dW(t)$$

converges in probability as $n \to \infty$. The limit is independent of the sequence of step processes converging to $\{X(t)\}_{t\geq 0}$ and is called the Itô integral of the process $\{X(t)\}_{t\geq 0}$ in the interval $[0, T]$. If $\{X(t)\}_{t\geq 0} \in \mathbb{L}^2$, the Itô integral just defined coincides (a.s.) with the one defined in Theorem 4.3.

For the proof of the previous theorem, see [1, Sec. 4.4]. We remark that Theorem 4.4 implies that all $\{\mathcal{F}(t)\}_{t\geq 0}$-adapted stochastic processes with a.s. continuous paths are Itô integrable. In fact, if $\{X(t)\}_{t\geq 0}$ has a.s. continuous paths, then for all $T > 0$, there exists $C_T(\omega)$ such that $\sup_{t\in[0,T]} |X(t, \omega)| \leq C_T(\omega)$ a.s. Hence

$$\int_0^T |X(s)|^2\, ds \leq T C_T^2(\omega), \quad \text{a.s.}$$

and thus Theorem 4.4 applies. The case of stochastic processes with a.s. continuous paths covers all the applications in the following chapters, hence we shall restrict to it from now on.

Definition 4.2. *We define $\mathcal{C}^0$ to be the space of all $\{\mathcal{F}(t)\}_{t\geq 0}$-adapted stochastic processes $\{X(t)\}_{t\geq 0}$ with a.s. continuous paths.*

In particular, if $\{X(t)\}_{t\geq 0}, \{Y(t)\}_{t\geq 0} \in \mathcal{C}^0$, then for all continuous functions f the process $\{f(t, X(t), Y(t))\}_{t\geq 0}$ belongs to $\mathcal{C}^0$ and thus it is Itô integrable.

The properties listed in Theorem 4.1 carry over to the Itô integral of a general stochastic process. For easy reference, we rewrite these properties in the following theorem.

Theorem 4.5. *Let $\{X(t)\}_{t\geq 0} \in \mathcal{C}^0$. Then the Itô integral*

$$I(t) = \int_0^t X(s) dW(s) \tag{4.9}$$

satisfies the following properties for all $t \geq 0$.

(0) $\{I(t)\}_{t\geq 0} \in \mathcal{C}^0$. *If* $\{X(t)\}_{t\geq 0} \in \mathbb{L}^2$, *then* $\{I(t)\}_{t\geq 0} \in \mathbb{L}^2$.

(i) **Linearity:** *For all stochastic processes* $\{X_1(t)\}_{t\geq 0}, \{X_2(t)\}_{t\geq 0} \in \mathcal{C}^0$ *and real constants* $c_1, c_2 \in \mathbb{R}$ *there holds*

$$\int_0^t (c_1 X_1(s) + c_2 X_2(s))dW(s) = c_1 \int_0^t X_1(s)dW(s) + c_2 \int_0^t X_2(s)dW(s).$$

(ii) **Martingale property:** *If* $\{X(t)\}_{t\geq 0} \in \mathbb{L}^2$, *the stochastic process* $\{I(t)\}_{t\geq 0}$ *is a martingale in the filtration* $\{\mathcal{F}(t)\}_{t\geq 0}$. *In particular,* $\mathbb{E}[I(t)] = \mathbb{E}[I(0)] = 0$, *for all* $t \geq 0$.

(iii) **Quadratic variation:** *For all* $T > 0$, *the quadratic variation of the stochastic process* $\{I(t)\}_{t\geq 0}$ *on the interval* $[0, T]$ *is independent of the sequence of partitions along which it is computed and it is given by*

$$[I, I](T) = \int_0^T X^2(s)\, ds. \tag{4.10}$$

(iv) **Itô's isometry:** *If* $\{X(t)\}_{t\geq 0} \in \mathbb{L}^2$, *then* $\mathrm{Var}[I(t)] = \mathbb{E}[I^2(t)] = \mathbb{E}[\int_0^t X^2(s)\, ds]$, *for all* $t \geq 0$.

Proof of (ii). By (iv) and the Schwartz inequality, $\mathbb{E}[I(t)] \leq \sqrt{\mathbb{E}[I^2(t)]} < \infty$. According to (3.20), it now suffices to show that

$$\mathbb{E}[I(t)\mathbb{I}_A] = \mathbb{E}[I(s)\mathbb{I}_A], \quad \text{for all } A \in \mathcal{F}(s).$$

Let $\{\{I_n(t)\}_{t\geq 0}\}_{n\in\mathbb{N}}$ be a sequence of Itô integrals of step processes which converges to $\{I(t)\}_{t\geq 0}$ in $L^2(\Omega)$, uniformly in compact intervals of time (see Theorem 4.3). Since $\{I_n(t)\}_{t\geq 0}$ is a martingale for each $n \in \mathbb{N}$, see Theorem 4.1, then

$$\mathbb{E}[I_n(t)\mathbb{I}_A] = \mathbb{E}[I_n(s)\mathbb{I}_A], \quad \text{for all } A \in \mathcal{F}(s).$$

Hence the claim follows if we show that $\mathbb{E}[I_n(t)\mathbb{I}_A] \to \mathbb{E}[I(t)\mathbb{I}_A]$, for all $t \geq 0$. Using the Schwarz inequality (3.3), we have

$$\begin{aligned}\mathbb{E}[(I_n(t) - I(t))\mathbb{I}_A] &\leq \sqrt{\mathbb{E}[(I_n(t) - I(t))^2]\mathbb{E}[\mathbb{I}_A]} \leq \|I_n(t) - I(t)\|_2\sqrt{\mathbb{P}(A)} \\ &\leq \|I_n(t) - I(t)\|_2 \to 0, \quad \text{as } n \to \infty,\end{aligned}$$

and the claim follows. □

Remark 4.1. Note carefully that the martingale property (ii) requires $\{X(t)\}_{t\geq 0} \in \mathbb{L}^2$. A stochastic process in $\mathcal{C}^0 \backslash \mathbb{L}^2$ is not a martingale in general (although it is a **local martingale**, see [1]).

Exercise 4.2. *Let $\{X(t)\}_{t\geq 0} \in \mathbb{L}^2$. Show that the double Itô integral*

$$J(t) = \int_0^t \left(\int_0^s X(\tau) dW(\tau) \right) dW(s), \quad t \geq 0,$$

is well defined. Write down the properties in Theorem 4.5 for $J(t)$.

Exercise 4.3. *Prove the following generalization of Itô's isometry. Let $\{X(t)\}_{t\geq 0}, \{Y(t)\}_{t\geq 0} \in \mathcal{C}^0 \cap \mathbb{L}^2$ and denote by $I_X(t), I_Y(t)$ their Itô integral over the interval $[0,t]$. Then*

$$\mathrm{Cov}(I_X(t), I_Y(t)) = \mathbb{E}[\int_0^t X(s)Y(s)\, ds].$$

We now state without proof the **martingale representation theorem**, which asserts that any martingale is an Itô integral (i.e., the logically opposite claim to (ii) in Theorem 4.5 holds). For the proof see Theorem 4.3.4 in [17].

Theorem 4.6. *Let $\{M(t)\}_{t\geq 0}$, with $M(t) \in L^2(\Omega)$ for all $t \geq 0$, be a martingale stochastic process relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$. Then there exists a stochastic process $\{\Gamma(t)\}_{t\geq 0} \in \mathbb{L}^2$ such that*

$$M(t) = M(0) + \int_0^t \Gamma(s) dW(s), \quad \textit{for all } t \geq 0.$$

Remark 4.2. Note carefully that the filtration used in the martingale representation theorem must be the one generated by the Brownian motion. Theorem 4.6 will be used in Chapter 6 to show the existence of hedging portfolios of European derivatives, see Theorem 6.2.

We conclude this section by introducing the "differential notation" for stochastic integrals. Instead of (4.9), we write

$$dI(t) = X(t)dW(t).$$

For instance, the identities (4.5), (4.8) are also expressed as

$$d(W^2(t)) = dt + 2W(t)dW(t), \quad d(tW(t)) = W(t)dt + tdW(t).$$

The quadratic variation (4.10) is expressed also as

$$dI(t)dI(t) = X^2(t)dt.$$

Note that this notation is in agreement with the one already introduced in Section 3.4, namely

$$dI(t)dI(t) = d\left(\int_0^t X^2(s)ds \right) = X^2(t)dt.$$

The differential notation is very useful to provide informal proofs in stochastic calculus. For instance, using $dI(t) = X(t)dW(t)$, and $dW(t)dW(t) = dt$, see (3.12), we obtain the following simple "proof" of Theorem 4.5(iii):

$$dI(t)dI(t) = X(t)dW(t)X(t)dW(t) = X^2(t)dW(t)dW(t) = X^2(t)dt.$$

4.4 Diffusion processes

Now that we know how to integrate along the paths of a Brownian motion, we can define a new class of stochastic processes.

Definition 4.3. *Given $\{\alpha(t)\}_{t\geq 0}, \{\sigma(t)\}_{t\geq 0} \in \mathcal{C}^0$, the stochastic process $\{X(t)\}_{t\geq 0} \in \mathcal{C}^0$ given by*

$$X(t) = X(0) + \int_0^t \sigma(s)dW(s) + \int_0^t \alpha(s)\, ds, \quad t \geq 0 \tag{4.11}$$

is called **diffusion process** *with* **rate of quadratic variation** $\{\sigma^2(t)\}_{t\geq 0}$ *and* **drift** $\{\alpha(t)\}_{t\geq 0}$.

We denote diffusion processes also as

$$dX(t) = \sigma(t)dW(t) + \alpha(t)dt. \tag{4.12}$$

Note that

$$dX(t)dX(t) = \sigma^2(t)dW(t)dW(t) + \alpha^2(t)dtdt + \sigma(t)\alpha(t)dW(t)dt$$

and thus, by (3.11), (3.12) and (3.14), we obtain

$$dX(t)dX(t) = \sigma^2(t)dt,$$

which means that the quadratic variation of the diffusion process (4.12) is given by

$$[X, X](t) = \int_0^t \sigma^2(s)\, ds, \quad t \geq 0.$$

Thus the stochastic process $\{\sigma^2(t)\}_{t\geq 0}$ measures the rate at which quadratic variations accumulates in time in the diffusion process $\{X(t)\}_{t\geq 0}$. Furthermore, assuming $\{\sigma(t)\}_{t\geq 0} \in \mathbb{L}^2$, we have

$$\mathbb{E}[\int_0^t \sigma(s)dW(s)] = 0.$$

Hence the term $\int_0^t \alpha(s)ds$ is the only one contributing to the evolution of the average of $\{X(t)\}_{t\geq 0}$, which is the reason to call $\alpha(t)$ the drift of the diffusion process (if $\alpha = 0$ and $\{\sigma(t)\}_{t\geq 0} \in \mathbb{L}^2$, the diffusion process is a martingale, as it follows by Theorem 4.5(ii)). Finally, the integration along the paths of the diffusion process (4.12) is defined as

$$\int_0^t Y(s)dX(s) := \int_0^t Y(s)\sigma(s)dW(s) + \int_0^t Y(s)\alpha(s)ds, \tag{4.13}$$

for all $\{Y(t)\}_{t\geq 0} \in \mathcal{C}^0$.

4.4.1 The product rule in stochastic calculus

Recall that if $f, g : \mathbb{R} \to \mathbb{R}$ are two differentiable functions, the product (or Leibnitz) rule of ordinary calculus states that

$$(fg)' = f'g + fg',$$

and thus

$$fg(t) = fg(0) + \int_0^t (g(s)df(s) + f(s)dg(s)).$$

Can this rule be true in stochastic calculus, i.e., when f and g are general diffusion processes? The answer is clearly no. In fact, letting for instance $f(t) = g(t) = W(t)$, Leibnitz's rule give us the relation $d(W^2(t)) = 2W(t)dW(t)$, while we have seen before that the correct formula in Itô's calculus is $d(W^2(t)) = 2W(t)dW(t) + t$. The correct product rule in Itô's calculus is the following.

Theorem 4.7. *Let $\{X_1(t)\}_{t\geq 0}$ and $\{X_2(t)\}_{t\geq 0}$ be the diffusion processes*

$$dX_i(t) = \sigma_i(t)dW(t) + \theta_i(t)dt.$$

Then $\{X_1(t)X_2(t)\}_{t\geq 0}$ is the diffusion process given by

$$d(X_1(t)X_2(t)) = X_2(t)dX_1(t) + X_1(t)dX_2(t) + \sigma_1(t)\sigma_2(t)dt. \tag{4.14}$$

Exercise 4.4 (•). *Prove the theorem in the case that α_i and σ_i are deterministic constants and $X_i(0) = 0$, for $i = 1, 2$.*

Recall that the correct way to interpret (4.14) is

$$X_1(t)X_2(t) = X_1(0)X_2(0) + \int_0^t X_2(s)dX_1(s) + \int_0^t X_1(s)dX_2(s) + \int_0^t \sigma_1(s)\sigma_2(s)ds, \tag{4.15}$$

where the integrals along the paths of the processes $\{X_i(t)\}_{t\geq 0}$ are defined as in (4.13). Note that all integrals in (4.15) are well-defined, since the integrand stochastic processes have a.s. continuous paths. We also remark that, since

$$\begin{aligned}
dX_1(t)dX_2(t) &= (\sigma_1(t)dW(t) + \alpha_1(t)dt)(\sigma_2(t)dW(t) + \alpha_2(t)dt) \\
&= \sigma_1(t)\sigma_2(t)dW(t)dW(t) + (\alpha_1(t)\sigma_2(t) + \alpha_2(t)\sigma_1(t))dW(t)dt + \alpha_1(t)\alpha_2(t)dtdt \\
&= \sigma_1(t)\sigma_2(t)dt,
\end{aligned}$$

then we may rewrite (4.14) as

$$d(X_1(t)X_2(t)) = X_2(t)dX_1(t) + X_1(t)dX_2(t) + dX_1(t)dX_2(t), \tag{4.16}$$

which is somehow easier to remember. Going back to the examples considered in Section 4.3, the Itô product rule gives

$$d(W^2(t)) = W(t)dW(t) + W(t)dW(t) + dW(t)dW(t) = 2W(t)dW(t) + dt,$$

$$d(tW(t)) = tdW(t) + W(t)dt + dW(t)dt = tdW(t) + W(t)dt,$$

in agreement with our previous calculations, see (4.5) and (4.8).

4.4.2 The chain rule in stochastic calculus

Next we consider the generalization to Itô's calculus of the chain rule. Let us first recall how the chain rule works in ordinary calculus. Assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable functions. Then

$$\frac{d}{dt} f(t, g(t)) = \partial_t f(t, g(t)) + \partial_x f(t, g(t)) \frac{d}{dt} g(t),$$

by which we derive

$$f(t, g(t)) = f(0, g(0)) + \int_0^t \partial_s f(s, g(s))\, ds + \int_0^t \partial_x f(s, g(s))\, dg(s).$$

Can this formula be true in stochastic calculus, i.e., when g is a diffusion process? The answer is clearly no. In fact by setting $f(t, x) = x^2$, $g(t) = W(t)$ and $t = T$ in the previous formula we obtain

$$W^2(T) = 2 \int_0^T W(t) dW(t), \quad \text{i.e.,} \quad \int_0^T W(t) dW(t) = \frac{W^2(T)}{2},$$

while the Itô integral of the Brownian motion is given by (4.5). The correct formula for the chain rule in stochastic calculus is given in the following theorem.

Theorem 4.8. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f = f(t, x)$, be a C^1 function such that $\partial_x^2 f$ is continuous and let $\{X(t)\}_{t \geq 0}$ be the diffusion process $dX(t) = \sigma(t) dW(t) + \alpha(t) dt$. Then* **Itô's formula** *holds:*

$$df(t, X(t)) = \partial_t f(t, X(t))\, dt + \partial_x f(t, X(t))\, dX(t) + \frac{1}{2} \partial_x^2 f(t, X(t))\, dX(t) dX(t), \tag{4.17}$$

i.e.,

$$df(t, X(t)) = \partial_t f(t, X(t))\, dt + \partial_x f(t, X(t))\, (\sigma(t) dW(t) + \alpha(t) dt) + \frac{1}{2} \partial_x^2 f(t, X(t)) \sigma^2(t)\, dt. \tag{4.18}$$

For instance, letting $X(t) = W(t)$ and $f(t, x) = x^2$, we obtain $d(W^2(t)) = W(t) dW(t) + \frac{1}{2} dt$, i.e., (4.5), while for $f(t, x) = tx$ we obtain $d(tW(t)) = W(t) dt + t dW(t)$, which is (4.8). In fact, the proof of Theorem 4.8 is similar to proof of (4.5) and (4.8). We omit the details (see [21, Theorem 4.4.1] for a sketch of the proof).

Recall that (4.18) is a shorthand for

$$f(t, X(t)) = f(0, X(0)) + \int_0^t (\partial_t f + \alpha(s) \partial_x f + \frac{1}{2} \sigma^2(s) \partial_x^2 f)(s, X(s))\, ds + \int_0^t \partial_x f(s, X(s)) dW(s).$$

All integrals in the right hand side of the previous equation are well defined, as the integrand stochastic processes have continuous paths. We conclude with the generalization of Itô's formula to functions of several random variables, which again we give without proof.

Theorem 4.9. *Let $f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $f = f(t, x)$ is twice continuously differentiable on the variable $x \in \mathbb{R}^N$. Let $\{X_1(t)\}_{t\geq 0}, \dots, \{X_N(t)\}_{t\geq 0}$ be diffusion processes and let $X(t) = (X_1(t), \dots, X_N(t))$. Then there holds:*

$$df(t, X(t)) = \partial_t f(t, X(t))\, dt + \sum_{i=1}^{N} \partial_{x^i} f(t, X(t))\, dX_i(t) + \frac{1}{2} \sum_{i,j=1}^{N} \partial_{x^i}\partial_{x^j} f(t, X(t))\, dX_i(t) dX_j(t). \tag{4.19}$$

For instance, for $N = 2$ and letting $f(t, x_1, x_2) = x_1 x_2$ into (4.19), we obtain the Itô product rule (4.16).

Remark 4.3. Let $\{X(t)\}_{t\geq 0}$, $\{Y(t)\}_{t\geq 0}$ be diffusion processes and define the complex-valued stochastic process $\{Z(t)\}_{t\geq 0}$ by $Z(t) = X(t) + iY(t)$. Then any stochastic process of the form $g(t, Z(t))$ can be written in the form $f(t, X(t), Y(t))$, where $f(t, x, y) = g(t, x + iy)$. Hence $dg(t, Z(t))$ can be computed using Theorem 4.9. An application to this formula is given in Exercise 4.9 below.

The following exercises help to get familiar with the rules of stochastic calculus.

Exercise 4.5 (•). *Let $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ be Brownian motions. Assume that there exists a constant $\rho \in [-1, 1]$ such that $dW_1(t)dW_2(t) = \rho dt$. Show that ρ is the correlation of the two Brownian motions at time t. Assuming that $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ are independent, compute $\mathbb{P}(W_1(t) > W_2(s))$, for all $s, t > 0$.*

Exercise 4.6 (•). *Consider the stochastic process $\{X(t)\}_{t\geq 0}$ defined by $X(t) = W(t)^3 - 3tW(t)$. Show that $\{X(t)\}_{t\geq 0}$ is a martingale and find a process $\{\Gamma(t)\}_{t\geq 0}$ adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$ such that*

$$X(t) = X(0) + \int_0^t \Gamma(s) dW(s).$$

(The existence of the process $\{\Gamma(t)\}_{t\geq 0}$ is ensured by Theorem 4.6.)

Exercise 4.7 (•). *Let $\{\theta(t)\}_{t\geq 0} \in \mathcal{C}^0$ and define the stochastic process $\{Z(t)\}_{t\geq 0}$ by*

$$Z(t) = \exp\left(-\int_0^t \theta(s) dW(s) - \frac{1}{2} \int_0^t \theta^2(s) ds \right).$$

Show that

$$Z(t) = -\int_0^t \theta(s) Z(s)\, dW(s).$$

Processes of the form considered in Exercise 4.7 are fundamental in mathematical finance. In particular, it is important to know whether $\{Z(t)\}_{t\geq 0}$ is a martingale. By Exercise 4.7 and Theorem 4.5(ii), $\{Z(t)\}_{t\geq 0}$ is a martingale if $\theta(t)Z(t) \in \mathbb{L}^2$, which is however difficult in general to verify directly. The following condition, known as **Novikov's condition**, is more useful in the applications, as it involves only the time-integral of the process $\{\theta(t)\}_{t\geq 0}$. The proof can be found in [13].

Theorem 4.10. *Let $\{\theta(t)\}_{t\geq 0} \in \mathcal{C}^0$ satisfy*

$$\mathbb{E}[\exp(\frac{1}{2}\int_0^T \theta(t)^2\,dt)] < \infty, \quad \textit{for all } T > 0. \tag{4.20}$$

Then the stochastic process $\{Z(t)\}_{t\geq 0}$ given by

$$Z(t) = \exp\left(-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\right).$$

is a martingale relative the filtration $\{F(t)\}_{t\geq 0}$.

In particular, the stochastic process $\{Z(t)\}_{t\geq 0}$ is a martingale when $\theta(t) = const$, hence we recover the result of Exercise 3.29. The following exercise extends the result of Exercise 4.7 to the case of several independent Brownian motions.

Exercise 4.8 ($\star$)**.** *Let $\{W_1(t)\}_{t\geq 0}, \dots \{W_N(t)\}_{t\geq 0}$ be independent Brownian motions and let $\{\mathcal{F}(t)\}_{t\geq 0}$ be a non-anticipating filtration for all of them. Let $\{\theta_1(t)\}_{t\geq 0}, \dots \{\theta_N(t)\}_{t\geq 0} \in \mathcal{C}^0$ be adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$ and set $\theta(t) = (\theta_1(t), \dots, \theta_N(t))$, $\|\theta(t)\| = \sqrt{\theta_1(t)^2 + \cdots + \theta_N(t)^2}$. Compute $dZ(t)$, where*

$$Z(t) = \exp\left(-\sum_{j=1}^N \int_0^t \theta_j(s)dW_j(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right).$$

Under which condition is $\{Z(t)\}_{t\geq 0}$ a martingale?

Exercise 4.9 ($\bullet$)**.** *Let $A : \mathbb{R} \to \mathbb{R}$ be continuous deterministic function of time. Show that the random variable*

$$I(t) = \int_0^t A(s)\,dW(s)$$

is normally distributed with zero expectation and variance $\int_0^t A(s)^2 ds$.

Exercise 4.10. *Show that the process $\{W^2(t) - t\}_{t\geq 0}$ is a martingale relative to $\{\mathcal{F}(t)\}_{t\geq 0}$, where $\{W(t)\}_{t\geq 0}$ is a Brownian motion and $\{\mathcal{F}(t)\}_{t\geq 0}$ a non-anticipating filtration thereof. Prove also the following logically opposite statement: assume that $\{X(t)\}_{t\geq 0}$ and $\{X^2(t) - t\}_{t\geq 0}$ are martingales relative to $\{\mathcal{F}(t)\}_{t\geq 0}$, $\{X(t)\}_{t\geq 0}$ has a.s. continuous paths and $X(0) = 0$ a.s.. Then $\{X(t)\}_{t\geq 0}$ is a Brownian motion with non-anticipating filtration $\{\mathcal{F}(t)\}_{t\geq 0}$.*

4.5 Girsanov's theorem

In this section we assume that the non-anticipating filtration of the Brownian motion coincides with $\{\mathcal{F}_W(t)\}_{t\geq 0}$. Let $\{\theta(t)\}_{t\geq 0} \in \mathcal{C}^0$ satisfy the Novikov condition (4.20). It follows by Theorem 4.10 that the positive stochastic process $\{Z(t)\}_{t\geq 0}$ given by

$$Z(t) = \exp\left(-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\right) \tag{4.21}$$

is a martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$. As $Z(0) = 1$, then $\mathbb{E}[Z(t)] = 1$ for all $t \geq 0$. Thus we can use the stochastic process $\{Z(t)\}_{t\geq 0}$ to generate a measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ as we did at the end of Section 3.6, namely $\widetilde{\mathbb{P}} : \mathcal{F} \to [0, 1]$ is given by

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z(T)\mathbb{I}_A], \quad A \in \mathcal{F}. \tag{4.22}$$

The relation between $\mathbb{E}$ and $\widetilde{\mathbb{E}}$ has been determined in Theorem 3.19, where we showed that

$$\widetilde{\mathbb{E}}[X] = \mathbb{E}[Z(t)X], \tag{4.23}$$

for all $t \geq 0$ and $\mathcal{F}_W(t)$-measurable random variables X, and

$$\widetilde{\mathbb{E}}[Y|\mathcal{F}_W(s)] = \frac{1}{Z(s)}\mathbb{E}[Z(t)Y|\mathcal{F}_W(s)] \tag{4.24}$$

for all $0 \leq s \leq t$ and random variables Y. We can now state and sketch the proof of **Girsanov**'s theorem, which is a fundamental result with deep applications in mathematical finance.

Theorem 4.11. *Define the stochastic process $\{\widetilde{W}(t)\}_{t\geq 0}$ by*

$$\widetilde{W}(t) = W(t) + \int_0^t \theta(s)ds, \tag{4.25}$$

i.e., $d\widetilde{W}(t) = dW(t) + \theta(t)dt$. Then $\{\widetilde{W}(t)\}_{t\geq 0}$ is a $\widetilde{\mathbb{P}}$-Brownian motion. Moreover $\{F_W(t)\}_{t\geq 0}$ is a non-anticipating filtration for the $\widetilde{\mathbb{P}}$-Brownian motion $\{\widetilde{W}(t)\}_{t\geq 0}$.

Sketch of the proof. We prove the theorem using the Lévy characterization of Brownian motions, see Theorem 3.18. Clearly, $\{\widetilde{W}(t)\}_{t\geq 0}$ starts from zero and has continuous paths a.s. Moreover we (formally) have $d\widetilde{W}(t)d\widetilde{W}(t) = dW(t)dW(t) = dt$. Hence it remains to show that the Brownian motion $\{\widetilde{W}(t)\}_{t\geq 0}$ is $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$. By Itô's product rule we have

$$\begin{aligned} d(\widetilde{W}(t)Z(t)) &= \widetilde{W}(t)dZ(t) + Z(t)d\widetilde{W}(s) + d\widetilde{W}(t)dZ(t) \\ &= (1 - \theta(t)\widetilde{W}(t))Z(t)dW(t), \end{aligned}$$

that is to say,

$$\widetilde{W}(t)Z(t) = \int_0^t (1 - \widetilde{W}(u)\theta(u))Z(u)dW(u).$$

It follows by Theorem 4.5(ii) that the stochastic process $\{Z(t)\widetilde{W}(t)\}_{t\geq 0}$ is a $\mathbb{P}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$, i.e.,

$$\mathbb{E}[Z(t)\widetilde{W}(t)|\mathcal{F}_W(s)] = Z(s)\widetilde{W}(s).$$

But according to (4.24),

$$\mathbb{E}[Z(t)\widetilde{W}(t)|\mathcal{F}_W(s)] = Z(s)\widetilde{\mathbb{E}}[\widetilde{W}(t)|\mathcal{F}_W(s)].$$

Hence $\widetilde{\mathbb{E}}[\widetilde{W}(t)|\mathcal{F}_W(s)] = \widetilde{W}(s)$, as claimed. □

Later we shall need also the multi-dimensional version of Girsanov's theorem. Let $\{W_1(t)\}_{t\geq 0}, \dots, \{W_N(t)\}_{t\geq 0}$ be independent Brownian motions and let $\{\mathcal{F}_W(t)\}_{t\geq 0}$ be their own generated filtration. Let $\{\theta_1(t)\}_{t\geq 0}, \dots, \{\theta_N(t)\}_{t\geq 0} \in \mathcal{C}^0$ be adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and set $\theta(t) = (\theta_1(t), \dots, \theta_N(t))$. We assume that the Novikov condition (4.20) is satisfied (with $\theta(t)^2 = \|\theta(t)\|^2 = \theta_1(t)^2 + \cdots + \theta_N(t)^2$). Then, as shown in Exercise 4.8, the stochastic process $\{Z(t)\}_{t\geq 0}$ given by

$$Z(t) = \exp\left(-\sum_{j=1}^{N}\int_0^t \theta_j(s)dW_j(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right)$$

is a martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$. It follows as before that the map $\widetilde{\mathbb{P}} : \mathcal{F} \to [0,1]$ given by

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z(T)\mathbb{I}_A], \quad A \in \mathcal{F} \tag{4.26}$$

is a new probability measure equivalent to $\mathbb{P}$ and the following N-dimensional generalization of Girsanov's theorem holds.

Theorem 4.12. *Define the stochastic processes* $\{\widetilde{W}_1(t)\}_{t\geq 0}, \dots, \{\widetilde{W}_N(t)\}_{t\geq 0}$ *by*

$$\widetilde{W}_k(t) = W_k(t) + \int_0^t \theta_i(s)ds, \quad k = 1, \dots, N. \tag{4.27}$$

Then $\{\widetilde{W}_1(t)\}_{t\geq 0}, \dots, \{\widetilde{W}_N(t)\}_{t\geq 0}$ *are independent Brownian motions in the probability measure* $\widetilde{\mathbb{P}}$. *Moreover the filtration* $\{\mathcal{F}_W(t)\}_{t\geq 0}$ *generated by* $\{W_1(t)\}_{t\geq 0}, \dots, \{W_N(t)\}_{t\geq 0}$ *is a non-anticipating filtration for* $\{\widetilde{W}_1(t)\}_{t\geq 0}, \dots, \{\widetilde{W}_N(t)\}_{t\geq 0}$.

4.6 Diffusion processes in financial mathematics

The purpose of this final section is to introduce some important examples of diffusion processes in financial mathematics. The analysis of the properties of such processes is the subject of Chapter 6.

Generalized geometric Brownian motion

Given two stochastic processes $\{\alpha(t)\}_{t\geq 0}, \{\sigma(t)\}_{t\geq 0} \in \mathcal{C}^0$, adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$, the stochastic process $\{S(t)\}_{t\geq 0}$ given by

$$S(t) = S(0)\exp\left(\int_0^t \alpha(s)ds + \int_0^t \sigma(s)dW(s)\right)$$

is called **generalized geometric Brownian motion** with **instantaneous mean of log-return** $\{\alpha(t)\}_{t\geq 0}$ and **instantaneous volatility** $\{\sigma(t)\}_{t\geq 0}$. When $\alpha(t) = \alpha \in \mathbb{R}$ and $\sigma(t) = \sigma > 0$ are deterministic constant, the process above reduces to the geometric Brownian motion, see (2.14). The generalized geometric Brownian motion provides a quite more general and realistic model for the dynamics of stock prices than the simple geometric Brownian motion. In the rest of these notes we assume that stock prices are modeled by geometric Brownian motions.

Since

$$S(t) = S(0)e^{X(t)}, \quad dX(t) = \alpha(t)dt + \sigma(t)dW(t),$$

then Itô's formula gives

$$\begin{aligned} dS(t) &= S(0)e^{X(t)}dX(t) + \frac{1}{2}S(0)e^{X(t)}dX(t)dX(t) \\ &= S(t)\alpha(t)dt + S(t)\sigma(t)dW(s) + \frac{1}{2}\sigma^2(t)S(t)dt \\ &= \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad \text{where } \mu(t) = \alpha(t) + \frac{1}{2}\sigma^2(t), \end{aligned}$$

hence a generalized geometric Brownian motion is a diffusion process in which the rate of quadratic variation and the drift depend on the process itself.

In the presence of several stocks, it is reasonable to assume that each of them introduced a new source of randomness in the market. Thus, when dealing with N stocks, we assume the existence of N Brownian motions $\{W_1(t)\}_{t\geq 0}, \dots \{W_N(t)\}_{t\geq 0}$, not necessarily independent, and model the evolution of the stocks prices $\{S_1(t)\}_{t\geq 0}\}, \dots, \{S_N(t)\}_{t\geq 0}$ by the following **N-dimensional generalized geometric Brownian motion**:

$$dS_k(t) = \left(\mu_k(t) + \sum_{j=1}^{N} \sigma_{kj}(t)dW_j(t)\right) S_k(t) \tag{4.28}$$

for some stochastic processes $\{\mu_k(t)\}_{t\geq 0}, \{\sigma_{kj}(t)\}_{t\geq 0} \in \mathcal{C}^0$, $j, k = 1, \dots, N$, adapted to the filtration generated by the Brownian motions.

Self-financing portfolios

Consider a portfolio $\{h_S(t), h_B(t)\}_{t\geq 0}$ invested in a 1+1-dimensional market. We assume that the price of the stock follows the generalized geometric Brownian motion

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \tag{4.29}$$

while the value of the risk-free asset is given by (2.15), i.e.,

$$dB(t) = B(t)R(t)dt. \tag{4.30}$$

Moreover we assume that the **market parameters** $\{\mu(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, $\{R(t)\}_{t\geq 0}$ have continuous paths a.s. and are adapted to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$. The value of the portfolio is given by

$$V(t) = h_S(t)S(t) + h_B(t)B(t). \tag{4.31}$$

We say that the portfolio is self-financing if purchasing more shares of one asset is possible only by selling shares of the other asset for an equivalent value (and not by infusing new cash into the portfolio), and, conversely, if any cash obtained by selling one asset is immediately re-invested to buy shares of the other asset (and not withdrawn from the portfolio). To translate this condition into a mathematical formula, assume that (h_S, h_B) is the investor position on the stock and the risk-free asset during the "infinitesimal" time interval $[t, t+\delta t)$. Let $V^-(t+\delta t)$ be the value of this portfolio immediately before the time $t+\delta t$ at which the position is changed, i.e.,

$$V^-(t+\delta t) = \lim_{u\to t+\delta t} h_S S(u) + h_B B(u) = h_S S(t+\delta t) + h_B B(t+\delta t),$$

where we used the continuity in time of the assets price. At the time $t+\delta t$, the investor sells/buys shares of the assets. Let (h'_S, h'_B) be the new position on the stock and the risk-free asset. Then the value of the portfolio at time $t+\delta t$ is given by

$$V(t+\delta t) = h'_S S(t+\delta t) + h'_B B(t+\delta t).$$

The difference $V(t+\delta t) - V^-(t+\delta t)$, if not zero, corresponds to cash withdrawn or added to the portfolio as a result of the change in the position on the assets. In a self-financing portfolio, however, this difference must be zero. We obtain

$$V(t+\delta t) - V^-(t+\delta t) = 0 \Leftrightarrow (h_S - h'_S)S(t+\delta t) + (h_B - h'_B)B(t+\delta t) = 0.$$

Hence, the change of the portfolio value in the interval $[t, t+\delta t]$ is given by

$$\delta V = V(t+\delta t) - V(t) = h'_S S(t+\delta t) + h'_B B(t+\delta t) - (h_S S(t) + h_B B(t)) = h_S \delta S + h_B \delta B,$$

where $\delta S = S(t+\delta t) - S(t)$, and $\delta B = B(t+\delta t) - B(t)$ are the changes of the assets value in the interval $[t, t+\delta t]$. This discussion leads to the following definition.

Definition 4.4. *A portfolio process $\{h_S(t), h_B(t)\}_{t\geq 0}$ invested in the $1+1$-dimensional market* (4.29)-(4.30) *is said to be* **self-financing** *if it is adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and if its value process $\{V(t)\}_{t\geq 0}$ satisfies*

$$dV(t) = h_S(t)dS(t) + h_B(t)dB(t). \tag{4.32}$$

We conclude with the important definition of hedging portfolio. Suppose that at time t a European derivative with pay-off Y at the time of maturity $T > t$ is sold for the price $\Pi_Y(t)$. An important problem in financial mathematics is to find a strategy for how the seller should invest the premium $\Pi_Y(t)$ of the derivative in order to **hedge** the derivative, i.e., to ensure that the portfolio value of the seller at time T is enough to pay-off the buyer of the derivative. We shall assume throughout that the seller can invest the premium of the derivative only on the 1+1 dimensional market consisting of the underlying stock and the risk-free asset.

Definition 4.5. *Consider the European derivative with pay-off Y and time of maturity T, where we assume that Y is $\mathcal{F}_W(T)$-measurable. A portfolio process $\{h_S(t), h_B(t)\}_{t\geq 0}$ invested in the underlying stock and the risk-free asset is said to be an* **hedging portfolio** *if* *(i)* $\{h_S(t), h_B(t)\}_{t\geq 0}$ *is adapted to* $\{\mathcal{F}_W(t)\}_{t\geq 0}$*;*

(ii) *The value of the portfolio satisfies* $V(T) = Y$*.*

In the next chapter we shall answer the following questions:

1) What is a reasonable "fair" price for the European derivative at time $t \in [0, T]$?

2) What investment strategy (on the underlying stock and the risk-free asset) should the seller undertake in order to hedge the derivative?

4.A Appendix: Solutions to selected problems

Exercise 4.4. Assume

$$X_i(t) = \alpha_i t + \sigma_i dW(t), \quad i = 1, 2,$$

for constants $\alpha_1, \alpha_2, \sigma_1, \sigma_2$. Then the right hand side of (4.15) is

$$\begin{aligned}
&\int_0^t [(\alpha_2 s + \sigma_2 W(s))\sigma_1 + (\alpha_1 s + \sigma_1 W(s))\sigma_2]dW(s) \\
&\quad + \int_0^t [(\alpha_2 s + \sigma_2 W(s))\alpha_1 + (\alpha_1 s + \sigma_1 W(s))\alpha_2 + \sigma_1\sigma_2]ds \\
&= \sigma_1 \int_0^t (\alpha_2 s + \sigma_2 W(s))dW(s) + \sigma_2 \int_0^t (\alpha_1 s + \sigma_1 W(s))dW(s) \\
&\quad + \alpha_1\alpha_2 \frac{t^2}{2} + \alpha_1\sigma_2 \int_0^t W(s)ds + \alpha_1\alpha_2 \frac{t^2}{2} + \sigma_1\alpha_2 \int_0^t W(s)ds + \sigma_1\sigma_2 t \\
&= \sigma_1\alpha_2 \int_0^t sdW(s) + 2\sigma_1\sigma_2 \int_0^t W(s)dW(s) + \sigma_2\alpha_1 \int_0^t sdW(s) \\
&\quad + \alpha_1\alpha_2 t^2 + \sigma_1\sigma_2 t + (\alpha_1\sigma_2 + \sigma_1\alpha_2) \int_0^t W(s)ds \\
&= 2\sigma_1\sigma_2 (\frac{W(t)^2}{2} - \frac{t}{2}) + \sigma_1\sigma_2 t + \alpha_1\alpha_2 t^2 + (\sigma_1\alpha_2 + \alpha_1\sigma_2)(\int_0^t sdW(s) + \int_0^t W(s)ds) \\
&= \sigma_1\sigma_2 W(t)^2 + \alpha_1\alpha_2 t^2 + (\sigma_1\alpha_2 + \alpha_1\sigma_2)tW(t) \\
&= (\alpha_1 t + \sigma_1 W(t))(\alpha_2 t + \sigma_2 W(t)) = X_1(t)X_2(t).
\end{aligned}$$

Exercise 4.5. We have

$$\mathrm{Cor}(W_1(t), W_2(t)) = \frac{\mathbb{E}[W_1(t)W_2(t)]}{\sqrt{\mathrm{Var}[W_1(t)]\mathrm{Var}[W_2(t)]}} = \frac{1}{t}\mathbb{E}[W_1(t)W_2(t)].$$

Hence we have to show that $\mathbb{E}[W_1(t)W_2(t)] = \rho t$. By Itô's product rule

$$d(W_1(t)W_2(t)) = W_1(t)dW_2(t) + W_2(t)dW_1(t) + dW_1(t)dW_2(t) = W_1(t)dW_2(t) + W_2(t)dW_1(t) + \rho dt.$$

Taking the expectation we find $\mathbb{E}[W_1(t)W_2(t)] = \rho t$, which concludes the first part of the exercise. As to the second part, the independent random variables $W_1(t), W_2(s)$ have the joint density

$$f_{W_1(t)W_2(s)}(x,y) = f_{W_1(t)}(x)f_{W_2(s)}(y) = \frac{1}{2\pi\sqrt{ts}}e^{-\frac{x^2}{2t}-\frac{y^2}{2s}}.$$

Hence

$$\mathbb{P}(W_1(t) > W_2(s)) = \frac{1}{2\pi\sqrt{ts}}\int_{x>y} e^{-\frac{x^2}{2t}-\frac{y^2}{2s}}\,dx\,dy = \frac{1}{2}.$$

Exercise 4.6. To solve the exercise we must prove that $dX(t) = \Gamma(t)dW(t)$, for some process $\{\Gamma(t)\}_{t\geq 0}$ adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$. In fact, by Itô's formula,

$$dX(t) = -3W(t)\,dt + (3W(t) - 3t)\,dW(t) + \frac{1}{2}6W(t)dW(t)dW(t) = 3(W(t)^2 - t)dW(t),$$

where in the last step we used that $dW(t)dW(t) = dt$.

Exercise 4.7. By Itô's formula, the stochastic process $\{Z(t)\}_{t\geq 0}$ satisfies $dZ(t) = -\theta(t)Z(t)dW(t)$, which is the claim.

Exercise 4.9. Since $\{I(t)\}_{t\geq 0}$ is a martingale, then $\mathbb{E}[I(t)] = \mathbb{E}[I(0)] = 0$. By Itô's isometry,

$$\mathrm{Var}[I(t)] = \mathbb{E}[Y(t)^2] = \mathbb{E}[\int_0^t A(s)^2\,ds] = \int_0^t A^2(s)\,ds,$$

since $A(t)$ is not random. To prove that $I(t)$ is normally distributed, it suffices to show that its characteristic function satisfies

$$\theta_{I(t)}(u) = e^{-\frac{u^2}{2}\int_0^t A(s)\,ds},$$

see Section 3.3. The latter is equivalent to

$$\mathbb{E}[e^{iuI(t)}] = e^{-\frac{u^2}{2}\int_0^t A(s)\,ds}, \quad \text{i.e.,} \quad \mathbb{E}[\exp(iuI(t) + \frac{u^2}{2}\int_0^t A(s)\,ds)] = 1.$$

Let $Z(t) = \exp(iuI(t) + \frac{u^2}{2}\int_0^t A(s)\,ds)$. If we show that $Z(t)$ is a martingale, we are done, because then $\mathbb{E}[Z(t)] = \mathbb{E}[Z(0)] = 1$. We write

$$Z(t) = e^{iX(t)+Y(t)} = f(X(t), Y(t)), \quad X(t) = u\int_0^t A(s)\,dW(s),\ Y(t) = \frac{u^2}{2}\int_0^t A^2(s)\,ds.$$

Then, by Theorem 4.9, using $dX(t)dY(t) = dY(t)dY(t) = 0$,

$$dZ(t) = ie^{iX(t)+Y(t)}dX(t) + e^{iX(t)+Y(t)}dY(t) - \frac{1}{2}e^{iX(t)+Y(t)}dX(t)dX(t) = iuZ(t)A(t)dW(t),$$

where we used that $dX(t)dX(t) = u^2A^2(t)dt$. Being an Itô's integral, the process $\{Z(t)\}_{t\geq 0}$ is a martingale, which completes the solution of the exercise.

Chapter 5

Stochastic differential equations and partial differential equations

Throughout this chapter, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given and $\{\mathcal{F}(t)\}_{t\geq 0}$ denotes a non-anticipating filtration for the given Brownian motion $\{W(t)\}_{t\geq 0}$. Given $T > 0$, we denote by $\mathcal{D}_T$ the open region in the (t, x)-plane given by

$$\mathcal{D}_T = \{t \in (0, T), x \in \mathbb{R}\} = (0, T) \times \mathbb{R}.$$

The closure and the boundary of $\mathcal{D}_T$ are given respectively by

$$\overline{\mathcal{D}_T} = [0, T] \times \mathbb{R}, \quad \partial\mathcal{D}_T = \{t = 0, x \in \mathbb{R}\} \cup \{t = T, x \in \mathbb{R}\}.$$

Similarly we denote $\mathcal{D}_T^+$ the open region

$$\mathcal{D}_T^+ = \{t \in (0, T), x > 0\} = (0, T) \times (0, \infty),$$

whose closure and boundary are given by

$$\overline{\mathcal{D}_T^+} = [0, T] \times [0, \infty), \quad \partial\mathcal{D}_T = \{t = 0, x \geq 0\} \cup \{t = T, x \geq 0\} \cup \{t \in [0, T], x = 0\}.$$

Moreover we shall employ the following notation for functions spaces.

- $C^k(\mathcal{D}_T)$ is the space of k-times continuously differentiable functions $u : \mathcal{D}_T \to \mathbb{R}$;
- $C^{1,2}(\mathcal{D}_T)$ is the space of functions $u \in C^1(\mathcal{D}_T)$ such that $\partial_x^2 u \in C(\mathcal{D}_T)$;
- $C^k(\overline{\mathcal{D}_T})$ is the space of functions $u \in C^k(\mathcal{D}_T)$ whose partial derivatives up to order k extend continuously on $\overline{\mathcal{D}_T}$. Similarly one defines $C^{1,2}(\overline{\mathcal{D}_T})$;
- $C_c^k(\mathbb{R}^n)$ is the space k-times continuously differentiable functions $u : \mathbb{R}^n \to \mathbb{R}$ with compact support. We also let $C_c^\infty(\mathbb{R}^n) = \cap_{k\in\mathbb{N}} C_c^k(\mathbb{R}^n)$

A continuous function u is **uniformly bounded** on $\mathcal{D}_T$ if there exists $C_T > 0$ such that $|u(t, x)| \leq C_T$, for all $(t, x) \in \mathcal{D}_T$. Unless otherwise stated, all functions are real-valued.

5.1 Stochastic differential equations

Definition 5.1. *Given $s \geq 0$, $\alpha, \beta \in C^0([s,\infty) \times \mathbb{R})$, and a deterministic constant $x \in \mathbb{R}$, we say that a stochastic process $\{X(t)\}_{t\geq s}$ is a global solution to the stochastic differential equation (SDE)*

$$dX(t) = \alpha(t, X(t))\, dt + \beta(t, X(t))\, dW(t) \tag{5.1}$$

with initial value $X(s,\omega) = x$ at time $t = s$, if $\{X(t)\}_{t\geq s} \in \mathcal{C}^0$ and

$$X(t) = x + \int_s^t \alpha(\tau, X(\tau))\, d\tau + \int_s^t \beta(\tau, X(\tau))\, dW(\tau), \quad t \geq s. \tag{5.2}$$

The initial value of a SDE can be a random variable instead of a deterministic constant, but we shall not need this more general case. Note also that the integrals in the right hand side of (5.2) are well-defined, as the integrand functions have continuous paths a.s. Of course one needs suitable assumptions on the functions α, β to ensure that there is a (unique) process $\{X(t)\}_{t\geq s}$ satisfying (5.2). The precise statement is contained in the following global existence and uniqueness theorem for SDE's, which is reminiscent of the analogous result for ordinary differential equations.

Theorem 5.1. *Assume that for each $T > s$ there exist constants $C_T, D_T > 0$ such that α, β satisfy*

$$|\alpha(t,x)| + |\beta(t,x)| \leq C_T(1 + |x|), \tag{5.3}$$

$$|\alpha(t,x) - \alpha(t,y)| + |\beta(t,x) - \beta(t,y)| \leq D_T|x - y|, \tag{5.4}$$

for all $t \in [s,T]$, $x, y \in \mathbb{R}$. Then there exists a unique global solution $\{X(t)\}_{t\geq s}$ of the SDE (5.1) with initial value $X(s) = x$. Moreover $\{X(t)\}_{t\geq s} \in \mathbb{L}^2$.

A proof of Theorem 5.1 can be found in [17, Theorem 5.2.1]. Note that the result proved in [17] is a bit more general than the one stated above, as it covers the case of a random initial value.

The solution of (5.1) with initial initial value x at time $t = s$ will be also denoted by $\{X(t; s, x)\}_{t\geq s}$. It can be shown that, under the assumptions of Theorem 5.1, the random variable $X(t; s, x)$ depends (a.s.) continuously on the initial conditions (s, x), see [1, Sec. 7.3].

Remark 5.1. The uniqueness statement in Theorem 5.1 is to be understood "up to null sets". Precisely, if $\{X_i(t)\}_{t\geq s}$, $i = 1, 2$ are two solutions with the same initial value x, then

$$\mathbb{P}(\sup_{t\in[s,T]} |X_1(t) - X_2(t)| > 0) = 0, \quad \text{for all } T > s.$$

Remark 5.2. If the assumptions of Theorem 5.1 are satisfied only up to a *fixed* time $T > 0$, then the solution of (5.1) could *explode* at some finite time in the future of T. For example, the stochastic process $\{X(t)\}_{0\leq t<T_*}$ given by $X(t) = \log(W(t) + e^x)$ solves (5.1) with $\alpha = -\exp(-2x)/2$ and $\beta = \exp(-x)$, but only up to the time $T_* = \inf\{t : W(t) = -e^x\} > 0$. In these notes we are only interested in global solutions of SDE's, hence we require (5.3)-(5.4) to hold for all $T > 0$.

Remark 5.3. The growth condition (5.3) alone is sufficient to prove the existence of a global solution to (5.1). The Lipschitz condition (5.4) is used to ensure uniqueness. By using a more general notion of solution (**weak solution**) and uniqueness (**pathwise uniqueness**), one can extend Theorem 5.1 to a larger class of SDE's, which include in particular the CIR process considered in Section 5.3; see [19] for details.

Exercise 5.1. *Within many applications in finance, the drift term $\alpha(t,x)$ is linear, an so it can be written in the form*

$$\alpha(t,x) = a(b-x), \quad a, b \text{ constant.} \tag{5.5}$$

A stochastic process $\{X(t)\}_{t\geq 0}$ is called **mean reverting** *if there exists a constant c such that $\mathbb{E}[X(t)] \to c$ as $t \to +\infty$. Most financial variables are required to satisfy the mean reversion property. Prove that the solution $\{X(t;s,x)\}_{t\geq 0}$ of (5.1) with linear drift (5.5) satisfies*

$$\mathbb{E}[X(t;s,x)] = xe^{-a(t-s)} + b(1-e^{-a(t-s)}). \tag{5.6}$$

Hence the process $\{X(t;s,x)\}_{t\geq 0}$ is mean reversing if and only if $a > 0$ and in this case the long time mean is given by $c = b$.

5.1.1 Linear SDE's

A SDE of the form

$$dX(t) = (a(t) + b(t)X(t))\,dt + (\gamma(t) + \sigma(t)X(t))\,dW(t), \quad X(s) = x, \tag{5.7}$$

where a, b, γ, σ are deterministic functions of time, is called a linear stochastic differential equation. We assume that for all $T > 0$ there exists a constant C_T such that

$$\sup_{t\in[s,T]} (|a(t)| + |b(t)| + |\gamma(t)| + |\sigma(t)|) < C_T,$$

and so by Theorem 5.1 there exists a unique global solution of (5.7). For example, the geometric Brownian motion (2.14) solves the linear SDE $dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)$, where $\mu = \alpha + \sigma^2/2$. Another example of linear SDE in finance is the Hull-White interest rate model, see Section 6.6. Linear SDE's can be solved explicitly, as shown in the following theorem.

Theorem 5.2. *The solution of (5.7) is given by $X(t) = Y(t)Z(t)$, where*

$$Z(t) = \exp\left(\int_s^t \sigma(\tau)dW(\tau) + \int_s^t (b(\tau) - \frac{\sigma(\tau)^2}{2})d\tau\right),$$

$$Y(t) = x + \int_s^t \frac{a(\tau) - \sigma(\tau)\gamma(\tau)}{Z(\tau)}\,d\tau + \int_s^t \frac{\gamma(\tau)}{Z(\tau)}\,dW(\tau).$$

Exercise 5.2 ($\star$)**.** *Proof Theorem 5.2.*

For example, in the special case in which the functions a, b, γ, σ are constant (independent of time), the solution of (5.7) with initial value $X(0) = x$ at time $t = 0$ is

$$X(t) = e^{\sigma W(t)+(b-\frac{\sigma^2}{2})t}\Big(x+(a-\gamma\sigma)\int_0^t e^{-\sigma W(\tau)-(b-\frac{\sigma^2}{2})\tau}d\tau+\gamma\int_0^t W(\tau)e^{-\sigma W(\tau)-(b-\frac{\sigma^2}{2})\tau}dW(\tau)\Big).$$

Exercise 5.3 (•). *Consider the linear SDE (5.7) with constant coefficients a, b, γ and $\sigma = 0$. Find the solution and show that $X(t; s, x) \in \mathcal{N}(m(t-s, x), \Delta(t-s)^2)$, where*

$$m(\tau, x) = xe^{-b\tau} + \frac{a}{b}(a - e^{-b\tau}), \quad \Delta(\tau)^2 = \frac{\gamma^2}{2b}(1 - e^{-2b\tau}). \tag{5.8}$$

Exercise 5.4 (•). *Find the solution $\{X(t)\}_{t\geq 0}$ of the linear SDE*

$$dX(t) = tX(t)\,dt + dW(t), \quad t \geq 0$$

with initial value $X(0) = 1$. Find $\mathrm{Cov}(X(s), X(t))$.

Exercise 5.5. *Compute $\mathrm{Cov}(W(t), X(t))$ and $\mathrm{Cov}(W^2(t), X(t))$, where $X(t) = X(t; s, x)$ is the stochastic process in Exercise 5.3.*

5.1.2 Markov property

It can be shown that, under the assumptions of Theorem 5.1, the solution $\{X(t; s, x)\}_{t\geq s}$ of (5.1) is a Markov process, see for instance [1, Th. 9.2.3]. Moreover when α, β in (5.1) are time-independent, $\{X(t; s, x)\}_{t\geq s}$ is a homogeneous Markov process. The fact that solutions of SDE's should satisfy the Markov property is quite intuitive, for, as shown in Theorem 5.1, the solution at time t is uniquely characterized by the initial value at time $s < t$. Consider for example the linear SDE

$$dX(t) = (a - bX(t))\,dt + \gamma dW(t), \quad t \geq s, \quad X(s) = x. \tag{5.9}$$

As shown in Exercise 5.5, the solution of (5.9) is given by

$$X(t; s, x) = xe^{b(s-t)} + \frac{a}{b}(1 - e^{b(s-t)}) + \int_s^t \gamma e^{b(u-t)}dW(u), \quad t \geq s,$$

and therefore $X(t; s, x) \in \mathcal{N}(m(t-s, x), \Delta(t-s)^2)$, where $m(\tau, x)$ and $\Delta(\tau)$ are given by (5.8). By Theorem 3.20, the transition density of the Markov process $\{X(t; s, x)\}_{t\geq 0}$ exists and is given by the pdf of the random variable $X(t; s, x)$, that is $p(t, s, x, y) = p_*(t - s, x, y)$, where

$$p_*(\tau, x, y) = e^{\frac{(y-m(\tau,x))^2}{2\Delta(\tau)^2}} \frac{1}{\sqrt{2\pi\Delta(\tau)^2}}.$$

The previous example rises the question of whether the Markov process solution of SDE's always admits a transition density. This problem is one of the subjects of Section 5.2.

5.1.3 Systems of SDE's

Occasionally in the next chapter we need to consider systems of several SDE's. All the results presented in this section extend *mutatis mutandis* to systems of SDE's, the difference being merely notational. For example, given two Brownian motions $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ and continuous functions $\alpha_1, \alpha_2, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22} : [s,\infty) \times \mathbb{R}^2 \to \mathbb{R}$, the relations

$$dX_i(t) = \alpha_i(t, X_1(t), X_2(t))\, dt + \sum_{j=1,2} \beta_{ij}(t, X_1(t), X_2(t)) dW_j(t), \tag{5.10a}$$

$$X_i(s) = x_i, \quad i = 1, 2 \tag{5.10b}$$

define a system of two SDE's on the stochastic processes $\{X_1(t)\}_{t\geq 0}$, $\{X_2(t)\}_{t\geq 0}$ with initial values $X(s) = x_1, X_2(s) = x_2$ at time s. As usual, the correct way to interpret the relations above is in the integral form:

$$X_i(t) = x_i + \int_s^t \alpha_i(\tau, X_1(\tau), X_2(\tau))\, d\tau + \sum_{j=1,2} \int_s^t \beta_{ij}(\tau, X_1(\tau), X_2(\tau)) dW_j(\tau) \quad i = 1, 2.$$

Upon defining the vector and matrix valued functions $\alpha = (\alpha_1, \alpha_2)^T$, $\beta = (\beta_{ij})_{i,j=1,2}$, and letting $X(t) = (X_1(t), X_2(t))$, $x = (x_1, x_2)$, $W(t) = (W_1(t), W_2(t))$, we can rewrite (5.10) as

$$dX(t) = \alpha(t, X(t))\, dt + \beta(t, X(t)) \cdot dW(t), \quad X(s) = x, \tag{5.11}$$

where $\cdot$ denotes the row by column matrix product. In fact, every system of any arbitrary number of SDE's can be written in the form (5.11). Theorem 5.1 continues to be valid for systems of SDE's, the only difference being that $|\alpha|$, $|\beta|$ in (5.3)-(5.4) stand now for the vector norm of α and for the matrix norm of β.

5.2 Partial differential equations

All financial variables are represented by stochastic processes solving (systems of) SDE's. In this context, a problem which recurs often is to find a function f such that the process $\{Y(t)\}_{t\geq 0}$, $Y(t) = f(t, X(t))$, is a martingale, where $\{X(t)\}_{t\geq 0}$ is the global solution of (5.1) with initial value $X(0) = x$. To this regard we have the following result.

Theorem 5.3. *Let $T > 0$ and $u \in C^{1,2}(\overline{\mathcal{D}_T})$. Assume that u satisfies*

$$\partial_t u + \alpha(t, x)\partial_x u + \frac{1}{2}\beta(t, x)^2 \partial_x^2 u = 0, \tag{5.12}$$

in the region $\mathcal{D}_T$. Assume also that α, β satisfy the conditions in Theorem 5.1 (with $s = 0$) and let $\{X(t)\}_{t\geq 0}$ be the unique global solution of (5.1) *with initial value $X(0) = x$. The stochastic process $\{u(t, X(t))\}_{t\in[0,T]}$ satisfies*

$$u(t, X(t)) = u(0, x) + \int_0^t \beta(t, X(t))\partial_x u(t, X(t))\, dW(t), \quad t \in [0, T]. \tag{5.13}$$

Moreover if $\partial_x u$ is uniformly bounded on $\mathcal{D}_T$, then the stochastic process $\{u(t,X(t))\}_{t\in[0,T]}$ is a martingale relative to the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$.

Proof. By Itô's formula we find

$$du(t,X(t)) = (\partial_t u + \alpha\partial_x u + \frac{\beta^2}{2}\partial_x^2 u)(t,X(t))\,dt + (\beta\partial_x u)(t,X(t))\,dW(t).$$

As u solves (5.12), then $du(t,X(t)) = (\beta\partial_x u)(t,X(t))\,dW(t)$, which is equivalent to (5.13) (as $u(0,X(0)) = u(0,x)$). Under the additional assumption that $\partial_x u$ is uniformly bounded on $\mathcal{D}_T$, there exists a constant $C_T > 0$ such that $|\partial_x u(t,x)| \leq C_T$ and so, due also to (5.3), the Itô integral in the right hand side of (5.13) is a martingale. This concludes the proof of the theorem. □

Definition 5.2. *The partial differential equation (PDE)* (5.12) *is called the* **(backward) Kolmogorov equation** *associated to the SDE* (5.1)*. We say that $u : \mathcal{D}_T \to \mathbb{R}$ is a* **strong solution** *of* (5.12) *in the region $\mathcal{D}_T$ if $u \in C^{1,2}(\overline{\mathcal{D}_T})$, u solves* (5.12) *for all $(t,x) \in \mathcal{D}_T$, and $\partial_x u(t,x)$ is uniformly bounded on $\mathcal{D}_T$. Similarly, replacing $\mathcal{D}_T$ with $\mathcal{D}_T^+$, one defines strong solutions of* (5.12) *in the region $\mathcal{D}_T^+$*

Note carefully that for a strong solution u of the Kolmogorov equation in the region $\mathcal{D}_T^+$, we require that u, $\partial_t u$, $\partial_x u$ and $\partial_x^2 u$ extend continuously on the axis $x = 0$. This assumption can be weakened, but we shall not do so. The statement of Theorem 5.3 rises the question of whether there exist strong solutions to the Kolmogorov PDE. This important problem is solved in the following theorem.

Theorem 5.4. *Assume that the hypothesis of Theorem 5.1 are satisfied and in addition assume that $\alpha, \beta \in C^2(\overline{\mathcal{D}_T})$ such that $\partial_x^i\alpha, \partial_x^i\beta$ are uniformly bounded on $\mathcal{D}_T$, for $i = 1, 2$ and for all $T > 0$. Let $g \in C^2(\mathbb{R})$ such that g' and g'' are uniformly bounded on $\mathbb{R}$. Define the function*

$$u_T(t,x) = \mathbb{E}[g(X(T;t,x))], \quad 0 \leq t < T. \tag{5.14}$$

Then the following holds.

(i) u_T is a strong solution of the Kolmogorov PDE

$$\partial_t u + \alpha(t,x)\partial_x u + \frac{1}{2}\beta(t,x)^2\partial_x^2 u = 0, \quad (t,x) \in \mathcal{D}_T, \tag{5.15}$$

with the **terminal condition**

$$\lim_{t\to T} u(t,x) = g(x), \quad \textit{for all } x \in \mathbb{R}. \tag{5.16}$$

(ii) The solution is unique in the following sense: if v is another strong solution and $\lim_{t\to T} v(t,x) = g(x)$, then $v = u_T$ in $\overline{\mathcal{D}}_T$.

Proof. For (i) see [17, Theorem 8.1.1]. We prove only (ii). Let v be a solution as stated in the theorem and set $Y(\tau) = v(\tau, X(\tau; t, x))$, for $t \leq \tau \leq T$. By Itô's formula and using that v solves (5.15) we find $dY(\tau) = \beta\partial_x v(\tau, X(\tau; t, x))dW(\tau)$. Hence

$$v(T, X(T; t, x)) - v(t, X(t; t, x)) = \int_t^T \beta\partial_x v(\tau, X(\tau; t, x))dW(\tau).$$

Moreover $v(T, X(T; t, x)) = g(X(T; t, x))$, $v(t, X(t; t, x)) = v(t, x)$ and in addition, by (5.3) and the fact that of $\partial_x v$ is uniformly bounded, the Itô integral in the right hand side is a martingale. Hence taking the expectation we find $v(t, x) = \mathbb{E}[g(T, X(T; t, x))] = u(t, x)$. □

Remark 5.4. It is often convenient to study the Kolmogorov PDE with an initial, rather than terminal, condition. To this purpose it suffices to make the change of variable $t \to T-t$ in (5.15). Letting $\bar{u}(t, x) = u(T - t, x)$, we now see that $\bar{u}$ satisfies the PDE

$$-\partial_t\bar{u} + \alpha(T - t, x)\partial_x\bar{u} + \frac{1}{2}\beta(T - t, x)^2\partial_x^2\bar{u} = 0, \quad 0 \leq t \leq T \tag{5.17}$$

with initial condition $\bar{u}(0, x) = g(x)$. Note that this is the equation considered in [17, Theorem 8.1.1]

Remark 5.5. Observe that in Theorem 5.4 we have a different solution for each fixed T. As $\partial_x u_T$ is uniformly bounded, Theorem 5.3 gives that the stochastic process $\{u_T(t, X(t))\}_{t\in[0,T]}$ is a martingale. Equation (5.14) is also called **Dynkin's formula** (which is a special case of the **Feynman-Kac** formula.)

Remark 5.6. It is possible to define other concepts of solution to the Kolmogorov PDE other than the strong one, e.g., weak solution, entropy solution, etc. In general these solutions are not uniquely characterized by their terminal value. In these notes we only consider strong solutions, which, as proved in Theorem 5.4, are uniquely determined by (5.16).

Exercise 5.6. *Consider the Kolmogorov PDE associated to the linear SDE* (5.7) *with constant coefficients and* $\sigma = 0$*:*

$$\partial_t u + (a - bx)\partial_x u + \frac{1}{2}\gamma\partial_x^2 u = 0, \quad (t, x) \in \mathcal{D}_T^+. \tag{5.18}$$

Find the strong solution of (5.18) *that satisfies* $u(T, x) = 1$. *HINT: Use the ansatz* $f(t, x) = e^{-xA(t,T)+B(t,T)}$.

The study of the Kolmogorov equation is also important to establish whether the solution of a SDE admits a transition density. In fact, it can be shown that when $\{X(t)\}_{t\geq s}$ admits a smooth transition density, then the latter coincides with the **fundamental solution** of the Kolmogorov equation. To state the precise result, let us denote by $\delta(x - y)$ the **δ-distribution** centered in $y \in \mathbb{R}$, i.e., the distribution satisfying

$$\int_{\mathbb{R}} \psi(x)\delta(x - y)\, dx = \psi(y), \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}).$$

A sequence of measurable functions $(g_n)_{n\in\mathbb{N}}$ is said to converge to $\delta(x-y)$ in the sense of distributions if

$$\lim_{n\to\infty}\int_{\mathbb{R}} g_n(x)\psi(x)\,dx \to \psi(y), \quad \text{as } n\to\infty, \text{ for all } \psi\in C_c^\infty(\mathbb{R}).$$

Theorem 5.5. *Assume the conditions in Theorem 5.1 are satisfied. Let $\{X(t;s,x)\}_{t\geq s}$ be the global solution of* (5.1) *with initial value $X(s)=x$; recall that this solution is a Markov stochastic process.*

(i) If $\{X(t;s,x)\}_{t\geq s}$ admits a transition density $p(t,s,x,y)$ which is C^1 in the variable s and C^2 in the variable x, then $p(t,s,x,y)$ solves the Kolmogorov PDE

$$\partial_s p + \alpha(s,x)\partial_x p + \frac{1}{2}\beta(s,x)^2\partial_x^2 p = 0, \quad 0<s<t,\ x\in\mathbb{R}, \tag{5.19}$$

with terminal value

$$\lim_{s\to t} p(t,s,x,y) = \delta(x-y). \tag{5.20}$$

(ii) If $\{X(t;s,x)\}_{t\geq 0}$ admits a transition density $p(t,s,x,y)$ which is C^1 in the variable t and C^2 in the variable y, then $p(t,s,x,y)$ solves the **Fokker-Planck** *PDE*[1]

$$\partial_t p + \partial_y(\alpha(t,y)p) - \frac{1}{2}\partial_y^2(\beta(t,y)^2 p) = 0, \quad t>s,\ x\in\mathbb{R}, \tag{5.21}$$

with initial value

$$\lim_{t\to s} p(t,s,x,y) = \delta(x-y). \tag{5.22}$$

Exercise 5.7. *Prove Theorem 5.5. HINT: See Exercises 6.8 and 6.9 in [21].*

Remark 5.7. The solution p of the problem (5.19)-(5.20) is called the **fundamental solution** for the Kolmogorov PDE, as any other solution can be reconstructed from it. For example for all functions g as in Theorem 5.4, the solution of (5.15) with the terminal condition (5.16) is given by

$$u_T(t,x) = \int_{\mathbb{R}} p(T,t,x,y)g(y)\,dy.$$

This can be verified either by a direct calculation or by using the interpretation of the fundamental solution as transition density. Similarly, p is the fundamental solution of the Fokker-Planck equation

Let us see an example of application of Theorem 5.5. First notice that when the functions α,β in (5.1) are time-independent, then the Markovian stochastic process $\{X(t;s,x)\}_{t\geq s}$ is homogeneous and therefore the transition density, when it exists, has the form $p(t,s,x,y) =$

[1]Also known as forward Kolmogorov PDE.

$p_*(t-s,x,y)$. By the change of variable $s \to t-s=\tau$ in (5.19), we find that $p_*(\tau,x,y)$ satisfies

$$-\partial_\tau p_* + \alpha(x)\partial_x p_* + \frac{1}{2}\sigma(x)^2\partial_x^2 p_* = 0, \tag{5.23}$$

as well as

$$\partial_\tau p_* + \partial_y(\alpha(y)p_*) - \frac{1}{2}\partial_y^2(\sigma(y)^2 p_*) = 0, \tag{5.24}$$

with the initial condition $p_*(0,x,y) = \delta(x-y)$. For example the Brownian motion is a Markov process with transition density (3.28). In this case, (5.23) and (5.24) both reduce to the heat equation $-\partial_\tau p_* + \frac{1}{2}\partial_x^2 p_* = 0$. It is straightforward to verify that (3.28) satisfies the heat equation for $(\tau,x) \in (0,\infty)\times\mathbb{R}$. Now we show that, as claimed in Theorem 5.5, the initial condition $p_*(0,x,y) = \delta(x-y)$ is also verified, that is

$$\lim_{\tau\to 0}\int_{\mathbb{R}} p_*(\tau,x,y)\psi(y)\,dy = \psi(x), \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}) \text{ and } x \in \mathbb{R}.$$

Indeed with the change of variable $y = x + \sqrt{\tau}z$, we have

$$\int_{\mathbb{R}} p_*(\tau,x,y)\psi(y)\,dy = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{z^2}{2}}\psi(x+\sqrt{\tau}z)\,dz \to \psi(0)\int_{\mathbb{R}} e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}} = \psi(0),$$

as claimed. Moreover, as $W(0) = 0$ a.s., Theorem 5.5 entails that the density of the Brownian motion is $f_{W(t)}(y) = p_*(t,0,y) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}$, which is of course correct.

Exercise 5.8. *Show that the transition density derived in the example at the end of Section 5.1 is the fundamental solution of the Kolmogorov equation for the linear SDE* (5.9)*.*

5.3 The CIR process

A **CIR process** is a stochastic process $\{X(t)\}_{t\geq s}$ satisfying the SDE

$$dX(t) = a(b - X(t))\,dt + c\sqrt{X(t)}\,dW(t), \quad X(s) = x > 0, \tag{5.25}$$

where a, b, c are constant ($c \neq 0$). CIR processes are used in finance to model the stock volatility in the Heston model (see Section 6.5.2) and the spot interest rate of bonds in the CIR model (see Section 6.6). Note that the SDE (5.25) is not of the form considered so far, as the function $\beta(t,x) = c\sqrt{x}$ is defined only for $x \geq 0$ and, more importantly, it is not Lipschitz continuous in a neighborhood of $x = 0$ as required in Theorem 5.1. Nevertheless, as already mentioned in Remark 5.3, it can be shown that (5.25) admits a unique global solution for all $x > 0$. Clearly the solution satisfies $X(t) \geq 0$ a.s., for all $t \geq 0$, otherwise the Itô integral in the right hand side of (5.25) would not even be defined. For future applications, it is important to know whether the solution can hit zero in finite time with positive probability. This question is answered in the following theorem, whose proof can be found for instance in [15, Exercise 37]).

Theorem 5.6. *Let $\{X(t)\}_{t\geq 0}$ be the CIR process with initial value $X(0) = x > 0$ at time $t = 0$. Define the (stopping*[2]*) time*

$$\tau_0^x = \inf\{t \geq 0 : X(t) = 0\}.$$

Then $\mathbb{P}(\tau_0^x < \infty) = 0$ if and only if $ab \geq c^2/2$, which is called **Feller's condition***.*

Exercise 5.9. *Prove Theorem 5.6 following the hints in [15, Exercise 37].*

The following theorem shows how to build a CIR process from a family of linear SDE's.

Theorem 5.7. *Let $\{W_1(t)\}_{t\geq 0}, \dots \{W_N(t)\}_{t\geq 0}$ be $N \geq 2$ independent Brownian motions and assume that $\{X_1(t)\}_{t\geq 0}, \dots, \{X_N(t)\}_{t\geq 0}$ solve*

$$dX_j(t) = -\frac{\theta}{2} X_j(t)\, dt + \frac{\sigma}{2}\, dW_j(t), \quad j = 1, \dots, N, \quad X_j(0) = x_j \in \mathbb{R}, \tag{5.26}$$

where θ, σ are deterministic constant. There exists a Brownian motion $\{W(t)\}_{t\geq 0}$ such that the stochastic process $\{X(t)\}_{t\geq 0}$ given by

$$X(t) = \sum_{j=1}^{N} X_j(t)^2$$

solves (5.25) *with $a = \theta$, $c = \sigma$ and $b = \frac{N\sigma^2}{4\theta}$.*

Proof. Let $X(t) = \sum_{j=1}^{N} X_j(t)^2$. Applying Itô's formula we find, after straightforward calculations,

$$dX(t) = (\frac{N\sigma^2}{4} - \theta X(t))\, dt + \sigma \sum_{j=1}^{N} X_j(t)\, dW_j(t).$$

Letting $a = \theta$, $c = \sigma$, $b = \frac{N\sigma^2}{4\theta}$ and

$$dW(t) = \sum_{j=1}^{N} \frac{X_j(t)}{\sqrt{X(t)}}\, dW_j(t),$$

we obtain that $X(t)$ satisfies

$$dX(t) = a(b - X(t))\, dt + c\sqrt{X(t)}\, dW(t).$$

Thus $\{X(t)\}_{t\geq 0}$ is a CIR process, provided we prove that $\{W(t)\}_{t\geq 0}$ is a Brownian motion. Clearly, $W(0) = 0$ a.s. and the paths $t \to W(t, \omega)$ are a.s. continuous. Hence to conclude that $\{W(t)\}_{t\geq 0}$ is a Brownian motion we must show that $dW(t)dW(t) = dt$, see

[2] See Definition 6.9 for the general definition of stopping time.

Theorem 3.18. We have

$$\begin{aligned} dW(t)dW(t) &= \frac{1}{X(t)}\sum_{i,j=1}^{N} X_i(t)X_j(t)dW_i(t)dW_j(t) = \frac{1}{X(t)}\sum_{i,j=1}^{N} X_i(t)X_j(t)\delta_{ij}dt \\ &= \frac{1}{X(t)}\sum_{j=1}^{N} X_j^2(t)dt = dt, \end{aligned}$$

where we used that $dW_i(t)dW_j(t) = \delta_{ij}dt$, since the Brownian motions are independent. □

Note that $N \geq 2$ implies the Feller condition $ab \geq c^2/2$, hence the CIR process constructed in the previous theorem does not hit zero, see Theorem 5.6. Note also that the solution of (5.26) is

$$X_j(t) = e^{-\frac{1}{2}\theta t}\left(x_j + \frac{\sigma}{2}\int_0^t e^{\frac{1}{2}\theta\tau}dW_j(\tau)\right).$$

It follows by Exercise 4.9 that the random variables $X_1(t), \dots, X_N(t)$ are normally distributed with

$$\mathbb{E}[X_j(t)] = e^{\frac{1}{2}\theta t}x_j, \quad \mathrm{Var}[X_j(t)] = \frac{\sigma^2}{4\theta}(1 - e^{-\frac{1}{2}\theta t}).$$

It follows by Exercise 3.17 that the CIR process constructed Theorem 5.7 is non-central χ^2 distributed. The following theorem shows that this is a general property of CIR processes.

Theorem 5.8. *Assume $ab > 0$. The CIR process starting at $x > 0$ at time $t = s$ satisfies*

$$X(t; s, x) = \frac{1}{2k}Y, \quad Y \in \chi^2(\delta, \beta),$$

where

$$k = \frac{2a}{(1 - e^{-a(t-s)})c^2}, \quad \delta = \frac{4ab}{c^2}, \quad \beta = 2kxe^{-a(t-s)}.$$

Sketch of the proof. As the CIR process is a homogeneous Markov process, it is enough to prove the claim for $s = 0$. Let $X(t) = X(t; 0, x)$ for short and denote $p(t, 0, x, y) = p_*(t, x, y)$ the density of $X(t)$. By Theorem 5.5, p_* solves the Fokker-Planck equation

$$\partial_t p_* + \partial_y(a(b - y)p_*) - \frac{1}{2}\partial_y^2(c^2 y\, p_*) = 0, \tag{5.27}$$

with initial datum $p_*(0, x, y) = \delta(x - y)$. Moreover, the characteristic function $\theta_{X(t)}(u) := h(t, u)$ of $X(t)$ is given by

$$h(t, u) = \mathbb{E}[e^{iuX(t)}] = \int_{\mathbb{R}} e^{iuy}p_*(t, x, y)\, dy,$$

and after straightforward calculations we derive the following equation on h

$$\partial_\tau h - iabuh + (au - \frac{c^2}{2}iu^2)\partial_u h = 0. \tag{5.28a}$$

The initial condition for equation (5.28a) is

$$h(0, y) = e^{ix}, \tag{5.28b}$$

which is equivalent to $p_*(0, x, y) = \delta(x - y)$. The initial value problem (5.28) can be solved with the **method of characteristics** (see [8] for an illustration of this method) and one finds that the solution is given by

$$h(t, u) = \theta_{X(t)}(u) = \frac{\exp\left(-\frac{\beta u}{2(u+ik)}\right)}{(1 - iu/k)^{\delta/2}}. \tag{5.29}$$

Hence $\theta_{X(t)}(u) = \theta_Y(\frac{u}{2k})$, where $\theta_Y(u)$ is the characteristic function of $Y \in \chi^2(\delta, \beta)$, see Table 3.1. This completes the proof. □

Exercise 5.10. *Derive* (5.28a) *and verify with Mathematica that* (5.29) *is the solution of the initial value problem* (5.28).

Finally we discuss briefly the question of existence of strong solutions to the Kolmogorov equation for the CIR process, which is

$$\partial_t u + a(b - x)\partial_x u + \frac{c^2}{2} x \partial_x^2 u = 0, \quad (t, x) \in \mathcal{D}_T^+, \quad u(T, x) = g(x). \tag{5.30}$$

Note carefully that the Kolmogorov PDE is now defined only for $x > 0$, as the initial value x in (5.25) must be positive. Now, if a strong solution of (5.30) exists, then it must be given by $u(t, x) = \mathbb{E}[g(X(T; t, x))]$ (this claim is proved exactly as in Theorem 5.4(iii)). Supposing $ab > 0$, then

$$u(t, x) = \mathbb{E}[g(X(T; t, x))] = \int_0^\infty p_*(T - t, x, y) g(y)\, dy,$$

where the density of $X(T; t, x)$ is given as in Theorem 5.8. Using the asymptotic behavior of p_* as $x \to 0^+$, it can be shown $u(t, x)$ is bounded near the axis $x = 0$ only if the Feller condition $ab \geq c^2/2$ is satisfied and in this case $\partial_t u, \partial_x u, \partial_x^2 u$ are also bounded. Hence u is the (unique) strong solution of (5.30) if and only if $ab \geq c^2/2$.

5.4 Finite different solutions of PDE's

The finite difference methods are techniques to find (numerically) approximate solutions to ordinary differential equations (ODEs) and partial differential equations (PDEs). They are based on the idea to replace the ordinary/partial derivatives with a finite difference quotient, e.g., $y'(x) \approx (y(x + h) - y(x))/h$. The various methods differ by the choice of the finite difference used in the approximation. We shall present a number of methods by examples. As this suffices for our future applications, we consider only linear equations.

5.4.1 ODEs

Consider the first order ODE

$$\frac{dy}{dt} = ay + bt, \quad y(0) = y_0, \quad t \in [0, T], \tag{5.31}$$

for some constants $a, b \in \mathbb{R}$ and $T > 0$. The solution is given by

$$y(t) = y_0 e^{at} + \frac{b}{a^2}(e^{at} - at - 1). \tag{5.32}$$

We shall apply three different finite difference methods to approximate the solution of (5.31). In all cases we divide the time interval $[0, T]$ into a uniform partition,

$$0 = t_0 < t_1 < \cdots < t_n = T, \quad t_j = j\frac{T}{n}, \ \Delta t = t_{j+1} - t_j = \frac{T}{n}$$

and define

$$y(t_j) = y_j, \quad j = 0, \ldots, n.$$

Forward Euler method

In this method we introduce the following approximation of dy/dt at time t:

$$\frac{dy}{dt}(t) = \frac{y(t + \Delta t) - y(t)}{\Delta t} + O(\Delta t),$$

i.e.,

$$y(t + \Delta t) = y(t) + \frac{dy}{dt}(t)\Delta t + O(\Delta t^2). \tag{5.33}$$

For Equation (5.31) this becomes

$$y(t + \Delta t) = y(t) + (ay(t) + bt)\Delta t + O(\Delta t^2).$$

Setting $t = t_j$, $\Delta T = T/n$, $t + \Delta t = t_j + T/n = t_{j+1}$ and neglecting second order terms we obtain

$$y_{j+1} = y_j + (ay_j + bt_j)\frac{T}{n}, \quad j = 0, \ldots, n - 1. \tag{5.34}$$

As y_0 is known, the previous iterative equation can be solved at any step j. This method is called *explicit*, because the solution at the step $j+1$ is given explicitly in terms of the solution at the step j. It is a simple matter to implement this method numerically, for instance using the following Matlab function:[3]

```
function [time,sol]=exampleODEexp(T,y0,n)
dt=T/n;
```

[3]The Matlab codes presented in this text are not optimized. Moreover the powerful vectorization tools of Matlab are not employed, so as to make the codes easily adaptable to other computer softwares and languages.

```
sol=zeros(1,n+1);
time=zeros(1,n+1);
a=1; b=1;
sol(1)=y0;
for j=2:n+1
sol(j)=sol(j-1)+(a*sol(j-1)+b*time(j-1))*dt;
time(j)=time(j-1)+dt;
end
```

Exercise 5.11. *Compare the approximate solution with the exact solution for increasing values of n. Compile a table showing the difference between the approximate solution and the exact solution at time T for increasing value of n.*

Backward Euler method

This method consists in approximating dy/dt at time t as

$$\frac{dy}{dt}(t) = \frac{y(t) - y(t - \Delta t)}{\Delta t} + O(\Delta t),$$

hence

$$y(t + \Delta t) = y(t) + \frac{dy}{dt}(t + \Delta t)\Delta t + O(\Delta t^2). \tag{5.35}$$

The iterative equation for (5.31) now is

$$y_{j+1} = y_j + (ay_{j+1} + bt_{j+1})\frac{T}{n}, \quad j = 0, \ldots, n-1. \tag{5.36}$$

This method is called implicit, because the solution at the step $j+1$ depends on the solution at both the step j *and* the step $j+1$ itself. Therefore implicit methods involve an extra computation, which is to find y_{j+1} in terms of y_j only. For the present example this is a trivial step, as we have

$$y_{j+1} = \big(1 - \frac{aT}{n}\big)^{-1}\big(y_j + bt_{j+1}\frac{T}{n}\big), \tag{5.37}$$

provided $n \neq aT$. Here is a Matlab function implementing the backward Euler method for the ODE (5.31):

```
function [time,sol]=exampleODEimp(T,y0,n)
dt=T/n;
sol=zeros(1,n+1);
time=zeros(1,n+1);
a=1; b=1;
sol(1)=y0;
for j=2:n+1
time(j)=time(j-1)+dt;
```

```
sol(j)=1/(1-a*dt)*(sol(j-1)+b*time(j)*dt);
end
```

Exercise 5.12. *Compare the approximate solution obtained with the backward Euler method with the exact solution and the approximate one obtained via the forward Euler method. Compile a table for increasing values of* n *as in Exercise 1.*

Central difference method

By a Taylor expansion,

$$y(t+\Delta) = y(t) + \frac{dy}{dt}(t)\Delta t + \frac{1}{2}\frac{d^2y}{dt^2}(t)\Delta t^2 + O(\Delta t^3), \tag{5.38}$$

and replacing Δt with $-\Delta t$,

$$y(t-\Delta) = y(t) - \frac{dy}{dt}(t)\Delta t + \frac{1}{2}\frac{d^2y}{dt^2}(t)\Delta t^2 + O(\Delta t^3). \tag{5.39}$$

Subtracting the two equations we obtain the following approximation for dy/dt at time t:

$$\frac{dy}{dt}(t) = \frac{y(t+\Delta t) - y(t-\Delta)}{2\Delta t} + O(\Delta t^2),$$

which is called central difference approximation. Hence

$$y(t+\Delta t) = y(t-\Delta t) + 2\frac{dy}{dt}(t)\Delta t + O(\Delta t^3). \tag{5.40}$$

Note that, compared to (5.33) and (5.35), we have gained one order in accuracy. The iterative equation for (5.31) becomes

$$y_{j+1} = y_{j-1} - 2(ay_j + bt_j)\frac{T}{n}, \quad j = 0, \ldots, n-1. \tag{5.41}$$

Note that the first step $j = 0$ requires y_{-1}. This is fixed by the backward method

$$y_{-1} = y_0 - \frac{T}{n}ay_0, \tag{5.42}$$

which is (5.36) for $j = -1$.

Exercise 5.13. *Write a Matlab function that implements the central difference method for* (5.31)*. Compile a table comparing the exact solution with the approximate solutions at time* T *obtained by the three methods presented above for increasing value of* n*.*

A second order ODE

Consider the second order ODE for the harmonic oscillator:

$$\frac{d^2y}{dt^2} = -\omega^2 y, \quad y(0) = y_0, \quad \dot{y}(0) = \widetilde{y}_0. \tag{5.43}$$

The solution to this problem is given by

$$y(t) = y_0 \cos(\omega t) + \frac{\widetilde{y}_0}{\omega} \sin(\omega t). \tag{5.44}$$

One can define forward/backward/central difference approximations for second derivatives in a way similar as for first derivatives. For instance, adding (5.38) and (5.39) we obtain the following central difference approximation for d^2y/dt^2 at time t:

$$\frac{d^2y}{dt^2}(t) = \frac{y(t+\Delta t) - 2y(t) + y(t-\Delta t)}{\Delta t^2} + O(\Delta t),$$

which leads to the following iterative equation for (5.43):

$$y_{j+1} = 2y_j - y_{j-1} - \left(\frac{T}{n}\right)^2 \omega^2 y_j, \quad j = 1, \ldots, n-1, \tag{5.45}$$

$$y_1 = y_0 + \widetilde{y}_0 \frac{T}{n}. \tag{5.46}$$

Note the approximate solution at the first node is computed using the forward method and the initial datum $\dot{y}(0) = \widetilde{y}_0$. The Matlab function solving this iteration is the following.

```
function [time,sol]=harmonic(w,T,y0,N)
dt=T/N;
sol=zeros(1,N+1);
time=zeros(1,N+1);
sol(1)=y0(1);
sol(2)=sol(1)+y0(2)*dt;
for j=3:N+1
sol(j)=2*sol(j-1)-sol(j-2)-dt^2*w^2*sol(j-1);
time(j)=time(j-1)+dt;
end
```

Exercise 5.14. *Compare the exact and approximate solutions at time T for increasing values of n.*

5.4.2 PDEs

In this section we present three finite difference methods to find approximate solutions to the one-dimensional heat equation

$$\partial_t u = \partial_x^2 u, \quad u(0,x) = u_0(x), \tag{5.47}$$

where u_0 is continuous. We refer to t as the time variable and to x as the spatial variable, since this is what they typically represent in the applications of the heat equation. As before, we let $t \in [0,T]$. As to the domain of the spatial variable x, we distinguish two cases

(i) x runs over the whole real line, i.e., $x \in (-\infty,\infty)$, and we are interested in finding an approximation to the solution $u \in C_b^{1,2}(\mathcal{D}_T)$.

(ii) x runs over a finite interval, say $x \in (x_{\min}, x_{\max})$, and we want to find an approximation of the solution $u \in C_b^{1,2}((0,T)\times(x_{\min},x_{\max}))$ which satisfies the boundary conditions[4]

$$u(t,x_{\min}) = u_L(t), \quad u(t,x_{\max}) = u_R(t), \quad t \in [0,T],$$

for some given continuous functions u_L, u_R. We also require $u_L(0) = u_0(x_{\min})$, $u_R(0) = u_0(x_{\max})$, so that the solution is continuous on the boundary.

In fact, for numerical purposes, problem (i) is a special case of problem (ii), for the domain $(-\infty,\infty)$ must be approximated by $(-A,A)$ for $A >> 1$ when we solve problem (i) in a computer. Note however that in the finite domain approximation of problem (i), *the boundary conditions at $x = \pm A$ cannot be prescribed freely!* Rather they have to be given by suitable approximations of the limit values at $x = \pm\infty$ of the solution to the heat equation on the real line.

By what we have just said we can focus on problem (ii). To simplify the discussion we assume that the domain of the x variable is given by $x \in (0,X)$ and we assign zero boundary conditions, i.e., $u_L = u_R = 0$. Hence we want to study the problem

$$\partial_t u = \partial_x^2 u, \quad (t,x) \in (0,T)\times(0,X), \tag{5.48a}$$

$$u(0,x) = u_0(x), \quad u(t,0) = u(t,X) = 0, \quad x \in [0,X],\ t \in [0,T]; \quad u_0(0) = u_0(X) = 0. \tag{5.48b}$$

We introduce the partition of the interval $(0,X)$ given by

$$0 = x_0 < x_1 < \cdots < x_m = X, \quad x_j = j\frac{X}{m}, \quad \Delta x = x_{j+1} - x_j = \frac{X}{m},$$

and the partition of the time interval $[0,T]$ given by

$$0 = t_0 < t_1 < \cdots < t_n = T, \quad t_i = i\frac{T}{m}, \quad \Delta t = t_{i+1} - t_i = \frac{T}{n},$$

We let

$$u_{i,j} = u(t_i, x_j), \quad i = 0,\dots,n, \quad j = 0,\dots,m.$$

Hence $u_{i,j}$ is a $n+1 \times m+1$ matrix. The i^{th} row contains the value of the approximate solution at each point of the spatial mesh at the fixed time t_i. For instance, the zeroth row

[4] These are called Dirichlet type boundary conditions. Other types of boundary conditions can be imposed, but the Dirichlet type is sufficient for our applications to the Black-Scholes PDE.

is the initial datum: $u_{0,j} = u_0(x_j)$, $i = 0, \dots m$. The columns of the matrix $u_{i,j}$ contain the values of the approximate solution at one spatial point for different times. For instance, the column $u_{i,0}$ are the values of the approximate solution at $x_0 = 0$ for different times t_i, while $u_{i,m}$ contains the values at $x_m = X$. By the given boundary conditions we then have

$$u_{i,0} = u_{i,m} = 0, \quad i = 0, \dots, n.$$

We define

$$d = \frac{\Delta t}{\Delta x^2} = \frac{T}{X^2}\frac{m^2}{n}. \tag{5.49}$$

Method 1: Forward in time, centered in space

In this method we use a forward difference approximation for the time derivative and a centered difference approximation for the second spatial derivative:

$$\partial_t u(t,x) = \frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} + O(\Delta t),$$

$$\partial_x^2 u(t,x) = \frac{u(t, x+\Delta x) - 2u(t,x) + u(t, x-\Delta x)}{\Delta x^2} + O(\Delta x).$$

We find

$$u(t+\Delta t, x) = u(t,x) + d(u(t, x+\Delta x) - 2u(t,x) + u(t, x-\Delta x)).$$

Hence we obtain the following iterative equation

$$u_{i+1,j} = u_{i,j} + d(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}), \quad i = 0, \dots, n-1, \ j = 1, \dots, n-1, \tag{5.50}$$

where we recall that $u_{0,j} = u_0(x_j)$, $u_{i,0} = u_{i,m+1} = 0$, $i = 0, \dots, n$, $j = 0, \dots m$. This method is completely explicit. A Matlab function solving the iteration (5.50) with the initial datum $u_0(x) = \exp(X^2/4) - \exp((x - X/2)^2)$ is the following.

```
function [time,space,sol]=heatexp(T,X,n,m)
dt=T/n; dx=X/m;
d=dt/dx^2
sol=zeros(n+1,m+1);
time=zeros(1,n+1);
space=zeros(1,m+1);
for i=2:n+1
time(i)=time(i-1)+dt;
end
for j=2:m+1
space(j)=space(j-1)+dx;
end
for j=1:m+1
sol(1,j)=exp(X^2/4)-exp((space(j)-X/2)^2);
```

```
end
sol(:,1)=0; sol(:,m+1)=0;
for i=2:n+1
for j=3:m+1
sol(i,j-1)=sol(i-1,j-1)+d*(sol(i-1,j)-2*sol(i-1,j-1)+sol(i-1,j-2));
end
end
```

To visualize the result it is convenient to employ an animation which plots the approximate solution at each point on the spatial mesh for some increasing sequence of times in the partition $\{t_0, t_1, \ldots, t_n\}$. This visualization can be achieved with the following simple Matlab function:

```
function anim(r,F,v)
N=length(F(:,1));
step=round(1+N*v/10);
figure
for i=1:step:N
plot(r,F(i,:));
axis([0 1 0 1/2]);
drawnow;
pause(0.3);
end
```

Upon running the command `anim(space,sol,v)`, the previous function will plot the approximate solutions at different increasing times with speed `v` (the speed `v` must be between 0 and 1).

Let us try the following: `[time,space,sol]=heatexp(1,1,2500,50)`. Hence we solve the problem on the unit square $(t,x) \in (0,1)^2$ on a mesh of $(n,m) = 2500 \times 50$ points. The value of the parameter (5.49) is

$$d = 1.$$

If we now try to visualize the solution by running `anim(space,sol,0.1)`, we find that the approximate solution behaves very strangely (it produces just random oscillations). However by increasing the number of time steps with `[time,space,sol]=heatexp(1,1,5000,50)`, so that

$$d = 0.5,$$

and visualize the solution, we shall find that the approximate solution converges quickly and smoothly to $u \equiv 0$, which is the equilibrium of our problem (i.e., the time independent solution of (5.48)). In fact, this is not a coincidence, for we have the following

Theorem 5.9. *The forward-centered method for the heat equation is unstable if $d \geq 1$ and stable for $d < 1$.*

The term unstable here refers to the fact that numerical errors, due for instance to the truncation and round-off of the initial datum on the spatial grid, will increase in time. On the other hand, stability of a finite difference method means that the error will remain small at all times. The stability condition $d < 1$ for the forward-centered method applied to the heat equation is very restrictive: it forces us to choose a very high number of points on the time partition. To avoid such a restriction, which could be very costly in terms of computation time, implicit methods are preferred, such as the one we present next.

Method 2: Backward in time, centered in space

In this method we employ the backward finite difference approximation for the time derivative and the central difference for the second spatial derivative (same as before). This results in the following iterative equation:

$$u_{i+1,j} = u_{i,j} + d(u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}), \quad i = 0, \dots, n-1, \ j = 1, \dots, m-1, \quad (5.51)$$

where we recall that $u_{0,j} = u_0(x_j)$, $u_{i,0} = u_{i,m+1} = 0$, $i = 0, \dots, n$, $j = 0, \dots m$. This method is implicit and we need therefore to solve for the solution at time $i+1$ in terms of the solution at time i. To this purpose we let

$$\mathbf{u}_i = (u_{i,0}, u_{i,1}, \dots, u_{i,m})^T$$

be the *column* vector containing the approximate solution at time t_i and rewrite (5.51) in matrix form as follows:

$$A\mathbf{u_{i+1}} = \mathbf{u_i}, \quad (5.52)$$

where A is the $m+1 \times m+1$ matrix with non-zero entries given by

$$A_{0,0} = A_{m,m} = 1, \quad A_{k,k} = 1 + 2d, \quad A_{k,k-1} = A_{k,k+1} = -d, \quad k = 1, \dots m-1.$$

The matrix A is invertible, hence we can invert (5.52) to express $\mathbf{u_{j+1}}$ in terms of $\mathbf{u_j}$ as

$$\mathbf{u_{i+1}} = A^{-1}\mathbf{u_i}. \quad (5.53)$$

This method is unconditionally stable, i.e., it is stable for all values of the parameter d. We can test this property by using the following Matlab function, which solves the iterative equation (5.53):

```
function [time,space,sol]=heatimp(T,X,n,m)
dt=T/n; dx=X/m;
d=dt/dx^2
sol=zeros(n+1,m+1);
time=zeros(1,n+1);
space=zeros(1,m+1);
A=zeros(m+1,m+1);
A(1,1)=1; A(m+1,m+1)=1;
```

```
for i=2:n+1
time(i)=time(i-1)+dt;
end
for j=2:m+1
space(j)=space(j-1)+dx;
end
for j=1:m+1
sol(1,j)=exp(X^2/4)-exp((space(j)-X/2)^2);
end
sol(:,1)=0; sol(:,m+1)=0;
for k=2:m
A(k,k-1)=-d;
A(k,k)=1+2*d;
A(k,k+1)=-d;
end
for i=2:n+1
sol(i,:)=sol(i-1,:)*transpose(inv(A));
end
```

If we now run `[time,space,sol]=heatexp(1,1,500,50)`, for which $d = 5$, and visualize the solution we shall obtain that the approximate solution behaves smoothly as expected, indicating that the instability problem of the forward-centered method has been solved.

Method 3: Crank-Nicholson

This is an implicit method with higher order of accuracy than the backward-centered method. It is obtained by simply averaging between methods 1 and 2 above, i.e.,

$$u_{i+1,j} = \frac{1}{2}u_{i+1,j} + \frac{1}{2}u_{i+1,j},$$

where the first term in the right hand side is computed with method 1 and the second term with method 2. Thus we obtain the following iterative equation

$$u_{i+1,j} = u_{i,j} + \frac{d}{2}[(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) + (u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1})]. \tag{5.54}$$

As the backward-centered method, the Crank-Nicholson method is also unconditionally stable.

Exercise 5.15. *Write* (5.54) *in matrix form and solve for the solution at the time step* $i+1$ *in terms of the solution at the time step* i.

Exercise 5.16. *Write a Matlab function that implements the Crank-Nicholson method.*

Exercise 5.17. *Compare methods 2 and 3.*

5.A Appendix: Solutions to selected problems

Exercise 5.3. The SDE in question is

$$dX(t) = (a - bX(t))\,dt + \gamma dW(t), \quad t \geq s, \quad X(s) = x.$$

Letting $Y(t) = e^{bt}X(t)$ and applying Itô's formula we find that $Y(t)$ satisfies

$$dY(t) = ae^{bt}\,dt + \gamma e^{bt}dW(t), \quad Y(s) = xe^{bs}.$$

Hence

$$Y(t) = xe^{bs} + a\int_s^t e^{bu}\,d\tau + \gamma\int_s^t e^{bu}\,dW(u)$$

and so

$$X(t;s,x) = xe^{b(s-t)} + \frac{a}{b}(1 - e^{b(s-t)}) + \int_s^t \gamma e^{b(u-t)}dW(u).$$

Taking the expectation we obtain immediately that $\mathbb{E}[X(t;s,x)] = m(t-s,x)$. Moreover by Exercise 4.9, the Itô integral in $X(t;s,x)$ is a normal random variable with zero mean and variance $\Delta(t-s)^2$, hence the claim follows.

Exercise 5.4. Letting $Y(t) = e^{-\frac{t^2}{2}}X(t)$, we find that $dY(t) = e^{-\frac{t^2}{2}}dW(t)$ and $Y(0) = 1$. Thus

$$X(t) = e^{\frac{t^2}{2}} + e^{\frac{t^2}{2}}\int_0^t e^{-\frac{u^2}{2}}\,dW(u).$$

Note that $X(t)$ is normally distributed with mean

$$\mathbb{E}[X(t)] = e^{\frac{t^2}{2}}.$$

It follows that $\mathrm{Cov}(X(s),X(t)) = \mathbb{E}[X(s)X(t)] - \mathbb{E}[X(s)]\mathbb{E}[X(t)]$ is

$$\mathrm{Cov}(X(s),X(t)) = e^{\frac{s^2+t^2}{2}}\mathbb{E}\Big[\int_0^s e^{-\frac{u^2}{2}}dW(u)\int_0^t e^{-\frac{u^2}{2}}dW(u)\Big].$$

Assume for example that $s \leq t$. Hence

$$\mathrm{Cov}(X(s),X(t)) = e^{\frac{s^2+t^2}{2}}\mathbb{E}\Big[\int_0^t \mathbb{I}_{[0,s]}e^{-\frac{u^2}{2}}dW(u)\int_0^t e^{-\frac{u^2}{2}}dW(u)\Big].$$

Using the result of Exercise 4.3 we have

$$\begin{aligned}\mathrm{Cov}(X(s),X(t)) &= e^{\frac{s^2+t^2}{2}}\int_0^t \mathbb{I}_{[0,s]}e^{-\frac{u^2}{2}}e^{-\frac{u^2}{2}}\,du = e^{\frac{s^2+t^2}{2}}\int_0^s e^{-u^2}\,du \\ &= \sqrt{\pi}e^{\frac{s^2+t^2}{2}}\int_0^{\sqrt{2}s} e^{-\frac{u^2}{2}}\frac{du}{\sqrt{2\pi}} = \sqrt{\pi}e^{\frac{s^2+t^2}{2}}(\Phi(\sqrt{2}s) - \frac{1}{2}).\end{aligned}$$

For general $s,t \geq 0$ we find

$$\mathrm{Cov}(X(s),X(t)) = \sqrt{\pi}e^{\frac{s^2+t^2}{2}}(\Phi(\sqrt{2}\min(s,t)) - \frac{1}{2}).$$

Chapter 6

The risk-neutral price

Throughout this chapter we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the Brownian motion $\{W(t)\}_{t\geq 0}$ are given. Furthermore, in order to avoid the need to repeatedly specify technical assumptions, we make the following conventions:

- All stochastic processes in this chapter are assumed to have a.s. continuous paths and so in particular they are integrable, both path by path and in the Itô sense. Of course one may relax this assumption, but for our applications it is general enough.

- All Itô integrals in this chapter are assumed to be martingales, which holds for instance when the integrand stochastic process is in the space $\mathbb{L}^2$.

6.1 Absence of arbitrage in 1+1 dimensional markets

The ultimate purpose of this section is to prove that a self-financing portfolio invested in a 1+1 dimensional market is not an arbitrage. We shall prove the result by using Theorem 3.16, i.e., by showing that there exists a measure, equivalent to $\mathbb{P}$, with respect to which the discounted value of the portfolio is a martingale. We first define such a measure. We have seen in Theorem 4.10 that, given a stochastic process $\{\theta(t)\}_{t\geq 0}$ satisfying the Novikov condition (4.20), the stochastic process $\{Z(t)\}_{t\geq 0}$ defined by

$$Z(t) = \exp\left(-\int_0^t \theta(s)\, dW(s) - \frac{1}{2}\int_0^t \theta(s)^2\, ds\right), \tag{6.1}$$

is a $\mathbb{P}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and that the map $\widetilde{\mathbb{P}} : \mathcal{F} \to [0,1]$ given by

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z(T)\mathbb{I}_A] \tag{6.2}$$

is a probability measure equivalent to $\mathbb{P}$, for all $T > 0$.

Definition 6.1. *Consider the 1+1 dimensional market*

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad dB(t) = B(t)R(t)dt,$$

where the market parameters $\{\mu(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, $\{R(t)\}_{t\geq 0}$ *are all adapted to* $\{\mathcal{F}_W(t)\}_{t\geq 0}$. *Assume that* $\sigma(t) > 0$ *almost surely for all times. Let* $\{\theta(t)\}_{t\geq 0}$ *be the stochastic process given by*

$$\theta(t) = \frac{\mu(t) - R(t)}{\sigma(t)}, \tag{6.3}$$

and define $\{Z(t)\}_{t\geq 0}$ *by* (6.1). *Assume that* $\{Z(t)\}_{t\geq 0}$ *is a martingale (e.g.,* $\{\theta(t)\}_{t\geq 0}$ *satisfies the Novikov condition* (4.20)*). The probability measure* $\widetilde{\mathbb{P}}$ *equivalent to* $\mathbb{P}$ *given by* (6.2) *is called the* **risk-neutral probability measure** *of the market at time* T.

Note that, by the definition (6.3) of the stochastic process $\{\theta(t)\}_{t\geq 0}$, we can rewrite $dS(t)$ as

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t), \tag{6.4}$$

where

$$d\widetilde{W}(t) = dW(t) + \theta(t)dt. \tag{6.5}$$

By Girsanov theorem, Theorem 4.11, the stochastic process $\{\widetilde{W}(t)\}_{t\geq 0}$ is a $\widetilde{\mathbb{P}}$-Brownian motion. Moreover, $\{\mathcal{F}_W(t)\}_{t\geq 0}$ is a non-anticipating filtration for $\{\widetilde{W}(t)\}_{t\geq 0}$. We also recall that a portfolio $\{h_S(t), h_B(t)\}_{t\geq 0}$ is self-financing if it is adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and if its value $\{V(t)\}_{t\geq 0}$ satisfies

$$dV(t) = h_S(t)dS(t) + h_B(t)dB(t), \tag{6.6}$$

see Definition 4.4.

Theorem 6.1. *Consider the 1+1 dimensional market*

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad dB(t) = B(t)R(t)dt, \tag{6.7}$$

where the market parameters $\{\mu(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, $\{R(t)\}_{t\geq 0}$ *are adapted to* $\{\mathcal{F}_W(t)\}_{t\geq 0}$. *Assume that* $\sigma(t) > 0$ *almost surely for all times. Then the following holds.*

(i) The discounted stock price $\{S^*(t)\}_{t\geq 0}$ *is a* $\widetilde{\mathbb{P}}$*-martingale in the filtration* $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

(ii) A portfolio process $\{h_S(t), h_B(t)\}_{t\geq 0}$ *adapted to* $\{\mathcal{F}_W(t)\}_{t\geq 0}$ *is self-financing if and only if its discounted value satisfies*

$$V^*(t) = V(0) + \int_0^t D(s)h_S(s)\sigma(s)S(s)d\widetilde{W}(s). \tag{6.8}$$

(iii) If $\{h_S(t), h_B(t)\}_{t\geq 0}$ *is a self-financing portfolio, then* $\{h_S(t), h_B(t)\}_{t\geq 0}$ *is not an arbitrage.*

Proof. (i) By (6.4) and $dD(t) = -D(t)R(t)dt$ we have

$$\begin{aligned} dS^*(t) &= S(t)dD(t) + D(t)dS(t) + dD(t)dS(t) \\ &= -S(t)R(t)D(t)\,dt + D(t)(R(t)S(t)\,dt + \sigma(t)S(t)\,d\widetilde{W}(t)) \\ &= D(t)\sigma(t)S(t)d\widetilde{W}(t), \end{aligned}$$

and so the discounted price $\{S^*(t)\}_{t\geq 0}$ of the stock is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$.
(ii) By (6.7) and $h_B(t)B(t) = V(t) - h_S(t)S(t)$, the value (4.32) of self-financing portfolios can be written as

$$dV(t) = h_S(t)S(t)[(\mu(t) - R(t))dt + \sigma(t)dW(t)] + V(t)R(t)dt. \tag{6.9}$$

Hence

$$dV(t) = h_S(t)S(t)\sigma(t)[dW(t) + \theta(t)dt] + V(t)R(t)dt = h_S(t)S(t)\sigma(t)d\widetilde{W}(t) + V(t)R(t)dt.$$

Thus the discounted portfolio value $V^*(t) = D(t)V(t)$ satisfies

$$\begin{aligned} dV^*(t) &= V(t)dD(t) + D(t)dV(t) + dD(t)dV(t) \\ &= -D(t)V(t)R(t)\,dt + D(t)h_S(t)S(t)\sigma(t)d\widetilde{W}(t) + D(t)V(t)R(t)dt \\ &= D(t)h_S(t)S(t)\sigma(t)d\widetilde{W}(t), \end{aligned}$$

which proves (6.8).
(iii) By (6.8), the discounted value of self-financing portfolios is a $\widetilde{\mathbb{P}}$- martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$. As $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent, (iii) follows by Theorem 3.16. □

Remark 6.1 (Arbitrage-free principle)**.** The absence of self-financing arbitrage portfolios in the 1+1 dimensional market (6.7) is consistent with the observations. In fact, even if arbitrage opportunities may exist in real markets, they are very rare and last for very short times, as they are quickly exploited by investors. In general, when a stochastic model for the price of an asset is introduced, we require that it should satisfy the **arbitrage-free principle**, namely that any self-financing portfolio invested in this asset and the risk-free asset should be no arbitrage. Theorem 6.1 shows that the generalized geometric Brownian motion satisfies the arbitrage-free principle, provided $\sigma(t) > 0$ a.s. for all times.

6.2 The risk-neutral pricing formula

Consider the European derivative with pay-off Y and time of maturity $T > 0$. We assume that Y is $\mathcal{F}_W(T)$-measurable. Suppose that the derivative is sold at time $t < T$ for the price $\Pi_Y(t)$. The first concern of the seller is to hedge the derivative, that is to say, to invest the amount $\Pi_Y(t)$ in such a way that the value of the seller portfolio at time T is enough to pay-off the buyer of the derivative. The purpose of this section is to define a theoretical price for the derivative which makes it possible for the seller to set-up such an hedging portfolio. We argue under the following assumptions:

1. the seller is only allowed to invest the amount $\Pi_Y(t)$ in the 1+1 dimensional market consisting of the underlying stock and the risk-free asset;

2. the investment strategy of the seller is self-financing.

It follows by Theorem 6.1 that the sought hedging portfolio is not an arbitrage. We may interpret this fact as a "fairness" condition on the price of the derivative $\Pi_Y(t)$. In fact, if the seller can hedge the derivative and still be able to make a risk-less profit on the underlying stock, this may be considered unfair for the buyer.

We thus consider the 1+1 dimensional market

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad dB(t) = B(t)R(t)dt,$$

where we assume that the market parameters $\{\mu(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, $\{R(t)\}_{t\geq 0}$ are adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and that $\sigma(t) > 0$ almost surely for all times. Let $\{h_S(t), h_B(t)\}_{t\geq 0}$ be a self-financing portfolio invested in this market and let $\{V(t)\}_{t\geq 0}$ be its value. By Theorem 6.1, the discounted value $\{V^*(t)\}_{t\geq 0}$ of the portfolio is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$, hence

$$D(t)V(t) = \widetilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}_W(t)].$$

Requiring the hedging condition $V(T) = Y$ gives

$$V(t) = \frac{1}{D(t)}\widetilde{\mathbb{E}}[D(T)Y|\mathcal{F}_W(t)].$$

Since $D(t)$ is $\mathcal{F}_W(t)$-measurable, we can move it inside the expectation and write the latter equation as

$$V(t) = \widetilde{\mathbb{E}}[Y\frac{D(T)}{D(t)}|\mathcal{F}_W(t)] = \widetilde{\mathbb{E}}[Y\exp(-\int_t^T R(s)\,ds)|\mathcal{F}_W(t)],$$

where we used the definition $D(t) = \exp(-\int_0^t R(s)\,ds)$ of the discounting process. Assuming that the derivative is sold at time t for the price $\Pi_Y(t)$, then the value of the seller portfolio at this time is precisely equal to the premium $\Pi_Y(t)$, which leads to the following definition.

Definition 6.2. *Let Y be a $\mathcal{F}_W(T)$-measurable random variable with finite expectation. The* **risk-neutral price** *(or* **fair price***) at time $t \in [0,T]$ of the European derivative with pay-off Y and time of maturity $T > 0$ is given by*

$$\Pi_Y(t) = \widetilde{\mathbb{E}}[Y\exp(-\int_t^T R(s)\,ds)|\mathcal{F}_W(t)], \tag{6.10}$$

i.e., it is equal to the value at time t of any self-financing hedging portfolio invested in the underlying stock and the bond.

Remark 6.2. Being defined as a conditional expectation, the risk-neutral price can rarely be computed explicitly. An exception to this is when the market parameters are deterministic, see Section 6.3, and for some simple stochastic models, see Section 6.5.

In the particular case of a standard European derivative, i.e., when $Y = g(S(T))$, for some measurable function g, the risk-neutral price becomes

$$\Pi_Y(t) = \widetilde{\mathbb{E}}[g(S(T))\exp(-\int_t^T R(s)\,ds)|\mathcal{F}_W(t)].$$

By (6.4) we have

$$S(T) = S(t) \exp\left(\int_t^T (R(s) - \frac{1}{2}\sigma^2(s))ds + \int_t^T \sigma(s)d\widetilde{W}(s)\right),$$

hence the risk-neutral price of a standard European derivative takes the form

$$\Pi_Y(t) = \widetilde{\mathbb{E}}[g(S(t)e^{\int_t^T (R(s)-\frac{1}{2}\sigma^2(s))ds+\int_t^T \sigma(s)d\widetilde{W}(s)}) \exp(-\int_t^T R(s)\, ds)|\mathcal{F}_W(t)]. \tag{6.11}$$

Since the risk-neutral price of the European derivative equals the value of self-financing hedging portfolios invested in a 1+1 dimensional market, then, by Theorem 6.1, the discounted risk-neutral price $\{\Pi_Y^*(t)\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$. In fact, this property follows directly also by Definition 6.10, as shown in the first part of the following theorem.

Theorem 6.2. *Consider the 1+1 dimensional market*

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad dB(t) = B(t)R(t)dt,$$

where we assume that $\{\mu(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, $\{R(t)\}_{t\geq 0}$ are adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and that $\sigma(t) > 0$ almost surely for all times. Assume that the European derivative on the stock with pay-off Y and time of maturity $T > 0$ is priced by (6.10) and let $\Pi_Y^*(t) = D(t)\Pi_Y(t)$ be the discounted price of the derivative. Then the following holds.

(i) The process $\{\Pi_Y^*(t)\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

(ii) There exists a stochastic process $\{\Delta(t)\}_{t\in[0,T]}$, adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$, such that

$$\Pi_Y^*(t) = \Pi_Y(0) + \int_0^t \Delta(s)d\widetilde{W}(s), \quad t \in [0,T]. \tag{6.12}$$

(iii) The portfolio $\{h_S(t), h_B(t)\}_{t\in[0,T]}$ given by

$$h_S(t) = \frac{\Delta(t)}{D(t)\sigma(t)S(t)}, \quad h_B(t) = (\Pi_Y(t) - h_S(t)S(t))/B(t) \tag{6.13}$$

is self-financing and replicates the derivative at any time, i.e., its value $V(t)$ is equal to $\Pi_Y(t)$ for all $t \in [0,T]$. In particular, $V(T) = \Pi_Y(T) = Y$, i.e., the portfolio is hedging the derivative.

Proof. (i) We have

$$\Pi_Y^*(t) = D(t)\Pi_Y(t) = \widetilde{\mathbb{E}}[\Pi_Y(T)D(T)|\mathcal{F}_W(t)] = \widetilde{\mathbb{E}}[\Pi_Y^*(T)|\mathcal{F}_W(t)],$$

where we used that $\Pi_Y(T) = Y$. Hence, for $s \leq t$, and using Theorem 3.13(iii),

$$\widetilde{\mathbb{E}}[\Pi_Y^*(t)|\mathcal{F}_W(s)] = \widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[\Pi_Y^*(T)|\mathcal{F}_W(t)]|\mathcal{F}_W(s)] = \widetilde{\mathbb{E}}[\Pi_Y^*(T)|\mathcal{F}_W(s)] = \Pi_Y^*(s).$$

This shows that the discounted price of the derivative is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$.
(ii) By (i) and (3.24) we have

$$Z(s)\Pi^*_Y(s) = Z(s)\widetilde{\mathbb{E}}[\Pi^*_Y(t)|\mathcal{F}_W(s)] = \mathbb{E}[Z(t)\Pi^*_Y(t)|\mathcal{F}_W(s)], \tag{6.14}$$

i.e., the stochastic process $\{Z(t)\Pi^*_Y(t)\}_{t\in[0,T]}$ is a $\mathbb{P}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$. Hence, by the martingale representation theorem, Theorem 4.6, there exists a stochastic process $\{\Gamma(t)\}_{t\in[0,T]}$ adapted to $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$ such that

$$Z(t)\Pi^*_Y(t) = \Pi_Y(0) + \int_0^t \Gamma(s)dW(s), \quad t \in [0,T],$$

i.e.,

$$d(Z(t)\Pi^*_Y(t)) = \Gamma(t)dW(t). \tag{6.15a}$$

On the other hand, by Itô's product rule,

$$\begin{aligned} d\Pi^*_Y(t) =& d(Z(t)\Pi^*_Y(t)/Z(t)) = d(1/Z(t))Z(t)\Pi^*_Y(t) + 1/Z(t)d(Z(t)\Pi^*_Y(t)) \\ &+ d(1/Z(t))d(Z(t)\Pi^*_Y(t)). \end{aligned} \tag{6.15b}$$

By Itô's formula and $dZ(t) = -\theta(t)Z(t)dW(t)$, we obtain

$$d(1/Z(t)) = -\frac{1}{Z(t)^2}dZ(t) + \frac{1}{Z(t)^3}dZ(t)dZ(t) = \frac{\theta(t)}{Z(t)}d\widetilde{W}(t). \tag{6.15c}$$

Hence

$$d(1/Z(t))d(Z(t)\Pi^*_Y(t)) = \frac{\theta(t)\Gamma(t)}{Z(t)}dt. \tag{6.15d}$$

Combining Equations (6.15) we have

$$d\Pi^*_Y(t) = \Delta(t)d\widetilde{W}(t), \quad \text{where} \;\; \Delta(t) = \theta(t)\Pi^*_Y(t) + \frac{\Gamma(t)}{Z(t)},$$

which proves (6.12).
(iii) It is clear that the portfolio $\{h_S(t), h_B(t)\}_{t\in[0,T]}$ given by (6.13) is adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and replicates the derivative. Furthermore (6.12) entails that $V^*(t) = \Pi^*_Y(t)$ satisfies (6.8), hence, by Theorem 6.1(ii), $\{h_S(t), h_B(t)\}_{t\in[0,T]}$ is a self-financing portfolio, and the proof is completed. □

Consider now the 1+1 dimensional market consisting of the European derivative and the risk-free asset. The value of a self-financing portfolio invested in this market satisfies

$$dV(t) = h_Y(t)d\Pi_Y(t) + h_B(t)dB(t), \quad V(t) = h_Y(t)\Pi_Y(t) + h_B(t)B(t),$$

where $h_Y(t)$ is the number of shares of the derivative in the portfolio. It follows by (6.12) that the discounted value of this portfolio satisfies

$$\begin{aligned} d(V^*(t)) &= -R(t)D(t)V(t)dt + D(t)h_Y(t)d\Pi_Y(t) + D(t)h_B(t)B(t)R(t)dt \\ &= -R(t)D(t)h_Y(t)\Pi_Y(t)dt + D(t)h_Y(t)d\Pi_Y(t) = -h_Y(t)d(D(t)\Pi_Y(t)) \\ &= -h_Y(t)\Delta(t)d\widetilde{W}(t). \end{aligned}$$

We infer that the discounted value process $\{V^*(t)\}_{t\geq 0}$ is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$. Hence, by Theorem 3.16, the portfolio is not an arbitrage and therefore the risk-neutral price model for European derivatives satisfies the arbitrage-free principle, see Remark 6.1.

6.3 Black-Scholes price of European derivatives

In this section we apply the results of Section 6.2 to the case when the market parameters $\{\mu(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, $\{R(t)\}_{t\geq 0}$ are deterministic constants. In particular, the price of the stock now follows the geometric Brownian motion (2.14), where $\alpha = \mu - \sigma^2/2$, i.e.,

$$S(t) = S(0)e^{\alpha t + \sigma W(t)} = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)} = S(0)e^{(r - \frac{\sigma^2}{2})t + \sigma \widetilde{W}(t)}, \tag{6.16}$$

where $\widetilde{W}(t) = W(t) + (\frac{\mu - r}{\sigma})t$ is a Brownian motion in the risk-neutral probability measure. We assume in the following that the European derivative is standard. Replacing $\sigma(s) = \sigma > 0$ and $R(s) = r > 0$ into (6.11), we obtain

$$\Pi_Y(t) = e^{-r\tau}\widetilde{\mathbb{E}}[g(S(t)e^{(r - \frac{1}{2}\sigma^2)\tau}e^{\sigma(\widetilde{W}(T) - \widetilde{W}(t))})|\mathcal{F}_W(t)],$$

where

$$\tau = T - t$$

is the time left to the expiration of the derivative. As $\mathcal{F}_W(t) = \mathcal{F}_{\widetilde{W}}(t)$, we obtain

$$\Pi_Y(t) = e^{-r\tau}\widetilde{\mathbb{E}}[g(S(t)e^{(r - \frac{1}{2}\sigma^2)\tau}e^{\sigma(\widetilde{W}(T) - \widetilde{W}(t))})|\mathcal{F}_{\widetilde{W}}(t)].$$

As the increment $\widetilde{W}(T) - \widetilde{W}(t)$ is independent of $\mathcal{F}_{\widetilde{W}}(t)$, the conditional expectation above is a pure expectation, see Theorem 3.14(i), and so

$$\Pi_Y(t) = e^{-r\tau}\mathbb{E}[g(S(t)e^{(r - \frac{1}{2}\sigma^2)\tau}e^{\sigma(W(T) - W(t))})]. \tag{6.17}$$

(Note that we don't need anymore to be in the risk-neutral world). Finally, since $W(T) - W(t) \in \mathcal{N}(0, \tau)$, we obtain

$$\Pi_Y(t) = \frac{e^{-r\tau}}{\sqrt{2\pi\tau}}\int_{\mathbb{R}} g(S(t)e^{(r - \frac{1}{2}\sigma^2)\tau}e^{\sigma y})e^{-\frac{y^2}{2\tau}}\,dy, \tag{6.18}$$

that is,

$$\Pi_Y(t) = v_g(t, S(t)), \tag{6.19a}$$

where the **Black-Scholes price function** $v_g : \overline{\mathcal{D}_T^+} \to \mathbb{R}$ is given by

$$v_g(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g(xe^{(r-\frac{1}{2}\sigma^2)\tau} e^{\sigma\sqrt{\tau}y}) e^{-\frac{y^2}{2}} \, dy. \tag{6.19b}$$

Definition 6.3. *Let $g : (0,\infty) \to \mathbb{R}$ be a C^2 function and assume that g', g'' are uniformly bounded. The stochastic process $\{\Pi_Y(t)\}_{t\in[0,T]}$ given by (6.19), is called the* **Black-Scholes price** *of the standard European derivative with pay-off $Y = g(S(T))$ and time of maturity $T > 0$.*

Remark 6.3. Our assumptions on the pay-off function g can be considerably weakened, but since they cover all real-world applications, we shall not do it. Note that, under our assumptions, $v_g \in C^{1,2}(\overline{\mathcal{D}_T^+})$ and $\partial_x v_g$ is uniformly bounded.

Remark 6.4. The fact that the Black-Scholes price of the derivative at time t is a deterministic function of $S(t)$, that is, $\Pi_Y(t) = v_g(t, S(t))$, is an important property for the applications. In fact, thanks to this property, at time t we may look at the price $S(t)$ of the stock in the market and compute explicitly the theoretical price $\Pi_Y(t)$ of the derivative. This theoretical value is, in general, different from the real market price. We shall discuss how to interpret this difference is Section 6.3.2. Moreover, as shown below, the formula (6.19) is equivalent to the Markov property of the geometric Brownian motion (6.16) in the risk-neutral probability measure $\widetilde{\mathbb{P}}$.

We can rewrite the Black-Scholes price function as $v_g(t,x) = h(T-t, x)$, where, by a change of variable in the integral on the right hand side of (6.19b),

$$h(\tau, x) = \int_{\mathbb{R}} g(y) q(\tau, x, y) \, dy,$$

where

$$q(\tau,x,y) = \frac{e^{-r\tau}\mathbb{I}_{y>0}}{y\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{1}{2\sigma^2\tau}\left(\log\frac{y}{x} - (r - \frac{1}{2}\sigma^2)\tau\right)^2\right].$$

Comparing this expression with (3.31), we see that we can write the function q as

$$q(\tau,x,y) = e^{-r\tau} p_*(\tau,x,y),$$

where p_* is the transition density of the geometric Brownian motion (6.16). In particular, the risk-neutral pricing formula of a standard European derivative when the market parameters are constant is equivalent to the identity

$$\widetilde{\mathbb{E}}[g(S(T))|\mathcal{F}_{\widetilde{W}}(t)] = \int_{\mathbb{R}} p_*(T-t, S(t), y) g(y) \, dy,$$

and thus, since $0 \leq t \leq T$ are arbitrary, it is equivalent to the Markov property of the geometric Brownian motion (6.16) in the risk-neutral probability measure $\widetilde{\mathbb{P}}$, see again Exercise 3.34. We shall generalize this discussion to markets with non-constant parameters in

Section 6.5. Note also that replacing $s = 0$, $t = \tau$, $\alpha = r - \sigma^2/2$ into (3.33), and letting $u(\tau, x) = e^{r\tau} h(\tau, x)$, we obtain that u satisfies

$$-\partial_\tau u + rx\partial_x u + \frac{1}{2}\sigma^2 x^2 \partial_x^2 u = 0, \quad u(0, x) = h(0, x) = v_g(T, x) = g(x).$$

Hence the function $h(\tau, x)$ satisfies

$$-\partial_\tau h + rx\partial_x h + \frac{1}{2}\sigma^2 x^2 \partial_x^2 h = rh, \quad h(0, x) = g(x).$$

As $v_g(t, x) = h(T - t, x)$, we obtain the following result.

Theorem 6.3. *The Black-Scholes price function v_g is the unique strong solution of the* **Black-Scholes** *PDE*

$$\partial_t v_g + rx\partial_x v_g + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v_g = rv_g, \quad (t, x) \in \mathcal{D}_T^+ \tag{6.20a}$$

with the **terminal condition**

$$v_g(T, x) = g(x). \tag{6.20b}$$

Exercise 6.1. *Write a Matlab code that computes the finite difference solution of the problem* (6.20). *Use the Crank-Nicholson method presented in Section 5.4.2.*

Remark 6.5. For the previous exercise one needs to fix the boundary condition at $x = 0$ for (6.20a). It is easy to show that the boundary value at $x = 0$ of the Black-Scholes price function is given by

$$v_g(t, 0) = g(0)e^{r(t-T)}, \quad \text{for all } t \in [0, T]. \tag{6.21}$$

In fact from one hand, letting $x = 0$ in (6.20a) we obtain that $v(t) = v_g(t, 0)$ satisfies $dv/dt = rv$, hence $v(t) = v(T)e^{r(t-T)}$. On the other hand, $v(T) = v_g(T, 0) = g(0)$. Thus (6.21) follows. For instance, in the case of a call, i.e., when $g(z) = (z - K)_+$, we obtain $v_g(t, 0) = 0$, for all $t \in [0, T]$, hence the risk-neutral price of a call option is zero when the price of the underlying stock tends to zero. That this should be the case is clear, for the call will never expire in the money if the price of the stock is arbitrarily small. For a put option, i.e., when $g(z) = (K - z)_+$, we have $v_g(t, 0) = Ke^{-r\tau}$, hence the risk-neutral price of a put option is given by the discounted value of the strike price when the price of the underlying stock tends to zero. This is also clear, since in this case the put option will certainly expire in the money, i.e., its value at maturity is K with probability one, and so the value at any earlier time is given by discounting its terminal value.

Next we compute the hedging portfolio of the derivative.

Theorem 6.4. *Consider a standard European derivative priced according to Definition 6.3. The portfolio $\{h_S(t), h_B(t)\}$ given by*

$$h_S(t) = \partial_x v_g(t, S(t)), \quad h_B(t) = (\Pi_Y(t) - h_S(t)S(t))/B(t)$$

is a self-financing hedging portfolio for the derivative.

Proof. According to Theorem 6.2, we have to show that the discounted value of the Black-Scholes price satisfies

$$d\Pi^*_Y(t) = D(t)S(t)\sigma\partial_x v_g(t, S(t))d\widetilde{W}(t).$$

A straightforward calculation, using $\Pi_Y(t) = v_g(t, S(t))$, Itô's formula and Itô's product rule, gives

$$\begin{aligned} d(D(t)\Pi_Y(t)) =& D(t)[\partial_t v_g(t,x) + rx\partial_x v_g(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v_g(t,x) - rv_g(t,x)]_{x=S(t)} \\ &+ D(t)\sigma S(t)\partial_x v_g(t, S(t))d\widetilde{W}(t). \end{aligned} \tag{6.22}$$

Since v_g solves the Black-Scholes PDE (6.20a), the result follows. □

Exercise 6.2. *Work out the details of the computation leading to* (6.22).

Exercise 6.3. *Find the risk-neutral price at time $t = 0$ of standard European derivatives assuming that the market parameters are deterministic functions of time.*

6.3.1 Black-Scholes price of European vanilla options

In this section we focus the discussion on call/put options, which are also called **vanilla options**. We thereby assume that the pay-off of the derivative is given by

$$Y = (S(T) - K)_+, \text{ i.e., } Y = g(S(T)),\ g(x) = (x - K)_+, \quad \text{for a call option,}$$

$$Y = (K - S(T))_+, \text{ i.e., } Y = g(S(T)),\ g(x) = (K - x)_+, \quad \text{for a put option.}$$

The function v_g given by (6.19b) will be denoted by c, for a call option, and by p, for a put option.

Remark 6.6. Strictly speaking, the pay-off functions for call/put options do not satisfy the regularity assumptions in Definition 6.3. For instance, $g(x) = (x - K)_+$ is not differentiable at $x = K$ and so the Black-Scholes price function $c(t, x)$ does not extend smoothly on the boundary $t = T$. We shall ignore this technicality and still refer to $c(t, x)$ as a strong solution of the Black-Scholes PDE for call options.

Theorem 6.5. *The Black-Scholes price at time t of a European call option with strike price $K > 0$ and maturity $T > 0$ is given by $c(t, S(t))$, where*

$$c(t, x) = x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \tag{6.23a}$$

$$d_2 = \frac{\log\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_1 = d_2 + \sigma\sqrt{\tau}, \tag{6.23b}$$

and where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{1}{2}y^2}\, dy$ is the standard normal distribution. The Black-Scholes price of the corresponding put option is given by $p(t, S(t))$, where

$$p(t, x) = \Phi(-d_2)Ke^{-r\tau} - \Phi(-d_1)x. \tag{6.24}$$

Moreover the **put-call parity identity** *holds:*

$$c(t, S(t)) - p(t, S(t)) = S(t) - Ke^{-r\tau}. \tag{6.25}$$

Proof. We derive the Black-Scholes price of call options only, the argument for put options being similar (see Exercise 6.24). We substitute $g(z) = (z - K)_+$ into the right hand side of (6.19b) and obtain

$$c(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(x e^{(r-\frac{1}{2}\sigma^2)\tau} e^{\sigma\sqrt{\tau}y} - K \right)_+ e^{-\frac{y^2}{2}}\, dy.$$

Now we use that $xe^{(r-\frac{1}{2}\sigma^2)\tau}e^{\sigma\sqrt{\tau}y} > K$ if and only if $y > -d_2$. Hence

$$c(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \left[\int_{-d_2}^{\infty} x e^{(r-\frac{1}{2}\sigma^2)\tau} e^{\sigma\sqrt{\tau}y} e^{-\frac{y^2}{2}} - K \int_{-d_2}^{\infty} e^{-\frac{y^2}{2}}\, dy \right].$$

Using $-\frac{1}{2}y^2 + \sigma\sqrt{\tau}y = -\frac{1}{2}(y - \sigma\sqrt{\tau})^2 + \frac{\sigma^2}{2}\tau$ and changing variable in the integrals we obtain

$$\begin{aligned}
c(t,x) &= \frac{e^{-r\tau}}{\sqrt{2\pi}} \left[x e^{r\tau} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2}\, dy - K \int_{-d_2}^{\infty} e^{-\frac{y^2}{2}}\, dy \right] \\
&= \frac{e^{-r\tau}}{\sqrt{2\pi}} \left[x e^{r\tau} \int_{-\infty}^{d_2+\sigma\sqrt{\tau}} e^{-\frac{1}{2}y^2}\, dy - K \int_{-\infty}^{d_2} e^{-\frac{y^2}{2}}\, dy \right] \\
&= s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2).
\end{aligned}$$

As to the put-call parity, we have

$$\begin{aligned}
c(t,x) - p(t,x) &= s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) - \Phi(-d_2)Ke^{-r\tau} + s\Phi(-d_1) \\
&= x(\Phi(d_1) + \Phi(-d_1)) - Ke^{-r\tau}(\Phi(d_2) + \Phi(-d_2)).
\end{aligned}$$

As $\Phi(z) + \Phi(-z) = 1$, the claim follows. □

Exercise 6.4. *Derive the Black-Scholes price $p(t, S(t))$ of European put options claimed in Theorem 6.5.*

Remark 6.7. The formulas (6.23)-(6.24) appeared for the first time in the seminal paper [2], where they were derived by a completely diffent argument than the one presented here.

As to the self-financing hedging portfolio for the call/put option, we have $h_S(t) = \partial_x c(t, S(t))$ for call options and $h_S(t) = \partial_x p(t, S(t))$ for put options, see Theorem 6.4, while the number of shares of the bond in the hedging portfolio is given by

$$h_B(t) = (c(t, S(t)) - S(t)\partial_x c(t, S(t)))/B(t), \quad \text{for call options,}$$

and

$$h_B(t) = (c(t, S(t)) - S(t)\partial_x p(t, S(t)))/B(t), \quad \text{for put options.}$$

Let us compute $\partial_x c$:

$$\partial_x c = \Phi(d_1) + x\Phi'(d_1)\partial_x d_1 - Ke^{-r\tau}\Phi'(d_2)\partial_x d_2.$$

As $\partial_x d_1 = \partial_x d_2 = \frac{1}{\sigma\sqrt{\tau}x}$, and $\Phi'(x) = e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$, we obtain

$$\partial_x c = \Phi(d_1) + \frac{1}{\sigma\sqrt{2\pi\tau}}\left(e^{-\frac{1}{2}d_1^2} - \frac{K}{x}e^{-r\tau}e^{-\frac{1}{2}d_2^2}\right).$$

Replacing $d_1 = d_2 + \sigma\sqrt{\tau}$ we obtain

$$\partial_x c = \Phi(d_2) + \frac{e^{-\frac{1}{2}d_2^2}}{\sigma\sqrt{2\pi\tau}}\left(e^{-\frac{1}{2}\sigma^2\tau - d_2\sigma\sqrt{\tau}} - \frac{K}{x}e^{-r\tau}\right).$$

Using the definition of d_2, the term within round brackets in the previous expression is easily found to be zero, hence

$$\partial_x c = \Phi(d_1).$$

By the put-call parity we find also

$$\partial_x p = \Phi(d_1) - 1 = \Phi(-d_1).$$

Note that $\partial_x c > 0$, while $\partial_x p < 0$. This agrees with the fact that call options are bought to protect a short position on the underlying stock, while put options are bought to protect a long position on the underlying stock.

Exercise 6.5 (•). *Consider a European derivative with maturity T and pay-off Y given by*

$$Y = k + S(T)\log S(T),$$

where $k > 0$ is a constant. Find the Black-Scholes price of the derivative at time $t < T$ and the hedging self-financing portfolio. Find the probability that the derivative expires in the money.

6.3.2 The greeks. Implied volatility and volatility curve

The Black-Scholes price of a call (or put) option derived in Theorem 6.5 depends on the price of the underlying stock, the time to maturity, the strike price, as well as on the (constant) market parameters r, σ (it does not depend on α). The partial derivatives of the price function c with respect to these variables are called **greeks**. We collect the most important ones (for call options) in the following theorem.

Theorem 6.6. *The price function c of call options satisfies the following:*

$$\Delta := \partial_x c = \Phi(d_1), \tag{6.26}$$

$$\Gamma := \partial_x^2 c = \frac{\phi(d_1)}{x\sigma\sqrt{\tau}}, \tag{6.27}$$

$$\rho := \partial_r c = K\tau e^{-r\tau}\Phi(d_2), \tag{6.28}$$

$$\Theta := \partial_t c = -\frac{x\phi(d_1)\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2), \tag{6.29}$$

$$\nu := \partial_\sigma c = x\phi(d_1)\sqrt{\tau} \quad \text{(called "vega").} \tag{6.30}$$

In particular:

- $\Delta > 0$, i.e., the price of a call is increasing on the price of the underlying stock;
- $\Gamma > 0$, i.e., the price of a call is convex on the price of the underlying stock;
- $\rho > 0$, i.e., the price of the call is increasing on the interest rate of the bond;
- $\Theta < 0$, i.e., the price of the call is decreasing in time;
- $\nu > 0$, i.e., the price of the call is increasing on the volatility of the stock.

Exercise 6.6. *Use the put-call parity to derive the greeks of put options.*

The greeks measure the sensitivity of options prices with respect to the market conditions. This information can be used to draw some important conclusions. Let us comment for instance on the fact that vega is positive. It implies that the wish of an investor with a long position on a call option is that the volatility of the underlying stock increased. As usual, since this might not happen, the investor portfolio is exposed to possible losses due to the decrease of the stock volatility (which makes the call option in the portfolio loose value). This exposure can be secured by adding variance swaps into the portfolio, see Section 6.5.3.

Exercise 6.7. *Prove that*

$$\lim_{\sigma\to 0^+} c(t,x) = (x - Ke^{-r\tau})_+, \quad \lim_{\sigma\to\infty} c(t,x) = x.$$

Implied volatility

Let us temporarily re-denote the Black-Scholes price of the call as

$$c(t, S(t), K, T, \sigma),$$

which reflects the dependence of the price on the parameters K, T, σ (we disregard the dependence on r). As shown in Theorem 6.6,

$$\partial_\sigma c(t, S(t), K, T, \sigma) = \text{vega} = \frac{S(t)}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \sqrt{\tau} > 0.$$

Hence the Black-Scholes price of the option is an increasing function of the volatility. Furthermore, by Exercise 6.7,

$$\lim_{\sigma\to 0^+} c(t, S(t), K, T) = (S(t) - Ke^{-r\tau})_+, \quad \lim_{\sigma\to+\infty} c(t, S(t), K, T) = S(t).$$

Therefore the function $c(t, S(t), K, T, \cdot)$ is a one-to-one map from $(0, \infty)$ into the interval $I = ((S(t) - Ke^{-r\tau})_+, S(t))$, see Figure 6.1. Now suppose that at some given *fixed* time t the real market price of the call is $\tilde{c}(t)$. Clearly, the option is always cheaper than the stock (otherwise we would buy directly the stock, and not the option) and typically we also have $\tilde{c}(t) > \max(0, S(t) - Ke^{-r\tau})$. The latter is always true if $S(t) < Ke^{-r\tau}$ (the price of options is positive), while for $S(t) > Ke^{-r\tau}$ this follows by the fact that $S(t) - Ke^{-r\tau} \approx S(t) - K$ and real calls are always more expansive than their intrinsic value. This being said, we can safely assume that $\tilde{c}(t) \in I$.

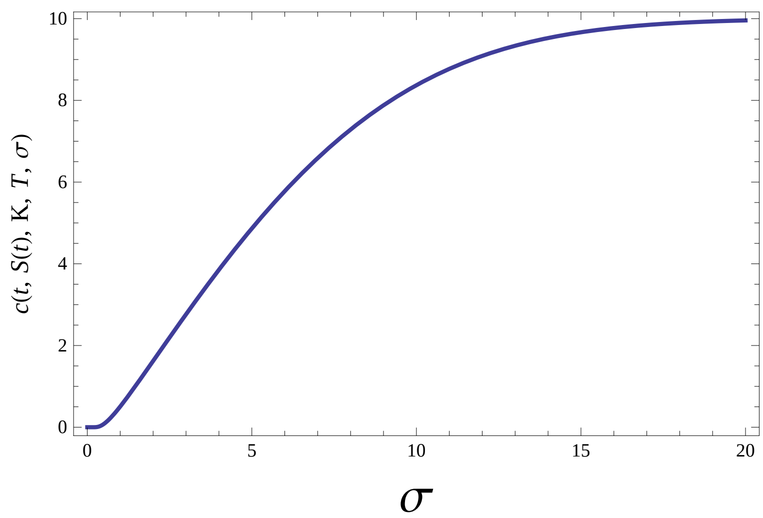


Figure 6.1: We fix $S(t) = 10$, $K = 12$, $r = 0.01$, $\tau = 1/12$ and depict the Black-Scholes price of the call as a function of the volatility. Note that in practice only the very left part of this picture is of interest, because typically $0 < \sigma < 1$.

Thus given the value of $\tilde{c}(t)$ there exists a unique value of σ, which depends on the fixed parameters T, K and which we denote by $\sigma_{\mathrm{imp}}(T, K)$, such that

$$c(t, S(t), K, T, \sigma_{\mathrm{imp}}(T, K)) = \tilde{c}(t).$$

$\sigma_{\mathrm{imp}}(T, K)$ is called the **implied volatility** of the option. The implied volatility must be computed numerically (for instance using Newton's method), since there is no close formula for it.

The implied volatility of an option (in this example of a call option) is a very important parameter and it is often quoted together with the price of the option. If the market followed exactly the assumptions in the Black-Scholes theory, then the implied volatility would be a constant, independent of T, K and equal to the volatility of the underlying asset. In this respect, $\sigma_{\mathrm{imp}}(T, K)$ may be viewed as a quantitative measure of how real markets deviate from ideal Black-Scholes markets. Furthermore, the implied volatility may be viewed as the market consensus on the future value of the volatility of the underlying stock. Recall in fact that in order for the Black-Scholes price of the option to be $c(t, S(t), K, T, \sigma_{\mathrm{imp}}(T, K))$, the volatility of the stock should be equal to $\sigma_{\mathrm{imp}}(T, K)$ in the time interval $[t, T]$ in the future. Hence by pricing the option at the price $\tilde{c}(t) = c(t, S(t), K, T, \sigma_{\mathrm{imp}}(T, K))$, the market is telling us that the buyers and sellers of the option believe that the volatility of the stock in the future will be $\sigma_{\mathrm{imp}}(T, K)$.

As a way of example, in Figure 6.2 the implied volatility is determined (graphically) for various Apple call options on May 12, 2014, when the stock was quoted at 585.54 dollars (closing price of the previous market day). All options expire on June 13, 2014 ($\tau = 1$ month $= 1/12$). The value $r = 0.01$ has been used, but the results do no change significantly

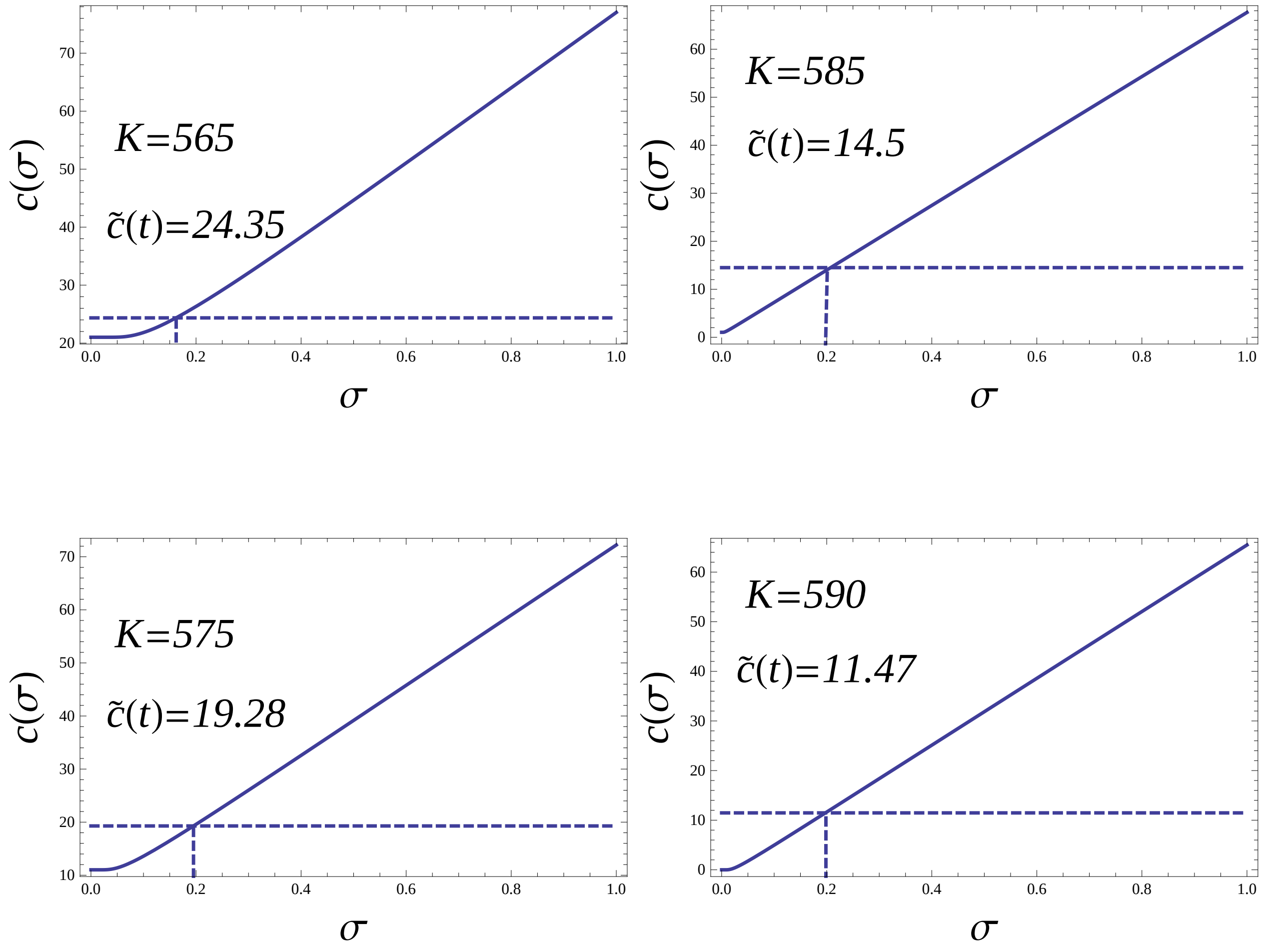
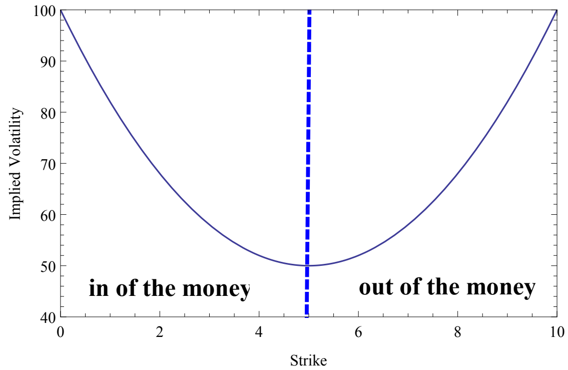


Figure 6.2: Implied volatility of various call options on the Apple stock

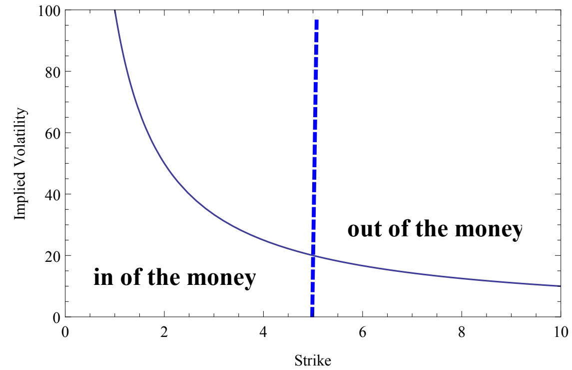
even assuming $r = 0.05$. In the pictures, K denotes the strike price and $\tilde{c}(t)$ the call price. We observe that the implied volatility is 20 % in three cases, while for the call with strike $K = 565$ dollars the implied volatility is a little smaller ($\approx 16\%$), which means that the latter call is slightly underpriced compare to the others.

Volatility curve

As mentioned before, the implied volatility depends on the parameters T, K. Here we are particularly interested in the dependence on the strike price, hence we re-denote the implied volatility as $\sigma_{\text{imp}}(K)$. If the market behaved exactly as in the Black-Scholes theory, then $\sigma_{\text{imp}}(K) = \sigma$ for all values of K, hence the graph of $K \to \sigma_{\text{imp}}(K)$ would be just a straight horizontal line. Given that real markets do not satisfy exactly the assumptions in the Black-Scholes theory, what can we say about the graph of the **volatility curve** $K \to \sigma_{\text{imp}}(K)$? Remarkably, it has been found that there exists recurrent convex shapes for the graph of volatility curves, which are known as **volatility smile** and **volatility skew**, see Figure 6.3.



(a) Smile



(b) Skew

Figure 6.3: Volatility smile and skew of a call option (not from real data!)

In the case of a volatility smile, the minimum of the graph is reached at the strike price $K \approx S(t)$, i.e., when the call is at the money. This behavior indicates that the more the call is far from being at the money, the more it will be overpriced. Volatility smiles have been recurrent in the market since after the crash in 1987 (Black Monday), indicating that this event led investors to be more cautious when trading on options that are in or out of the money. Volatility skews tell us whether investors prefer to trade on call or put options.

Devising mathematical models of volatility and asset prices able to reproduce volatility curves is an active research topic in financial mathematics. We discuss the most popular volatility models in the Section 6.5.

6.4 European derivatives on a dividend-paying stock

In this section we consider Black-Scholes markets with a dividend-paying stock. This means that at some time $t_0 \in (0,T)$ the price of the stock decreases of a fraction $a \in (0,1)$ of its price immediately before t_0, the difference being deposited in the account of the shareholders[1]. Letting $S(t_0^-) = \lim_{t \to t_0^-} S(t)$, we then have

$$S(t_0) = S(t_0^-) - aS(t_0^-) = (1-a)S(t_0^-). \tag{6.31}$$

We assume that on each of the intervals $[0, t_0)$, $[t_0, T]$, the stock price follows a geometric Brownian motion, namely,

$$S(s) = S(t)e^{\alpha(s-t)+\sigma(W(s)-W(t))}, \quad t \in [0, t_0), \ s \in [t, t_0) \tag{6.32}$$

$$S(s) = S(u)e^{\alpha(s-u)+\sigma(W(s)-W(u))}, \quad u \in [t_0, T], \ s \in [u, T]. \tag{6.33}$$

[1] The dividend is expressed in percentage of the price of the stock. For instance, $a = 0.03$ means that the dividend paid is 3%.

Theorem 6.7. *Consider the standard European derivative with pay-off $Y = g(S(T))$ and maturity T. Let $\Pi_Y^{(a,t_0)}(t)$ be the Black-Scholes price of the derivative at time $t \in [0,T]$ assuming that the underlying stock pays the dividend $aS(t_0^-)$ at time $t_0 \in (0,T)$. Then*

$$\Pi_Y^{(a,t_0)}(t) = \begin{cases} v_g(t,(1-a)S(t)), & \text{for } t < t_0, \\ v_g(t,S(t)), & \text{for } t \geq t_0, \end{cases}$$

where $v_g(t,x)$ is the Black-Scholes pricing function in the absence of dividends, which is given by (6.19b).

Proof. Using $\frac{S(T)}{S(t)} = e^{\alpha\tau+\sigma(W(T)-W(t))}$, we can rewrite (6.17) in the form

$$\Pi_Y(t) = e^{-r\tau}\mathbb{E}[g(S(T)e^{(r-\frac{\sigma^2}{2}-\alpha)\tau}]. \tag{6.34}$$

Taking the limit $s \to t_0^-$ in (6.32) and using the continuity of the paths of the Brownian motion we find

$$S(t_0^-) = S(t)e^{\alpha(t_0-t)+\sigma(W(t_0)-W(t))}, \quad t \in [0,t_0).$$

Replacing in (6.31) we obtain

$$S(t_0) = (1-a)S(t)e^{\alpha(t_0-t)+\sigma(W(t_0)-W(t))}, \quad t \in [0,t_0).$$

Hence, letting $(s,u) = (T,t_0)$ and $(s,u) = (T,t)$ into (6.33), we find

$$S(T) = \begin{cases} (1-a)S(t)e^{\alpha\tau+\sigma(W(T)-W(t))} & \text{for } t \in [0,t_0), \\ S(t)e^{\alpha\tau+\sigma(W(T)-W(t))} & \text{for } t \in [t_0,T]. \end{cases} \tag{6.35}$$

By the definition of Black-Scholes price in the form (6.34) and denoting $G = (W(T) - W(t))/\sqrt{\tau}$, we obtain

$$\Pi_Y^{(a,t_0)}(t) = e^{-r\tau}\mathbb{E}[g((1-a)S(t)e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}], \quad \text{for } t \in [0,t_0),$$

$$\Pi_Y^{(a,t_0)}(t) = e^{-r\tau}\mathbb{E}[g(S(t)e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}], \quad \text{for } t \in [t_0,T].$$

As $G \in \mathcal{N}(0,1)$, the result follows. □

We conclude that for $t \geq t_0$, i.e., after the dividend has been paid, the Black-Scholes price function of the derivative is again given by (6.19b), while for $t < t_0$ it is obtained by replacing x with $(1-a)x$ in (6.19b). To see the effect of this change, suppose that the derivative is a call option; let $c(t,x)$ be the Black-Scholes price function in the absence of dividends and $c_a(t,x)$ be the price function in the case that a dividend is paid at time t_0. Then, according to Theorem 6.7,

$$c_a(t,x) = \begin{cases} c(t,(1-a)x), & \text{for } t < t_0, \\ c(t,x), & \text{for } t \geq t_0. \end{cases}$$

Since $\partial_x c > 0$ (see Theorem 6.6), it follows that $c_a(t,x) < c(t,x)$, for $t < t_0$, that is to say, the payment of a dividend makes the call option on the stock less valuable (i.e., cheaper) than in the absence of dividends until the dividend is paid.

Exercise 6.8 (?). *Give an intuitive explanation for the property just proved for call options on a dividend paying stock.*

Exercise 6.9 (•). *A standard European derivative pays the amount* $Y = (S(T) - S(0))_+$ *at time of maturity* T*. Find the Black-Scholes price* $\Pi_Y(0)$ *of this derivative at time* $t = 0$ *assuming that the underlying stock pays the dividend* $(1 - e^{-rT})S(\frac{T}{2}-)$ *at time* $t = \frac{T}{2}$*. Compute the probability of positive return for a constant portfolio which is short 1 share of the derivative and short* $S(0)e^{-rT}$ *shares of the risk-free asset.*

Exercise 6.10. *Derive the Black-Scholes price of the derivative with pay-off* $Y = g(S(T))$*, assuming that the underlying pays a dividend at each time* $t_1 < t_2 < \cdots < t_M \in [0, T]$*. Denote by* a_i *the dividend paid at time* t_i*,* $i = 1, \dots, M$*.*

6.5 Local and Stochastic volatility models

In this and the next section we present a method to compute the risk-neutral price of European derivatives when the market parameters are not deterministic functions. We first assume in this section that the interest rate of the money market is constant, i.e., $R(t) = r$, which is quite reasonable for derivatives with short maturity such as options; stochastic interest rate models are important for pricing derivatives with very long time of maturity, such us zero-coupon bonds, which are briefly discussed below in Section 6.6 (see also [21, Section 6.5]). Assuming that the derivative is a standard European derivative with pay-off function g, the risk-neutral price formula (6.1) becomes

$$\Pi_Y(t) = e^{-r\tau}\widetilde{\mathbb{E}}[g(S(T))|\mathcal{F}_W(t)], \quad \tau = T - t. \tag{6.36}$$

Motivated by our earlier results on the Black-Scholes price, and Remark 6.4, we attempt to re-write the risk-neutral price formula in the form

$$\Pi_Y(t) = v_g(t, S(t)) \quad \text{for all } t \in [0, T], \text{ for all } T > 0, \tag{6.37}$$

for some function $v_g : \overline{\mathcal{D}_T^+} \to (0, \infty)$, which we call the pricing function of the derivative. By (6.36), this is equivalent to

$$\widetilde{\mathbb{E}}[g(S(T))|\mathcal{F}_W(t)] = e^{r\tau} v_g(t, S(t)) \tag{6.38}$$

i.e., to the property that $\{S(t)\}_{t\geq 0}$ is a Markov process in the risk-neutral probability measure $\widetilde{\mathbb{P}}$, relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$. At this point it remains to understand for which stochastic processes $\{\sigma(t)\}_{t\geq 0}$ does the generalized geometric Brownian motion (6.4) satisfies this Markov property. We have seen in Section 5.1 that this holds in particular when $\{S(t)\}_{t\geq 0}$ satisfies a (system of) stochastic differential equation(s), see Section 5.1. Next we discuss two examples which encompass most of the volatility models used in the applications: Local volatility models and Stochastic volatility models.

6.5.1 Local volatility models

A **local volatility model** is a special case of the generalized geometric Brownian motion in which the instantaneous volatility of the stock $\{\sigma(t)\}_{t\geq 0}$ is assumed to be a deterministic function of the stock price $S(t)$. Given a measurable function $\beta : [0,\infty)\times[0,\infty)\to(0,\infty)$, we then let

$$\sigma(t)S(t) = \beta(t, S(t)), \tag{6.39}$$

into (6.4), so that the stock price process $\{S(t)\}_{t\geq 0}$ satisfies the SDE

$$dS(t) = rS(t)\,dt + \beta(t, S(t))d\widetilde{W}(t), \quad S(0) = S_0 > 0. \tag{6.40}$$

We assume that this SDE admits a unique global solution, which is true in particular under the assumptions of Theorem 5.1. To this regard we observe that the drift term $\alpha(t,x) = rx$ in (6.40) satisfies both (5.3) and (5.4), hence these conditions restrict only the form of the function $\beta(t,x)$. In the following we shall also assume that the solution $\{S(t)\}_{t\geq 0}$ of (6.40) is non-negative a.s. for all $t > 0$. Note however that the stochastic process solution of (6.40) will in general hit zero with positive probability at any finite time. For example, letting $\beta(t,x) = \sqrt{x}$, the stock price is given by a CIR process (5.25) with $b = 0$ and so, according to Theorem 5.6, $S(t) = 0$ with positive probability for all $t > 0$.

Theorem 6.8. *Let $g \in C^2([0,\infty))$ (except possibly on finitely many points) such that g', g'' are uniformly bounded and assume that the Kolmogorov PDE*

$$\partial_t u + rx\partial_x u + \frac{1}{2}\beta(t,x)^2\partial_x^2 u = 0 \quad (t,x) \in \mathcal{D}_T^+, \tag{6.41}$$

associated to (6.40) *admits a (necessarily unique) strong solution in the region $\mathcal{D}_T^+$ satisfying $u(T,x) = g(x)$. Let also*

$$v_g(t,x) = e^{-r\tau}u(t,x).$$

Then we have the following.

(i) v_g satisfies

$$\partial_t v_g + rx\partial_x v_g + \frac{1}{2}\beta(t,x)^2\partial_x^2 v_g = rv_g \quad (t,x) \in \mathcal{D}_T^+, \tag{6.42}$$

and the terminal condition

$$v_g(T,x) = g(x). \tag{6.43}$$

(ii) The price of the European derivative with pay-off $Y = g(S(T))$ and maturity $T > 0$ is given by (6.37).

(iii) The portfolio given by

$$h_S(t) = \partial_x v_g(t, S(t)), \quad h_B(t) = (\Pi_Y(t) - h_S(t)S(t))/B(t)$$

is a self-financing hedging portfolio.

(iv) The put-call parity holds.

Proof. (i) It is straightforward to verify that v_g satisfies (6.42).
(ii) Let $X(t) = v_g(t, S(t))$. By Itô's formula we find

$$dX(t) = (\partial_t v_g(t, S(t)) + rS(t)\partial_x v_g(t, S(t)) + \frac{1}{2}\beta(t, S(t))^2 \partial_x^2 v_g(t, S(t)))dt \\ + \beta(t, S(t))\partial_x v_g(t, S(t))d\widetilde{W}(t).$$

Hence

$$d(e^{-rt}X(t)) = e^{-rt}(\partial_t v_g + rx\partial_x v_g + \frac{1}{2}\beta(t, x)^2 \partial_x^2 v_g - rv_g)(t, S(t))dt \\ + e^{-rt}\beta(t, S(t))\partial_x v_g(t, S(t))d\widetilde{W}(t).$$

As $v_g(t, x)$ satisfies (6.42), the drift term in the right hand side of the previous equation is zero. Hence

$$e^{-rt}v_g(t, S(t)) = v_g(t, S_0) + \int_0^t e^{-ru}\beta(u, S(u))\partial_x v_g(u, S(u))d\widetilde{W}(u). \quad (6.44)$$

It follows that[2] the stochastic process $\{e^{-rt}v_g(t, S(t))\}_{t\geq 0}$ is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$, i.e.,

$$\widetilde{\mathbb{E}}[e^{-rt_2}v_g(t_2, S(t_2))|\mathcal{F}_W(t_1)] = e^{-rt_1}v_g(t_1, S(t_1)), \quad \text{for all } 0 \leq t_1 \leq t_2 \leq T.$$

Letting $t_1 = t$, $t_2 = T$ and using the boundary condition (6.43), we find

$$v_g(t, S(t)) = e^{-r\tau}\widetilde{\mathbb{E}}[g(S(T))|\mathcal{F}_W(t)],$$

which proves (6.37).
(iii) Replacing $\Pi_Y(t) = v_g(t, S(t))$ into (6.44), we find

$$\Pi_Y^*(t) = \Pi_Y(0) + \int_0^t e^{-ru}\beta(u, S(u))\partial_x v_g(u, S(u))d\widetilde{W}(u).$$

Hence the claim on the hedging portfolio follows by Theorem 6.2.
(iv) Let $c(t, x)$ be the solution of (6.42) with terminal condition $g(x) = (x - K)_+$ and $p(t, x)$ be the solution with terminal condition $g(x) = (K - x)_+$, where $K > 0$ is the strike price of the call/put option. As (6.42) is linear, $c - p$ solves (6.42) with the terminal condition $(x - K)_+ - (K - x)_+ = (x - K) := h(x)$. The (unique) strong solution v_h of (6.42) is given by $v_h(t, x) = x - Ke^{-r(T-t)}$, hence

$$c(t, x) - p(t, x) = x - Ke^{-r(T-t)},$$

which is the put-call parity. □

Clearly, a closed formula for the solution of (6.41) is rarely available, hence to compute the price of the derivative one needs to rely on numerical methods, such as those discussed in Section 5.4.

[2] Recall that we assume that Itô's integrals are martingales!

Example: The CEV model

For the **constant elasticity variance (CEV)** model, we have $\beta(t,S(t)) = \sigma S(t)^\delta$, where $\sigma > 0$, $\delta > 0$ are constants. The SDE for the stock price becomes

$$dS(t) = rS(t)dt + \sigma S(t)^\delta d\widetilde{W}(t), \quad S(0) = S_0 > 0. \tag{6.45}$$

Note that for $\delta = 1$ one recovers the Black-Scholes model. For $\delta \neq 1$, we can construct the solution of (6.45) using a CIR process, as shown in the following exercise.

Exercise 6.11. *Given σ, r and $\delta \neq 1$, define*

$$a = 2r(\delta - 1),\ c = -2\sigma(\delta-1),\ b = \frac{\sigma^2}{2r}(2\delta - 1),\ \theta = -\frac{1}{2(\delta-1)}.$$

Let $\{X(t)\}_{t\geq 0}$ be the CIR process

$$dX(t) = a(b - X(t))\,dt + c\sqrt{X(t)}d\widetilde{W}(t), \quad X(0) = x > 0.$$

Show that $S(t) = X(t)^\theta$ solves (6.45) with $S_0 = x^\theta$.

It follows by Exercise 6.11, and by Feller's condition $ab \geq c^2/2$ for the positivity of the CIR process, that the solution of (6.45) remains strictly positive a.s. if $\delta \geq 1$, while for $0 < \delta < 1$, the stock price hits zero in finite time with positive probability.

The Kolmogorov PDE (6.41) associated to the CEV model is

$$\partial_t u + rx\partial_x u + \frac{\sigma^2}{2}x^{2\delta}\partial_x^2 u = 0,\ (t,x) \in \mathcal{D}_T^+.$$

Given a terminal value g at time T as in Theorem 5.4, the previous equation admits a unique solution. However a fundamental solution, in the sense of Theorem 5.5, exists only for $\delta > 1$, as otherwise the stochastic process $\{S(t)\}_{t\geq 0}$ hits zero at any finite time with positive probability and therefore the density of the random variable $S(t)$ has a discrete part. The precise form of the (generalized) density $f_{S(t)}(x)$ in the CEV model is known for all values of δ and are given for instance in [16]. An exact formula for call options can be found in [20].

6.5.2 Stochastic volatility models

For local volatility models, the stock price and the instantaneous volatility are both stochastic processes. However there is only one source of randomness which drives both these processes, namely a single Brownian motion $\{W(t)\}_{t\geq 0}$. The next level of generalization consists in assuming that the stock price and the volatility are driven by two different sources of randomness.

Definition 6.4. *Let $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ be two independent Brownian motions and $\{\mathcal{F}_W(t)\}_{t\geq 0}$ be their own generated filtration. Let $\rho \in [-1, 1]$ be a deterministic constant* *and $\mu, \eta, \beta : [0, \infty)^3 \to \mathbb{R}$ be continuous functions. A* ***stochastic volatility model*** *is a pair of (non-negative) stochastic diffusion processes $\{S(t)\}_{t\geq 0}$, $\{v(t)\}_{t\geq 0}$ satisfying the following system of SDE's:*

$$dS(t) = \mu(t, S(t), v(t))S(t)\, dt + \sqrt{v(t)}S(t)\, dW_1(t), \tag{6.46}$$

$$dv(t) = \eta(t, S(t), v(t))\, dt + \beta(t, S(t), v(t))\sqrt{v(t)}(\rho\, dW_1(t) + \sqrt{1-\rho^2}\, dW_2(t)). \tag{6.47}$$

We see from (6.46) that $\{v(t)\}_{t\geq 0}$ is the instantaneous variance of the stock price $\{S(t)\}_{t\geq 0}$. Moreover the process $\{W^{(\rho)}(t)\}_{t\geq 0}$ given by

$$W^{(\rho)}(t) = \rho\, W_1(t) + \sqrt{1-\rho^2}\, W_2(t)$$

is a Brownian motion satisfying

$$dW_1(t)dW^{(\rho)}(t) = \rho\, dt;$$

in particular the two Brownian motions $\{W_1(t)\}_{t\geq 0}$, $\{W^{(\rho)}\}_{t\geq 0}$ are not independent, as their cross variation is not zero (in fact, by Exercise 4.5, ρ is the correlation of the two Brownian motions). Hence in a stochastic volatility model the stock price and the volatility are both stochastic processes driven by two correlated Brownian motions. We assume that $\{S(t)\}_{t\geq 0}$ is non-negative and $\{v(t)\}_{t\geq 0}$ is positive a.s. for all times, although we refrain from discussing under which general conditions this property holds (we will present an example below).

Our next purpose is to introduce a risk-neutral probability measure such that the discounted price of the stock is a martingale. As we have two Brownian motions in this model, we shall apply the two-dimensional Girsanov Theorem 4.12 to construct such a probability measure. Precisely, let $r > 0$ be the constant interest rate of the money market and $\gamma : [0, \infty)^3 \to \mathbb{R}$ be a continuous function We define

$$\theta_1(t) = \frac{\mu(t, S(t), v(t)) - r}{\sqrt{v(t)}}, \quad \theta_2(t) = \gamma(t, S(t), v(t)), \quad \theta(t) = (\theta_1(t), \theta_2(t)).$$

Given $T > 0$, we introduce the new probability measure $\widetilde{\mathbb{P}}^{(\gamma)}$ equivalent to $\mathbb{P}$ by $\widetilde{\mathbb{P}}^{(\gamma)}(A) = \mathbb{E}[Z(T)\mathbb{I}_A]$, for all $A \in \mathcal{F}$, where

$$Z(t) = \exp\left(-\int_0^t \theta_1(s)\, dW_1(s) - \int_0^t \theta_2(s)\, dW_2(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2\, ds\right).$$

Then by Theorem 4.12, the stochastic processes

$$\widetilde{W}_1(t) = W_1(t) + \int_0^t \theta_1(s)\, ds, \quad \widetilde{W}_2^{(\gamma)}(t) = W_2(t) + \int_0^t \gamma(s)\, ds$$

are two $\widetilde{\mathbb{P}}^{(\gamma)}$-independent Brownian motions. Moreover (6.46)-(6.47) can be rewritten as

$$dS(t) = rS(t)dt + \sqrt{v(t)}S(t)d\widetilde{W}_1(t), \tag{6.48a}$$

$$\begin{aligned} dv(t) = [&\eta(t, S(t), v(t)) - \sqrt{v(t)}\psi(t, S(t), v(t))\beta(t, S(t), v(t))]dt \\ &+ \beta(t, S(t), v(t))\sqrt{v(t)}d\widetilde{W}^{(\rho,\gamma)}, \end{aligned} \tag{6.48b}$$

where $\{\psi(t,S(t),v(t))\}_{t\geq 0}$ is the $\{\mathcal{F}_W(t)\}_{t\geq 0}$-adapted stochastic process given by

$$\psi(t,S(t),v(t)) = \frac{\mu(t,S(t),v(t))-r}{\sqrt{v(t)}}\rho + \gamma(t,S(t),v(t))\sqrt{1-\rho^2} \tag{6.49}$$

and where

$$\widetilde{W}^{(\rho,\gamma)}(t) = \rho\widetilde{W}_1(t) + \sqrt{1-\rho^2}\widetilde{W}_2^{(\gamma)}(t).$$

Note that the $\widetilde{\mathbb{P}}^{(\gamma)}$-Brownian motions $\{\widetilde{W}_1(t)\}_{t\geq 0}$, $\{\widetilde{W}^{(\rho,\gamma)}(t)\}_{t\geq 0}$ satisfy

$$d\widetilde{W}_1(t)d\widetilde{W}^{(\rho,\gamma)}(t) = \rho dt, \quad \text{for } \rho\in[-1,1]. \tag{6.50}$$

It follows immediately that the discounted price $\{e^{-rt}S(t)\}_{t\geq 0}$ is a $\widetilde{\mathbb{P}}^{(\gamma)}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$. Hence *all* probability measures $\widetilde{\mathbb{P}}^{(\gamma)}$ are equivalent risk-neutral probability measures.

Remark 6.8 (Incomplete markets)**.** As the risk-neutral probability measure is not uniquely defined, the market under discussion is said to be **incomplete**. Within incomplete markets there is no unique value for the price of derivatives. The stochastic process $\{\psi(t)\}_{t\geq 0}$ is called the **market price of volatility risk** and reduces to (6.3) for $\gamma\equiv 0$ (or $\rho=1$).

Consider now the standard European derivative with pay-off $Y=g(S(T))$ at time of maturity T. For stochastic volatility models it is reasonable to assume that the risk-neutral price $\Pi_Y(t)$ of the derivative is a local function of the stock price *and* of the instantaneous variance, i.e., we make the following *ansatz* which generalizes (6.37):

$$\Pi_Y(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}[g(S(T))|\mathcal{F}_W(t)] = v_g(t,S(t),v(t)) \tag{6.51}$$

for all $t\in[0,T]$, for all $T>0$ and for some measurable pricing v_g. Of course, as in the case of local volatility model, (6.51) is motivated by the Markov property of solutions to systems of SDE's. In fact, it is useful to consider a more general European derivative with pay-off Y given by

$$Y = h(S(T),v(T)),$$

for some function $h:[0,\infty)^2\to\mathbb{R}$, i.e., the pay-off of the derivative depends on the stock value *and* on the instantaneous variance of the stock at the time of maturity. We have the following analogue of Theorem 6.8.

Theorem 6.9. *Assume that the functions $\eta(t,x,y)$, $\beta(t,x,y)$, $\psi(t,x,y)$ in (6.48) are such that the PDE*

$$\partial_t u + rx\partial_x u + A\partial_y u + \frac{1}{2}yx^2\partial_x^2 u + \frac{1}{2}\beta^2 y\partial_y^2 u + \rho\beta xy\partial_{xy}^2 u = 0, \tag{6.52a}$$

$$A = \eta - \sqrt{y}\beta\psi, \quad (t,x,y)\in(0,T)\times(0,\infty)^2 \tag{6.52b}$$

admits a unique strong solution u satisfying $u(0,x,y)=h(x,y)$. Then the risk-neutral price of the derivative with pay-off $Y=h(S(T),\sigma(T))$ and maturity T is given by

$$\Pi_Y(t) = v_h(t,S(t),\sigma(t))$$

where the pricing function v_h is given by $v_h(t,x,y)=e^{-r\tau}u(t,x,y)$, $\tau=T-t$.

Exercise 6.12. *Prove the theorem. Hint: use Itô's formula in two dimensions, see Theorem 4.9, and the argument in the proof of Theorem 6.8.*

As for the local volatility models, a closed formula solution of (6.52) is rarely available and the use numerical methods to price the derivative becomes essential.

Heston model

The most popular stochastic volatility model is the **Heston** model, which is obtained by the following substitutions in (6.46)-(6.47):

$$\mu(t, S(t), v(t)) = \alpha_0, \quad \beta(t, x, y) = c, \quad \eta(t, x, y) = a(b - y),$$

where μ_0, a, b, c are constant. Hence the stock price and the volatility dynamics in the Heston model are given by the following stochastic differential equations:

$$dS(t) = \mu_0 S(t)\, dt + \sqrt{v(t)} S(t) dW_1(t), \tag{6.53a}$$

$$dv(t) = a(b - v(t))dt + c\sqrt{v(t)} dW^{(\rho)}(t). \tag{6.53b}$$

Note in particular that the volatility in the Heston model is CIR process, see (5.25). The condition $2ab > c^2$ ensures that $v(t)$ is strictly positive (almost surely). To pass to the risk neutral world we need to fix a risk-neutral probability measure, that is, we need to fix the market price of volatility risk function ψ in (6.49). In the Heston model it is assumed that

$$\psi(t, x, y) = \lambda\sqrt{y},$$

for some constant $\lambda \in \mathbb{R}$, which leads to the following form of the pricing PDE (6.52):

$$\partial_t u + rx\partial_x u + (k - my)\partial_y u + \frac{1}{2} y x^2 \partial_x^2 u + \frac{c^2}{2} y \partial_y^2 u + \rho c x y \partial_{xy}^2 u = ru, \tag{6.54}$$

where the constant k, m are given by $k = ab$, $m = (a + c\lambda)$.

The general solution of (6.54) with terminal datum $u(T, x, y) = h(x, y)$ is not known. However in the case of a call option (i.e., $h(x, y) = g(x) = (x - K)_+$) an explicit formula for the Fourier transform of the solution is available, see [12]. The existence of such formula, which permits to compute the price of call options by very efficient numerical methods, is one of the main reasons for the popularity of the Heston model.

6.5.3 Variance swaps

Variance swaps are financial derivatives[3] on the realized annual variance of an asset (or index). We first describe how the realized annual variance is computed from the historical data of the asset price. Let $T > 0$ (measure in days) and consider the partition

$$0 = t_0 < t_1 < \cdots < t_n = T, \quad t_j = \frac{jT}{n}$$

[3]More precisely, forward contracts, see Section 6.7.

of the interval $[0,T]$. Assume for instance that the asset is a stock and let $S(t_j) = S_j$ be the stock price at time t_j. Here $S_1, \dots S_n$ are historical data for the stock price and *not* random variables (i.e., the interval $[0,T]$ lies in the past of the present time). The **realized annual variance** of the stock in the interval $[0,T]$ along this partition is defined as

$$\sigma^2_{1\text{year}}(n,T) = \frac{\kappa}{T}\sum_{j=0}^{n-1}\left(\log\left(\frac{S_{j+1}}{S_j}\right)\right)^2,$$

where κ is the number of trading days in one year (typically, $\kappa = 252$). A **variance swap** stipulated at time $t = 0$, with maturity T and strike variance K is a contract between two parties which, at the expiration date, entails the exchange of cash given by $N(\sigma^2_{1\text{year}} - K)$, where N (called **variance notional**) is a conversion factor from units of variance to units of currency. In particular, the holder of the long position on the swap is the party who receives the cash in the case that the realized annual variance at the expiration date is larger than the strike variance. Variance swaps are traded over the counter and they are used by investors to protect their exposure to the volatility of the asset. For instance, suppose that an investor has a position on an asset which is profitable if the volatility of the stock price increases (e.g., the investor owns call options on the stock). Then it is clearly important for the investor to secure such position against a possible decrease of the volatility. To this purpose the investor opens a short position on a variance swap with another investor who is exposed to the opposite risk.

Let us now discuss variance swaps from a mathematical modeling point of view. We assume that the stock price follows the generalized geometric Brownian motion

$$S(t) = S(0)\exp\left(\int_0^t \alpha(s)ds + \int_0^t \sigma(s)dW(s)\right).$$

Moreover, for modeling purposes, it is convenient to assume that $n \to \infty$ and to introduce, as an (unbiased) estimate for the realized annual variance in the future time interval $[0,T]$, the random variable

$$Q_T = \frac{\kappa}{T}[\log S, \log S](T) = \frac{\kappa}{T}\int_0^T \sigma^2(t)\,dt.$$

In fact, by the definition of quadratic variation, it follows that

$$\mathbb{E}\left[\left(\frac{\kappa}{T}\sum_{j=0}^{n-1}\left(\log\left(\frac{S_{j+1}}{S_j}\right)\right)^2 - Q_T\right)^2\right] \to 0, \quad \text{as } n \to \infty.$$

A variance swap can thus be defined as the (non-standard) European derivative with pay-off $Y = Q_T - K$. Assuming that the interest rate of the bond is constant, $R(t) = r > 0$, the risk-neutral value of a variance swap is given by

$$\Pi_Y(t) = e^{-r\tau}\widetilde{\mathbb{E}}[Q_T - K|\mathcal{F}_W(t)]. \tag{6.55}$$

In particular, at time $t = 0$, i.e., when the contract is stipulated, we have

$$\Pi_Y(0) = e^{-rT}\widetilde{\mathbb{E}}[Q_T - K]. \tag{6.56}$$

where we used that $\mathcal{F}_W(0)$ is a trivial σ-algebra, and therefore the conditional expectation with respect to $\mathcal{F}_W(0)$ is a pure expectation. As none of the two parties in a variance swap has a privileged position on the contract, there is no premium associated to variance swaps, that is to say, the fair value of a variance swap is zero[4]. The value K_* of the variance strike which makes the risk-neutral price of a variance swap equal to zero at time $t = 0$, i.e., $\Pi_Y(0) = 0$, is called the **fair variance strike**. By (6.56) we find

$$K_* = \frac{\kappa}{T}\int_0^T \widetilde{\mathbb{E}}[\sigma^2(t)]\,dt, \tag{6.57}$$

To compute K_* explicitly, we need to fix a stochastic model for the variance process $\{\sigma^2(t)\}_{t\geq 0}$. Let us consider the Heston model

$$d\sigma^2(t) = a(b - \sigma^2(t))dt + c\sigma(t)d\widetilde{W}(t), \tag{6.58}$$

where a, b, c are positive constants satisfying $2ab > c^2$ and where $\{\widetilde{W}(t)\}_{t\geq 0}$ is a Brownian motion in the risk-neutral measure. To compute the fair variance strike of a swap using the Heston model we use that

$$\widetilde{\mathbb{E}}[\sigma^2(t)] = abt - a\int_0^t \widetilde{\mathbb{E}}[\sigma^2(s)]\,ds,$$

which implies $\frac{d}{dt}\widetilde{\mathbb{E}}[\sigma^2(t)] = ab - a\widetilde{\mathbb{E}}[\sigma^2(t)]$ and so

$$\widetilde{\mathbb{E}}[\sigma^2(t)] = b + (\sigma_0^2 - b)e^{-at}, \quad \sigma_0^2 = \widetilde{\mathbb{E}}[\sigma^2(0)] = \sigma^2(0). \tag{6.59}$$

Replacing into (6.57) we obtain

$$K_* = \kappa\left[b + \frac{\sigma_0^2 - b}{aT}(1 - e^{-aT})\right].$$

Exercise 6.13. *Assume $R = r > 0$, $\alpha =$ constant. Moreover, given $\sigma_0 > 0$, let $\sigma(t) = \sigma_0\sqrt{S(t)}$, which is an example of CEV model. Compute the fair strike of the variance swap.*

Exercise 6.14 (•). *Assume that the price $S(t)$ of a stock follows a generalized geometric Brownian motion with instantaneous volatility $\{\sigma(t)\}_{t\geq 0}$ given by the Heston model $d\sigma^2(t) = a(b - \sigma^2(t))\,dt + c\sigma(t)\,d\widetilde{W}(t)$, where $\{\widetilde{W}(t)\}_{t\geq 0}$ is a Brownian motion in the risk-neutral probability measure and a, b, c are constants such that $2ab > c^2 > 0$. A volatility call option with strike K and maturity T is a financial derivative with pay-off*

$$Y = N\left(\sqrt{\frac{\kappa}{T}\int_0^T \sigma^2(t)\,dt} - K\right)_+,$$

[4] This is a general property of forward contracts, see Section 6.7.

where κ is the number of trading days in one year and N is a dimensional constant that converts units of volatility into units of currency. Assuming that the risk-neutral price $\Pi_Y(t)$ of this derivative has the form

$$\Pi_Y(t) = f(t, \sigma^2(t), \int_0^t \sigma(s)^2 \, ds)$$

and that the interest rate of the risk-free asset is constant, find the partial differential equation and the terminal value satisfied by the pricing function f.

6.6 Interest rate models

In this section we discuss an example of stochastic model for the interest rate $\{R(t)\}_{t\geq 0}$. The general theory of interest rate models is among the most studied and important for the applications. For a throughout discussion on this topic, see [3]. Here we just briefly consider the Cox-Ingersoll-Ross (CIR) model, where $\{R(t)\}_{t\geq 0}$ is given by the CIR process

$$dR(t) = a(b - R(t))dt + c\sqrt{R(t)}d\widetilde{W}(t), \quad R(0) = r_0, \tag{6.60}$$

where $\{\widetilde{W}(t)\}_{t\in[0,T]}$ is a Brownian motion in the risk-neutral probability measure and r_0, a, b, c are positive constants such that $2ab > c^2$. In particular, the interest rate $R(t)$ is always strictly positive. Our purpose is to compute the value at time t of a zero-coupon bond when the interest rate is given by the CIR model. A zero-coupon bond with **face value** K and maturity time T is a financial derivative which promises to pay to its owner the amount K at time T. The term "zero-coupon" refers to the fact that no intermediate payments are made to the owner; in particular, a zero-coupon bond is a European derivative and therefore its risk-neutral price is given by

$$\Pi_Y(t) = \widetilde{\mathbb{E}}[K \exp(-\int_t^T R(s) \, ds)|\mathcal{F}_W(t)] = KB(t,T),$$

where

$$B(t,T) = \widetilde{\mathbb{E}}[\exp(-\int_t^T R(s) \, ds)|\mathcal{F}_W(t)] \tag{6.61}$$

is the risk-neutral value of a zero-coupon bond with face value 1. To compute $B(t,T)$ under a CIR interest rate model, we make the *ansatz*

$$B(t,T) = f(t, R(t)), \tag{6.62}$$

for some smooth function $f : [0,T] \times (0,\infty) \to \mathbb{R}$, which we want to find. Note that the *ansatz* (6.62) does *not* correspond to a classical Markov property for the interest rate, as the random variable $\exp(-\int_t^T R(s) \, ds)$ need not be, in general, a function of $R(T)$.

Theorem 6.10. *When the interest rate $\{R(t)\}_{t>0}$ follows the CIR model (6.60), the value*

$B(t,T)$ of the zero-coupon bond is given by (6.62) *with*

$$f(t,x) = e^{-xC(T-t)-A(T-t)}, \tag{6.63a}$$

where

$$C(\tau) = \frac{\sinh(\gamma\tau)}{\gamma\cosh(\gamma\tau) + \frac{1}{2}a\sinh(\gamma\tau)} \tag{6.63b}$$

$$A(\tau) = -\frac{2ab}{c^2}\log\left[\frac{\gamma e^{\frac{1}{2}a\tau}}{\gamma\cosh(\gamma\tau) + \frac{1}{2}a\sinh(\gamma\tau)}\right] \tag{6.63c}$$

and

$$\gamma = \frac{1}{2}\sqrt{a^2 + 2c^2}. \tag{6.63d}$$

Proof. Using Itô's formula and product rule, together with (6.60), we obtain

$$\begin{aligned} d(D(t)f(t,R(t)) = D(t)[&\partial_t f(t,R(t)) + a(b - R(t))\partial_x f(t,R(t)) \\ &+ \frac{c^2}{2}R(t)\partial_x^2 f(t,R(t)) - R(t)f(t,R(t))]dt \\ &+ D(t)\partial_x f(t,R(t))c\sqrt{R(t)}d\widetilde{W}(t). \end{aligned}$$

Hence, imposing that f be a solution of the PDE

$$\partial_t f + a(b-x)\partial_x f + \frac{c^2}{2}x\partial_x^2 f = xf, \quad (t,x) \in \mathcal{D}_T^+, \tag{6.64a}$$

we obtain that the stochastic process $\{D(t)f(t,R(t))\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$. Imposing additionally the terminal condition

$$f(T,x) = 1, \quad \text{for all } x > 0, \tag{6.64b}$$

we obtain

$$D(t)f(t,R(t)) = \widetilde{\mathbb{E}}[D(T)f(T,R(T))|\mathcal{F}_W(t)] = \widetilde{\mathbb{E}}[D(T)|\mathcal{F}_W(t)],$$

hence

$$f(t,R(t)) = \widetilde{\mathbb{E}}[D(T)/D(t)|\mathcal{F}_W(t)],$$

and thus (6.62) is verified. It can be shown that (5.32) is the solution of the terminal value problem (6.64). □

Exercise 6.15. *Derive the solution of the problem* (6.64). *HINT: Use the ansatz* (6.63a).

6.7 Forwards and Futures

6.7.1 Forwards

A **forward contract** with **delivery price** K and maturity (or delivery) time T on an asset $\mathcal{U}$ is a type of financial derivative stipulated by two parties in which one of the parties promises to sell and deliver to the other party the asset $\mathcal{U}$ at time T in exchange for the cash K. As opposed to option contracts, both parties in a forward contract are *obliged* to fulfill their part of the agreement. Forward contracts are traded over the counter and most commonly on commodities or currencies. Let us give two examples.

Example of forward contract on a commodity. Consider a farmer who grows wheat and a miller who needs wheat to produce flour. Clearly, the farmer interest is to sell the wheat for the highest possible price, while the miller interest is to pay the least possible for the wheat. The price of the wheat depends on many economical and non-economical factors (such as whether conditions, which affect the quality and quantity of harvests) and it is therefore quite volatile. The farmer and the miller then stipulate a forward contract on the wheat in the winter (before the plantation, which occurs in the spring) with expiration date in the end of the summer (when the wheat is harvested), in order to lock its future trading price beforehand.

Example of forward contract on a currency. Suppose that a car company in Sweden promises to deliver a stock of 100 cars to another company in the United States in exactly one month. Suppose that the price of each car is fixed in Swedish crowns, say 100.000 crowns. Clearly the American company will benefit by an increase of the exchange rate crown/dollars and will be damaged in the opposite case. To avoid possible high losses, the American company by a forward contract on 100×100.000=ten millions Swedish crowns expiring in one month which gives the company the right *and* the obligation to buy ten millions crowns for a price in dollars agreed upon today.

Remark 6.9. As it is clear from the examples above, one of the purposes of forward contracts is to share risks.

The delivery price of a forward contract is agreed by the two parties after a careful analysis of several factors that may influence the future value of the asset (including logistic factors, such as the cost of delivery, storage, etc.). For this reason, the delivery price K in a forward contract may also be viewed as a pondered estimation for the price of the asset at the time T in the future. In this respect, K is also called the **forward price** of $\mathcal{U}$. More precisely, the T-forward price of the asset $\mathcal{U}$ at time t is the strike price of a forward contract on $\mathcal{U}$ with maturity T stipulated at time t; the current, actual price $\Pi(t)$ of the asset is also called the **spot** price.

Remark 6.10. As the consensus on the forward price is limited to the participants of the forward contract, it is unlikely to be accepted by all investors as a good estimation for the price of the asset at time T. The delivery price of futures contracts on the asset, which we define in Section 6.7.2, gives a better and more commonly accepted estimation for the future value of an asset.

Let us apply the risk-neutral pricing theory introduced in Section 6.2 to derive a mathematical model for the forward price of an asset. Let $f(t,\Pi(t),K,T)$ be the value at time t of a forward contract on an asset with price $\{\Pi(t)\}_{t\in[0,T]}$, maturity T and delivery price K. The pay-off for the party agreeing to buy the asset is given by

$$Y = (\Pi(T) - K),$$

while the pay-off for the party selling the asset is $(K - \Pi(T))$.

Remark 6.11. Note that one of the two parties in a forward contract is always going to incur in a loss. If this loss is very large, then this party could become insolvent, i.e., unable to fulfill the contract, and then both parties will end up loosing. In a futures contract this is prevented by the mechanism of margine accounts, see Section 6.7.2.

As both parties in a forward contract have the same rights/obligations, none of them pays a premium to stipulate the contract, and so $f(t, \Pi(t), K, T) = 0$. Assuming that the price $\{\Pi(t)\}_{t\geq 0}$ of the underlying asset follows a generalized geometric Brownian motion with strictly positive volatility, the risk-neutral value of the forward contract for the two parties is

$$\begin{aligned} f(t, \Pi(t), K, T) &= \pm\widetilde{\mathbb{E}}[(\Pi(T) - K)D(T)/D(t)|\mathcal{F}_W(t)] \\ &= \pm\left(\frac{1}{D(t)}\widetilde{\mathbb{E}}(\Pi(T)D(T)|\mathcal{F}_W(t)) - K\widetilde{\mathbb{E}}[\exp(-\int_t^T R(s)\,ds)|\mathcal{F}_W(t)]\right). \end{aligned}$$

As the discounted price $\{\Pi^*(t)\}_{t\geq 0}$ of the underlying asset is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$, then $\widetilde{\mathbb{E}}(\Pi(T)D(T)|\mathcal{F}_W(t)) = D(t)\Pi(t)$. Letting

$$B(t, T) = \widetilde{\mathbb{E}}[\exp(-\int_t^T R(s)\,ds)|\mathcal{F}_W(t)], \tag{6.65}$$

the value of the forward contract becomes

$$f(t, \Pi(t), K, T) = \pm(\Pi(t) - KB(t, T)).$$

This leads to the following definition.

Definition 6.5. *Assume that the price $\{\Pi(t)\}_{t\geq 0}$ of an asset and the value of the bond satisfy*

$$d\Pi(t) = \alpha(t)\Pi(t)dt + \sigma(t)\Pi(t)dW(t), \quad dB(t) = B(t)R(t)dt,$$

where $\{\alpha(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, and $\{R(t)\}_{t\geq 0}$ are adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and $\sigma(t) > 0$ almost surely for all times. The **risk-neutral T-forward price** *at time t of the asset is the $\{\mathcal{F}_W(t)\}_{t\geq 0}$-adapted stochastic process $\{\mathrm{For}_T(t)\}_{t\in[0,T]}$ given by*

$$\mathrm{For}_T(t) = \frac{\Pi(t)}{B(t, T)}, \quad t \in [0, T]$$

where $B(t, T)$ is given by (6.65).

Remark 6.12. The value $B(t, T)$ is the risk-neutral price at time t of a European derivative with pay-off 1 at the time of maturity T. This type of derivative is called a zero-coupon bond and is discussed in more details in Section 6.6.

Note that the forward price increases with respect to the time left to delivery, $\tau = T - t$, i.e., the longer we delay the delivery of the asset, the more we have to pay for it. This is intuitive, as the seller of the asset is loosing money by not selling the asset on the spot (due to its devaluation compared to the bond value). As a way of example, suppose that the interest rate of the bond is a deterministic constant, $R(t) = r > 0$. Then the forward price becomes

$$\mathrm{For}_T(t) = e^{r\tau}\Pi(t),$$

in which case we find that the spot price of an asset is the discounted value of the forward price. When the asset is a commodity (e.g., corn), the forward price is also inflated by the cost of storage. Letting $c > 0$ be the cost to storage one share of the asset for one year, then the forward price of the asset, for delivery in τ years in the future, is $e^{c\tau}e^{r\tau}\Pi(t)$.

6.7.2 Futures

Futures contracts are standardized forward contracts, i.e., rather than being traded over the counter, they are negotiated in regularized markets. Perhaps the most interesting role of futures contracts is that they make trading on commodities possible for anyone. To this regard we remark that commodities, e.g. crude oil, wheat, etc, are most often sold through long term contracts, such as forward and futures contracts, and therefore they do not usually have an “official spot price”, but only a future delivery price (commodities “spot markets” exist, but their role is marginal for the discussion in this section).

Futures markets are markets in which the objects of trading are futures contracts. Unlike forward contracts, all futures contracts in a futures market are subject to the same regulation, and so in particular all contracts on the same asset with the same delivery time T have the same delivery price, which is called the T-**future price** of the asset and which we denote by $\mathrm{Fut}_T(t)$. Thus $\mathrm{Fut}_T(t)$ is the delivery price in a futures contract on the asset with time of delivery T and which is stipulated at time $t < T$. Futures markets have been existing for more than 300 years and nowadays the most important ones are the Chicago Mercantile Exchange (CME), the New York Mercantile Exchange (NYMEX), the Chicago Board of Trade (CBOT) and the International Exchange Group (ICE).

In a futures market, anyone (after a proper authorization) can stipulate a futures contract. More precisely, holding a position in a futures contract in the futures market consists in the agreement to receive as a cash flow the change in the future price of the underlying asset during the time in which the position is held. Notice that the cash flow may be positive or negative. In a long position the cash flow is positive when the future price goes up and it is negative when the future price goes down , while a short position on the same contract receives the opposite cash flow. Moreover, in order to eliminate the risk of insolvency, the cash flow is distributed in time through the mechanism of the **margin account**. More precisely, assume that at $t = 0$ we open a long position in a futures contract expiring at time T. At the same time, we need to open a margin account which contains a certain amount of cash (usually, 10 % of the current value of the T-future price for each contract opened). At $t = 1$ day, the amount $\mathrm{Fut}_T(1) - \mathrm{Fut}_T(0)$ will be added to the account, if it positive, or

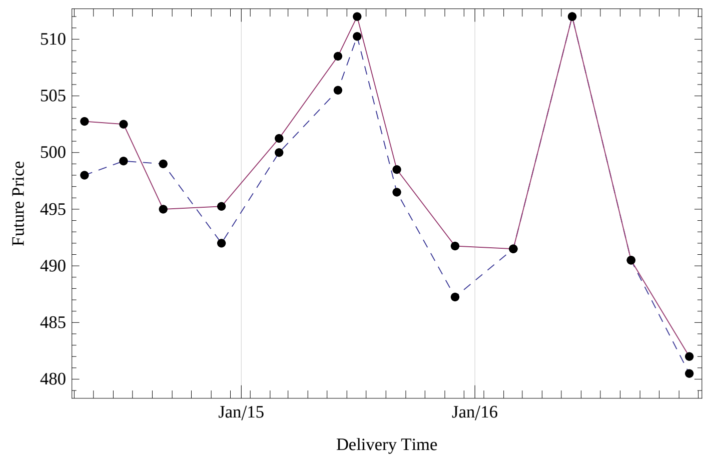


Figure 6.4: Futures price of corn on May 12, 2014 (dashed line) and on May 13, 2014 (continuous line) for different delivery times

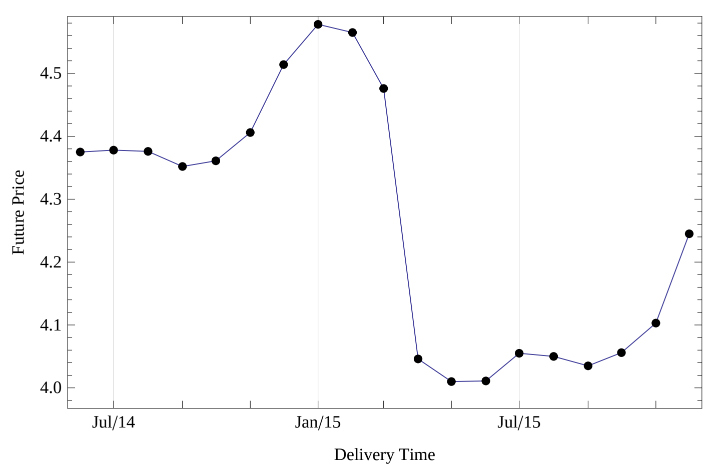


Figure 6.5: Futures price of natural gas on May 13, 2014 for different delivery times

withdrawn, if it is negative. The position can be closed at any time $t < T$ (multiple of days), in which case the total amount of cash flown in the margin account is

$$(\mathrm{Fut}_T(t) - \mathrm{Fut}_T(t-1)) + (\mathrm{Fut}_T(t-1) - \mathrm{Fut}_T(t-2)) + \\ \cdots + (\mathrm{Fut}_T(1) - \mathrm{Fut}_T(0)) = (\mathrm{Fut}_T(t) - \mathrm{Fut}_T(0)).$$

(In fact, if the margin account becomes too low, and the investor does not add new cash to it, the position will be automatically closed by the exchange market). If a long position is held up to the time of maturity, then the holder of the long position should buy the underlying asset. However in the majority of cases futures contracts are **cash settled** and not **physically settled**, i.e., the delivery of the underlying asset rarely occurs, and the equivalent value in cash is paid instead.

Remark 6.13. Since a futures contract can be closed at any time prior to expiration, future contracts are *not* European style derivatives.

Our next purpose is to derive a mathematical model for the future price of an asset. Our guiding principle is that the **1+1 dimensional futures market** consisting of a futures contract and a risk-free asset should not admit self-financing arbitrage portfolios. Consider a portfolio invested in $h(t)$ shares of the futures contract and $h_B(t)$ shares of the risk-free asset at time t. We assume that $\{h(t), h_B(t)\}_{t\in[0,T]}$ is adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and suppose that $\{\mathrm{Fut}_T(t)\}_{t\in[0,T]}$ is an Itô's process. Since futures contracts have zero-value, the value of the portfolio at time t is $V(t) = h_B(t)B(t) + h(t)C(t)$, where $C(t)$ is the cash-flow generated by each futures contract up to time t. For a self-financing portfolio we require that any positive cash-flow in the interval $[t, t+dt]$ should be invested to buy shares of the bond and that, conversely, any negative cash flow should be settled by issuing shares of the bond (i.e., by borrowing money). Since the cash-flow generated in the interval $[t, t+dt]$ is given by $dC(t) = h(t)d\mathrm{Fut}_T(t)$, the value of a self-financing portfolio invested in the 1+1 dimensional futures market must satisfy

$$dV(t) = h(t)d\mathrm{Fut}_T(t) + R(t)V(t)dt,$$

or equivalently

$$dV^*(t) = h(t)D(t)d\mathrm{Fut}_T(t). \tag{6.66}$$

Now, we have seen that a simple condition ensuring that a portfolio is not an arbitrage is that its discounted value be a martingale in the risk-neutral measure relative to the filtration generated by the Brownian motion. By (6.66), the latter condition is achieved by requiring that $d\mathrm{Fut}_T(t) = \Delta(t)d\widetilde{W}(t)$, for some stochastic process $\{\Delta(t)\}_{t\in[0,T]}$ adapted to $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$. In particular, it is reasonable to impose that

(i) $\{\mathrm{Fut}_T(t)\}_{t\in[0,T]}$ should be a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

Furthermore, it is clear that the future price of an asset at the expiration date T should be equal to its spot price at time T, and so we impose that

(ii) $\text{Fut}_T(T) = \Pi(T)$.

It follows by Exercise 3.31 that the conditions **(i)-(ii)** determine a unique stochastic process $\{\text{Fut}_T(t)\}_{t\in[0,T]}$, which is given in the following definition.

Definition 6.6. *Assume that the price $\{\Pi(t)\}_{t\geq 0}$ of an asset and the value of the bond satisfy*

$$d\Pi(t) = \alpha(t)\Pi(t)dt + \sigma(t)\Pi(t)dW(t), \quad dB(t) = B(t)R(t)dt,$$

*where $\{\alpha(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, and $\{R(t)\}_{t\geq 0}$ are adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and $\sigma(t) > 0$ almost surely for all times. The T-**Future price** at time t of the asset is the $\{\mathcal{F}_W(t)\}_{t\geq 0}$-adapted stochastic process $\{\text{Fut}_T(t)\}_{t\in[0,T]}$ given by*

$$\text{Fut}_T(t) = \widetilde{\mathbb{E}}[\Pi(T)|\mathcal{F}_W(t)], \quad t \in [0,T].$$

We now show that our goal to make the futures market arbitrage-free has been achieved.

Theorem 6.11. *There exists a stochastic process $\{\Delta(t)\}_{t\in[0,T]}$ adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ such that*

$$\text{Fut}_T(t) = \text{Fut}_T(0) + \int_0^t \Delta(s)d\widetilde{W}(s). \tag{6.67}$$

Moreover, any $\{\mathcal{F}_W(t)\}_{t\geq 0}$-adapted self-financing portfolio $\{h(t), h_B(t)\}_{t\in[0,T]}$ invested in the 1+1 dimensional futures market is not an arbitrage.

Proof. The second statement follows immediately by the first one, since (6.66) and (6.67) imply that the value of a self-financing portfolio invested in the 1+1 dimensional futures market is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$. To prove (6.67), we first notice that, by (3.24),

$$Z(s)\widetilde{\mathbb{E}}[\text{Fut}_T(t)|\mathcal{F}_W(s)] = \mathbb{E}[Z(t)\text{Fut}_T(t)|\mathcal{F}_W(s)].$$

By the martingale property of the future price, the left hand side is $Z(s)\text{Fut}_T(s)$. Hence

$$Z(s)\text{Fut}_T(s) = \mathbb{E}[Z(t)\text{Fut}_T(t)|\mathcal{F}_W(s)],$$

that is to say, the process $\{Z(t)\text{Fut}_T(t)\}_{t\in[0,T]}$ is a $\mathbb{P}$-martingale relative to the filtration $\{F_W(t)\}_{t\in[0,T]}$. By the martingale representation theorem, Theorem 4.6, there exists a stochastic process $\{\Gamma(t)\}_{t\in[0,T]}$ adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ such that

$$Z(t)\text{Fut}_T(t) = \text{Fut}_T(0) + \int_0^t \Gamma(s)dW(s).$$

We now proceed as in the proof of Theorem 6.2, namely we write

$$d\text{Fut}_T(t) = d(Z(t)\text{Fut}_T(t)Z(t)^{-1})$$

and apply Itô's product rule and Itô's formula to derive that (6.67) holds with

$$\Delta(t) = \theta(t)\text{Fut}_T(t) + \frac{\Gamma(t)}{Z(t)}.$$

□

Theorem 6.12. *The* **Forward-Future spread** *of an asset, i.e., the difference between its forward and future price, satisfies*

$$\mathrm{For}_T(t) - \mathrm{Fut}_T(t) = \frac{1}{\widetilde{\mathbb{E}}[D(T)|\mathcal{F}_W(t)]}\Big\{\widetilde{\mathbb{E}}[D(T)\Pi(T)|\mathcal{F}_W(t)] - \widetilde{\mathbb{E}}[D(T)|\mathcal{F}_W(t)]\widetilde{\mathbb{E}}[\Pi(T)|\mathcal{F}_W(t)]\Big\}. \tag{6.68}$$

Moreover, if the interest rate $\{R(t)\}_{t\in[0,T]}$ *is a deterministic function of time (e.g., a deterministic constant), then* $\mathrm{For}_T(t) = \mathrm{Fut}_T(t)$*, for all* $t \in [0,T]$.

Proof. The last claim follows directly by (6.68). In fact, when the interest rate of the bond is deterministic, the discounting process is also deterministic and thus in particular $D(T)$ is $\mathcal{F}_W(t)$-measurable. Hence, the term in curl brackets in (6.68) satisfies

$$\Big\{\dots\Big\} = D(T)\widetilde{\mathbb{E}}[\Pi(T)|\mathcal{F}_W(t)] - D(T)\widetilde{\mathbb{E}}[\Pi(T)|\mathcal{F}_W(t)] = 0.$$

As to (6.68), we compute

$$\begin{aligned}\frac{\Pi(t)}{B(t,T)} - \widetilde{\mathbb{E}}[\Pi(T)|\mathcal{F}_W(t)] &= \frac{D(t)\Pi(t)}{\widetilde{\mathbb{E}}[D(T)|\mathcal{F}_W(t)]} - \widetilde{\mathbb{E}}[\Pi(T)|\mathcal{F}_W(t)]\\ &= \frac{\widetilde{\mathbb{E}}[D(T)\Pi(T)|\mathcal{F}_W(t)]}{\widetilde{\mathbb{E}}[D(T)|\mathcal{F}_W(t)]} - \widetilde{\mathbb{E}}[\Pi(T)|\mathcal{F}_W(t)],\end{aligned}$$

where for the last equality we used that $\{\Pi^*(t)\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$. The result follows. □

Note that, as opposed to the forward price of the asset, the future price need not be increasing with the time left to delivery.

6.8 Multi-dimensional markets

In this section we consider $N+1$ dimensional stock markets. We denote the stocks prices by

$$\{S_1(t)\}_{t\geq 0}, \dots, \{S_N(t)\}_{t\geq 0}$$

and assume the following dynamics

$$dS_k(t) = \left(\mu_k(t)\,dt + \sum_{j=1}^{N}\sigma_{kj}(t)dW_j(t)\right)S_k(t), \tag{6.69}$$

for some stochastic processes $\{\mu_k(t)\}_{t\geq 0}$, $\{\sigma_{kj}(t)\}_{t\geq 0}$, $j,k = 1,\dots,N$, adapted to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$ generated by the Brownian motions $\{W_1(t)\}_{t\geq 0}, \dots \{W_N(t)\}_{t\geq 0}$. Moreover we assume that the Brownian motions are independent, in particular

$$dW_j(t)dW_k(t) = 0, \quad \text{for all } j \neq k, \tag{6.70}$$

see Exercise 3.21. Finally we denote by $\{R(t)\}_{t\geq 0}$ the interest rate of the money market, which we assume to be adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

Now, given stochastic processes $\{\theta_k(t)\}_{t\geq 0}$, $k = 1, \dots, N$, adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$, and satisfying the Novikov condition (4.20), the stochastic process $\{Z(t)\}_{t\geq 0}$ given by

$$Z(t) = \exp\left(-\sum_{k=1}^{N}\left(\int_0^t \frac{1}{2}\theta_k^2(s)\,ds + \int_0^t \theta_k(s)\,dW_k(s)\right)\right) \tag{6.71}$$

is a martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$ (see Exercise 4.8). Since $\mathbb{E}[Z(t)] = \mathbb{E}[Z(0)] = 1$, for all $t \geq 0$, we can use the stochastic process $\{Z(t)\}_{t\geq 0}$ to define a risk-neutral probability measure associated to the $N+1$ dimensional stock market, as we did in the one dimensional case, see Definition 6.1.

Definition 6.7. *Let $T > 0$ and assume that the* **market price of risk equations**

$$\mu_j(t) - R(t) = \sum_{k=1}^{N} \sigma_{jk}(t)\theta_k(t), \quad j = 1, \dots, N, \tag{6.72}$$

admit a solution $(\theta_1(t), \dots, \theta_N(t))$, for all $t \in [0, T]$. Define the stochastic process $\{Z(t)\}_{t\geq 0}$ as in (6.71)*. Then the measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ given by*

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z(T)\mathbb{I}_{\mathbb{A}}]$$

is called the risk-neutral probability measure of the market at time T.

Note that, as opposed to the one dimensional case, the risk-neutral measure just defined need not be unique, as the market price of risk equations may admit more than one solution. For each risk-neutral probability measure $\widetilde{\mathbb{P}}$ we can apply the multidimensional Girsanov theorem 4.12 and conclude that the stochastic processes $\{\widetilde{W}_1(t)\}_{t\geq 0}, \dots \{\widetilde{W}_N(t)\}_{t\geq 0}$ given by

$$\widetilde{W}_k(t) = W_k(t) + \int_0^t \theta_k(s)\,ds$$

are $\widetilde{\mathbb{P}}$-independent Brownian motions. Moreover these Brownian motions are $\widetilde{\mathbb{P}}$-martingales relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$.

Now let $\{h_{S_1}(t)\}_{t\geq 0}, \dots, \{h_{S_N}(t)\}_{t\geq 0}$ be $\{\mathcal{F}_W(t)\}_{t\geq 0}$-adapted stochastic processes representing the number of shares of the stocks in a portfolio invested in the $N+1$ dimensional stock market. The portfolio is self-financing if its value satisfies

$$dV(t) = \sum_{k=1}^{N} h_{S_k}(t)dS_k(t) + R(t)\left(V(t) - \sum_{k=1}^{N} h_{S_k}(t)S_k(t)\right)dt.$$

Theorem 6.13. *Assume that a risk-neutral probability $\widetilde{\mathbb{P}}$ exists, i.e., the equations* (6.72) *admit a solution. Then the discounted value of any self-financing portfolio invested in the $N+1$ dimensional market is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$. In particular (by Theorem 3.16) there exists no self-financing arbitrage portfolio invested in the $N+1$ dimensional stock market.*

Proof. The discounted value of the portfolio satisfies

$$\begin{aligned}
dV^*(t) &= D(t)\left(\sum_{j=1}^N h_{S_j}(t)S_j(t)(\alpha_j(t)-R(t))\,dt + \sum_{j,k=1}^N h_{S_j}(t)S_j(t)\sigma_{jk}(t)dW_k(t)\right)\\
&= D(t)\left(\sum_{j=1}^N h_{S_j}(t)S_j(t)\sum_{k=1}^N \sigma_{jk}(t)\theta_k(t)dt + \sum_{j,k=1}^N h_{S_j}(t)S_j(t)\sigma_{jk}(t)dW_k(t)\right)\\
&= D(t)\sum_{j=1}^N h_{S_j}(t)S_j(t)\sum_{k=1}^N \sigma_{jk}(t)d\widetilde{W}_k(t).
\end{aligned}$$

All Itô's integrals in the last line are $\widetilde{\mathbb{P}}$-martingales relative to $\{\mathcal{F}_W(t)\}_{t\geq 0}$. The result follows. □

Exercise 6.16. *Work out the details of the computations omitted in the proof of the previous theorem.*

Now we show that the existence of a risk-neutral probability measure is necessary for the absence of self-financing arbitrage portfolios in $N+1$ dimensional stock markets.

Let $N=3$ and assume that the market parameters are constant. Let $R(t)=r>0$, $(\mu_1,\mu_2,\mu_3)=(2,3,2)$ and let the volatility matrix be given by

$$\sigma_{ij} = \begin{pmatrix} 1 & 2 & 0\\ 2 & 4 & 0\\ 1 & 2 & 0\end{pmatrix}.$$

Thus the stocks prices satisfy

$$\begin{aligned}
dS_1(t) &= (2dt + dW_1(t) + 2dW_2(t))S_1(t),\\
dS_2(t) &= (3dt + 2dW_1(t) + 4dW_2(t))S_2(t),\\
dS_3(t) &= (2dt + dW_1(t) + 2dW_2(t))S_3(t).
\end{aligned}$$

The market price of risk equations are

$$\begin{aligned}
\theta_1 + 2\theta_2 &= 2-r\\
2\theta_1 + 4\theta_2 &= 3-r\\
\theta_1 + 2\theta_2 &= 2-r.
\end{aligned}$$

This system is solvable if and only if $r=1$, in which case there exist infinitely many solutions given by

$$\theta_1\in\mathbb{R},\quad \theta_2=\frac{1}{2}(1-\theta_1).$$

Hence for $r=1$ there exists at least one (in fact, infinitely many) risk-neutral probability measures, and thus the market is free of arbitrage. To construct an arbitrage portfolio when $0<r<1$, let

$$h_{S_1}(t)=\frac{1}{S_1(t)},\quad h_{S_2}(t)=-\frac{1}{S_2(t)},\quad h_{S_3}(t)=\frac{1}{S_3(t)}.$$

The value $\{V(t)\}_{t\geq 0}$ of this portfolio satisfies

$$\begin{aligned} dV(t) &= h_{S_1}(t)dS_1(t) + h_{S_2}(t)dS_2(t) + h_{S_3}(t)dS_3(t) \\ &\quad + r(V(t) - h_{S_1}(t)S_1(t) - h_{S_2}(t)S_2(t) - h_{S_3}(t)S_3(t))dt \\ &= rV(t)dt + (1-r)dt. \end{aligned}$$

Hence

$$V(t) = V(0)e^{rt} + \frac{1}{r}(1-r)(e^{rt}-1)$$

and this portfolio is an arbitrage, because for $V(0) = 0$ we have $V(t) > 0$, for all $t > 0$. Similarly one can find an arbitrage portfolio for $r > 1$.

Next we address the question of **completeness** of $N+1$ dimensional stock markets, i.e., the question of whether any European derivative can be hedged in this market. Consider a European derivative on the stocks with pay-off Y and time of maturity T. For instance, for a standard European derivative, $Y = g(S_1(T), \dots S_N(T))$, for some measurable function g. The risk-neutral price of the derivative is

$$\Pi_Y(t) = \widetilde{\mathbb{E}}[Y \exp(-\int_t^T R(s)\,ds)|\mathcal{F}(t)],$$

and coincides with the value at time t of any self-financing portfolio invested in the $N+1$ dimensional market. The question of existence of an hedging portfolio is answered by the following theorem.

Theorem 6.14. *Assume that the volatility matrix $(\sigma_{jk}(t))_{j,k=1,\dots N}$ is invertible, for all $t \geq 0$. There exist stochastic processes $\{\Delta_1(t)\}_{t\in[0,T]}, \dots \{\Delta_N(t)\}_{t\in[0,T]}$, adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$, such that*

$$D(t)\Pi_Y(t) = \Pi_Y(0) + \sum_{k=1}^N \int_0^t \Delta_k(s)d\widetilde{W}_k(s), \quad t \in [0,T]. \tag{6.73}$$

Let $(Y_1(t), \dots, Y_N(t))$ be the solution of

$$\sum_{k=1}^N \sigma_{jk}(t)Y_k(t) = \frac{\Delta_j(t)}{D(t)}. \tag{6.74}$$

Then the portfolio $\{h_{S_1}(t), \dots, h_{S_N}(t), h_B(t)\}_{t\in[0,T]}$ given by

$$h_{S_j}(t) = \frac{Y_j(t)}{S_j(t)}, \quad h_B(t) = (\Pi_Y(t) - \sum_{j=1}^N h_{S_j}(t)S_j(t))/B(t) \tag{6.75}$$

is self-financing and replicates the derivative at any time, i.e., its value $V(t)$ is equal to $\Pi_Y(t)$ for all $t \in [0,T]$. In particular, $V(T) = \Pi_Y(T) = Y$, i.e., the portfolio is hedging the derivative.

The proof of this theorem is conceptually very similar to that of Theorem 6.2 and is therefore omitted (it makes use of the multidimensional version of the martingale representation theorem). Notice that, having assumed that the volatility matrix is invertible, *the risk-neutral probability measure of the market is unique.* We now show that the uniqueness of the risk-neutral probability measure is necessary to guarantee completeness. In fact, let $r = 1$ in the example considered before and pick the following solutions of the market price of risk equations:

$$(\theta_1, \theta_2) = (0, 1/2), \quad \text{and} \quad (\theta_1, \theta_2) = (1, 0)$$

(any other pair of solutions would work). The two corresponding risk-neutral probability measures, denoted respectively by $\widetilde{\mathbb{P}}$ and $\widehat{\mathbb{P}}$, are given by

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}[\widetilde{Z}\mathbb{I}_A] \quad \widehat{\mathbb{P}}(A) = \mathbb{E}[\widehat{Z}\mathbb{I}_A], \quad \text{for all } A \in \mathcal{F},$$

where

$$\widetilde{Z} = e^{-\frac{1}{8}T - \frac{1}{2}W_2(T)}, \quad \widehat{Z} = e^{-\frac{1}{2}T - W_1(T)}.$$

Let $A = \{\omega : \frac{1}{2}W_2(T, \omega) - W_1(T, \omega) > \frac{3}{8}T\}$. Hence

$$\widehat{Z}(\omega) < \widetilde{Z}(\omega), \quad \text{for } \omega \in A$$

and thus $\widehat{\mathbb{P}}(A) < \widetilde{\mathbb{P}}(A)$. Consider a financial derivative with pay-off $Q = \mathbb{I}_A/D(T)$. If there existed an hedging, self-financing portfolio for such derivative, then, since the discounted value of such portfolio is a martingale in both risk-neutral probability measures, we would have

$$V(0) = \widetilde{\mathbb{E}}(QD(T)), \quad \text{and} \quad V(0) = \widehat{\mathbb{E}}(QD(T)). \tag{6.76}$$

But

$$\widehat{\mathbb{E}}(QD(T)) = \widehat{\mathbb{E}}(\mathbb{I}_A) = \widehat{\mathbb{P}}(A) < \widetilde{\mathbb{P}}(A) = \widetilde{\mathbb{E}}(\mathbb{I}_A) = \widetilde{\mathbb{E}}(QD(T))$$

and thus (6.76) cannot be verified.

We conclude this section with an example of application of Theorem 6.14.

An example of option on two stocks

Let us consider a $2+1$ dimensional stock market with constant parameters such that the volatility matrix is invertible. Let $K, T > 0$ and consider a standard European derivative with pay-off

$$Y = \left(\frac{S_1(T)}{S_2(T)} - K\right)_+$$

at time of maturity T. Let us find the risk-neutral price $\Pi_Y(t)$ of the derivative at time $t \in [0, T)$. Letting $r > 0$ be the interest rate of the bond, $(\sigma_{jk})_{j,k=1,2}$ be the volatility matrix of the stocks and $\sigma_j = (\sigma_{j1}, \sigma_{j2})$, $j = 1, 2$, we have

$$S_j(t) = S_j(0)e^{(r - \frac{|\sigma_j|^2}{2})t + \sigma_j \cdot \widetilde{W}(t)},$$

where $\widetilde{W}(t) = (\widetilde{W}_1(t), \widetilde{W}_2(t))$ and $\cdot$ denotes the standard scalar product of vectors. Hence, with $\tau = T - t$,

$$\Pi_Y(t) = e^{-r\tau}\widetilde{\mathbb{E}}\left[\left(\frac{S_1(T)}{S_2(T)} - K\right)_+ |\mathcal{F}(t)\right]$$
$$= e^{-r\tau}\widetilde{\mathbb{E}}\left[\left(\frac{S_1(t)}{S_2(t)} e^{(\frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2})\tau + (\sigma_1 - \sigma_2)\cdot(\widetilde{W}(T) - \widetilde{W}(t))} - K\right)_+ |\mathcal{F}(t)\right].$$

Now we write

$$(\sigma_1 - \sigma_2)\cdot(\widetilde{W}(T) - \widetilde{W}(t)) = \sqrt{\tau}[(\sigma_{11} - \sigma_{21})G_1 + (\sigma_{12} - \sigma_{22})G_2] = \sqrt{\tau}(X_1 + X_2),$$

where $G_j = (W_j(T) - W_j(t))/\sqrt{\tau} \in \mathcal{N}(0,1)$, $j = 1, 2$, hence $X_j \in \mathcal{N}(0, (\sigma_{1j} - \sigma_{2j})^2)$, $j = 1, 2$. In addition, X_1, X_2 are independent random variables, hence, as shown in Section 3.3, $X_1 + X_2$ is normally distributed with zero mean and variance $(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2 = |\sigma_1 - \sigma_2|^2$. It follows that

$$\Pi_Y(t) = e^{-r\tau}\widetilde{\mathbb{E}}\left[\left(\frac{S_1(t)}{S_2(t)} e^{(\frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2})\tau + \sqrt{\tau}|\sigma_1 - \sigma_2|G)} - K\right)_+\right],$$

where $G \in \mathcal{N}(0,1)$. Hence, letting

$$\hat{r} = \frac{|\sigma_1 - \sigma_2|^2}{2} - \left(\frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2}\right)$$

and $a = e^{(\hat{r} - r)\tau}$, we have

$$\Pi_Y(t) = a e^{-\hat{r}\tau}\mathbb{E}\left[\left(\frac{S_1(t)}{S_2(t)} e^{(\hat{r} - \frac{|\sigma_1 - \sigma_2|^2}{2})\tau + \sqrt{\tau}|\sigma_1 - \sigma_2|G} - K\right)_+\right]$$

Up to the multiplicative parameter a, this is the Black-Scholes price of a call on a stock with price $S_1(t)/S_2(t)$, volatility $|\sigma_1 - \sigma_2|$ and for an interest rate of the bond given by $\hat{r}$. Hence, Theorem 6.5 gives

$$\Pi_Y(t) = a\left(\frac{S_1(t)}{S_2(t)}\Phi(d_+) - K e^{-\hat{r}\tau}\Phi(d_-)\right) := v(t, S_1(t), S_2(t)),$$

where

$$d_\pm = \frac{\log\frac{S_1(t)}{KS_2(t)} + (\hat{r} \pm \frac{|\sigma_1 - \sigma_2|^2}{2})\tau}{|\sigma_1 - \sigma_2|\sqrt{\tau}}.$$

As to the self-financing hedging portfolio, it can be shown, by an argument similar to the one used in the 1+1 dimensional case (see Theorem 6.4), that one such portfolio is given by $h_{S_j}(t) = \frac{\partial v}{\partial x_j}(t, S_1(t), S_2(t))$, $j = 1, 2$. Therefore, recalling the delta function of the standard European call (see Theorem 6.6), we obtain

$$h_{S_1}(t) = \frac{a}{S_2(t)}\Phi(d_+), \quad h_{S_2}(t) = -\frac{aS_1(t)}{S_2(t)^2}\Phi(d_+). \tag{6.77a}$$

As usual

$$h_B(t) = (\Pi_Y(t) - \sum_{j=1}^{2} h_{S_j}(t)S_j(t))/B(t). \tag{6.77b}$$

Exercise 6.17. *Show that the portfolio* (6.77) *is self-financing and hedges the derivative.*

6.9 Introduction to American derivatives

Before giving the precise definition of fair price for American derivatives, we shall present some general properties of these contracts. American derivatives can be exercised at any time prior or including maturity T. Let $Y(t)$ be the pay-off resulting from exercising the derivative at time $t \in (0, T]$. We call $Y(t)$ the **intrinsic value** of the derivative. We consider only standard American derivatives, for which we have $Y(t) = g(S(t))$, for some measurable function $g : \mathbb{R} \to \mathbb{R}$. For instance, $g(x) = (x - K)_+$ for American calls and $g(x) = (K - x)_+$ for American puts. We denote by $\widetilde{\Pi}_Y(t)$ the price of the American derivative with intrinsic value $Y(t)$ and by $\Pi_Y(t)$ the price of the European derivative with pay-off $Y = Y(T)$ at maturity time T. Two obvious properties of American derivatives are the following:

(i) $\widetilde{\Pi}_Y(t) \geq \Pi_Y(t)$, for all $t \in [0, T]$. In fact an American derivative gives to its owner all the rights of the corresponding European derivative plus one: the option of early exercise. Thus it is clear that the American derivative cannot be cheaper than the European one.

(ii) $\widetilde{\Pi}_Y(t) \geq Y(t)$, for all $t \in [0, T]$. If not, an arbitrage opportunity would arise by purchasing the American derivative and exercising it immediately.

Any reasonable definition of fair price for American derivatives must satisfy (i)-(ii).

Definition 6.8. *A time $t \in (0, T]$ is said to be an* **optimal exercise time** *for the American derivative with intrinsic value $Y(t)$ if $\widetilde{\Pi}_Y(t) = Y(t)$.*

Hence by exercising the derivative at an optimal exercise time t, the buyer takes full advantage of the derivative: the resulting pay-off equals the value of the derivative. On the other hand, if $\widetilde{\Pi}_Y(t) > Y(t)$ and the buyer wants to close the (long) position on the American derivative, then the optimal strategy is to sell the derivative, thereby cashing the amount $\widetilde{\Pi}_Y(t)$.

Theorem 6.15. *Assume (i) holds and let $C(t)$ be the price of an American call at time $t \in [0, T]$. Assume further that the underlying stock price follows a generalized geometric Brownian motion and that the interest rate $R(t)$ of the money market is strictly positive for all times. Then $C(t) > Y(t)$, for all $t \in [0, T)$. In particular it is never optimal to exercise the call prior to maturity.*

Proof. We can assume $S(t) \geq K$, as for $S(t) < K$ the claim is obvious (since $C(t) \geq 0$).

Denoting by $D(t) = \exp(-\int_0^t R(s)ds)$ the discounting process, the price of the European call is $\widetilde{\mathbb{E}}[(S(T)-K)_+D(T)/D(t)|\mathcal{F}_W(t)]$, hence by (i):

$$\begin{aligned}C(t) &\geq \widetilde{\mathbb{E}}[(S(T)-K)_+D(T)/D(t)|\mathcal{F}_W(t)] \geq \widetilde{\mathbb{E}}[(S(T)-K)D(T)/D(t)|\mathcal{F}_W(t)]\\ &= \widetilde{\mathbb{E}}[S(T)D(T)/D(t)|\mathcal{F}_W(t)] - K\widetilde{\mathbb{E}}[D(T)/D(t)|\mathcal{F}_W(t)] > D(t)^{-1}\widetilde{\mathbb{E}}[S^*(T)|\mathcal{F}_W(t)] - K\\ &= S(t) - K,\end{aligned}$$

where we used $D(T)/D(t) < 1$ (by the positivity of the interest rate $R(t)$) and the martingale property of the discounted price $\{S^*(t)\}_{t\in[0,T]}$ of the stock. □

It follows that under the assumptions of the previous theorem, American and European call options have the same value.

Remark 6.14. The result is valid for general standard American derivatives with convex pay-off function, see [21, Section 8.5].

Remark 6.15. A notable exception to the assumed conditions in Theorem 6.15 is when the underlying stock pays a dividend. In this case it can be shown that it is optimal to exercise the American call immediately before the dividend is payed, provided the price of the stock is sufficiently high, see Section 6.9.2 below.

Definition 6.9. *Let $T \in (0,\infty)$. A random variable $\tau : \Omega \to [0,T]$ is called a* **stopping time** *for the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$ if $\{\tau \leq t\} \in \mathcal{F}_W(t)$, for all $t \in [0,T]$. We denote by Q_T the set of all stopping times for the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$.*

Think of τ as the time at which some random event takes place. Then τ is a stopping time if the occurrence of the event before or at time t can be inferred by the information available up to time t (no future information is required). For the applications that we have in mind, τ will be the optimal exercise time of an American derivative, which marks the event that the price of the derivative equals its intrinsic value.

From now on we assume that the market has constant parameters and $r > 0$. Hence the price of the stock is given by the geometric Brownian motion

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})t+\sigma\widetilde{W}(t)}.$$

We recall that in this case the price $\Pi(0,T)$ at time $t = 0$ of a European derivative with pay-off $Y = g(S(T))$ at maturity time $T > 0$ is given by

$$\Pi_Y(0,T) = \widetilde{\mathbb{E}}[e^{-rT}g(S(0)e^{(r-\frac{\sigma^2}{2})T+\sigma\widetilde{W}(T)})]$$

Now, if the writer of the American derivative were sure that the buyer would exercise at the time $u \in (0,T]$, then the fair price of the American derivative at time $t = 0$ would be equal to $\Pi_Y(0,u)$. As the writer cannot anticipate when the buyer will exercise, we would be tempted to define the price of the American derivative at time zero as $\max\{\Pi_Y(0,u), 0 \leq u \leq T\}$. However this definition would actually be unfair, as it does not take into account the fact that the exercise time is a stopping time, i.e., it is random and it cannot be inferred using future information. This leads us to the following definition.

Definition 6.10. *In a market with constant parameters, the fair price at time $t = 0$ of the standard American derivative with intrinsic value $Y(t) = g(S(t))$ and maturity $T > 0$ is given by*

$$\widetilde{\Pi}(0) = \max_{\tau \in Q_T} \widetilde{\mathbb{E}}[e^{-r\tau} g(S(0)e^{(r-\frac{\sigma^2}{2})\tau + \sigma \widetilde{W}(\tau)})] \tag{6.78}$$

It is not possible in general to find an closed formula for the price of an American derivative. A notable exception is the price of perpetual American put options, which we discuss next.

6.9.1 Perpetual American put options

An American put option is called perpetual if it never expires, i.e., $T = +\infty$. This is of course an idealization, but perpetual American puts are very useful to visualize the structure of general American put options. In this section we follow closely the discussion on [21, Section 8.3]. Definition 6.78 becomes the following.

Definition 6.11. *Let Q be the set of all stopping times for the filtration $\{\mathcal{F}_W(t)\}_{t \geq 0}$, i.e., $\tau \in Q$ iff $\tau : \Omega \to [0, \infty]$ is a random variable and $\{\tau \leq t\} \in \mathcal{F}_W(t)$, for all $t \geq 0$. The fair price at time $t = 0$ of the perpetual American put with intrinsic value $Y(t) = (K - S(t))_+$ is*

$$\widetilde{\Pi}(0) = \max_{\tau \in Q} \widetilde{\mathbb{E}}[e^{-r\tau}(K - S(\tau))_+] = \max_{\tau \in Q} \widetilde{\mathbb{E}}[e^{-r\tau}(K - S(0)e^{(r-\frac{\sigma^2}{2})\tau + \sigma \widetilde{W}(\tau)})_+]. \tag{6.79}$$

Theorem 6.16. *There holds*

$$\widetilde{\Pi}(0) = v_{L_*}(S(0)), \tag{6.80}$$

where

$$v_L(x) = \begin{cases} K - x & 0 \leq x \leq L \\ (K - L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}} & x > L \end{cases}$$

and

$$L_* = \frac{2r}{2r + \sigma^2} K.$$

Before we prove the theorem, some remarks are in order:

(i) $L_* < K$;

(ii) For $S(0) \leq L_*$ we have $\widetilde{\Pi}(0) = v_{L_*}(S(0)) = K - S(0) = (K - S(0))_+$. Hence when $S(0) \leq L_*$ it is optimal to exercise the derivative.

(iii) We have $\widetilde{\Pi}(0) > (K - S(0))_+$ for $S(0) > L_*$. In fact

$$v'_{L_*}(x) = -\frac{2r}{\sigma^2}\left(\frac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}-1} \frac{K - L_*}{L_*}$$

hence $v'_{L_*}(L_*) = -1$. Moreover

$$v''_{L_*}(x) = \frac{2r}{\sigma^2}(\frac{2r}{\sigma^2} + 1)\left(\frac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}-2} \frac{K - L_*}{L_*^2},$$

which is always positive. Thus the graph of $v_{L_*}(x)$ always lies above $k - x$ for $x > L_*$. It follows that it is not optimal to exercise the derivative if $S(0) > L_*$.

(iv) In the perpetual case, any time is equivalent to $t = 0$, as the time left to maturity is always infinite. Hence

$$\widetilde{\Pi}(t) = v_{L_*}(S(t)).$$

In conclusion the theorem is saying us that the buyer of the derivative should exercise as soon as the stock price falls below the threshold L_*. In fact we can reformulate the theorem in the following terms:

Theorem 6.17. *The maximum of* $\widetilde{\mathbb{E}}[e^{-r\tau}(K - S(\tau))_+]$ *over all possible* $\tau \in Q$ *is achieved at* $\tau = \tau_*$*, where*

$$\tau_* = \min\{t \geq 0 : S(t) = L_*\}.$$

Moreover $\widetilde{\mathbb{E}}[e^{-r\tau_*}(K - S(\tau_*))_+] = v_{L_*}(S(0))$.

For the proof of Theorem 6.16 we need the optional sampling theorem:

Theorem 6.18. *Let* $\{X(t)\}_{t\geq 0}$ *be an adapted process and* τ *a stopping time. Let* $t \wedge \tau = \min(t, \tau)$. *If* $\{X(t)\}_{t\geq 0}$ *is a martingale/supermatingale/submartingale, then* $\{X(t\wedge\tau)\}_{t\geq 0}$ *is also a martingale/supermartingale/submartingale.*

We can now prove Theorem 6.16. We divide the proof in two steps, which correspond respectively to Theorem 8.3.5 and Corollary 8.3.6 in [21].

Step 1: *The stochastic process* $\{e^{-r(t\wedge\tau)}v_{L_*}(S(t\wedge\tau))\}_{t\geq 0}$ *is a super-martingale for all* $\tau \in Q$. *Moreover for* $S(0) > L_*$ *the stochastic process* $\{e^{-r(t\wedge\tau_*)}v_{L_*}(S(t \wedge \tau_*))\}_{t\geq 0}$ *is a martingale.* By Itô's formula,

$$d(e^{-rt}v_{L_*}(S(t)t)) = e^{-rt}[-rv_{L_*}(S(t)) + rS(t)v'_{L_*}(S(t)) + \frac{1}{2}\sigma^2 S(t)^2 v''_{L_*}(S(t))]dt \\ + e^{-rt}\sigma S(t)v'_{L_*}(S(t))d\widetilde{W}(t).$$

The drift term is zero for $S(t) > L_*$ and it is equal to $-rK\,dt$ for $S(t) \leq L_*$. Hence

$$e^{-rt}v_{L_*}(S(t)) = v_{L_*}(S(0)) - rK\int_0^t e^{-ru}\mathbb{I}_{S(u)\leq L_*}(u)\,du + \int_0^t e^{-ru}\sigma S(u)v'_{L_*}(S(u))d\widetilde{W}(u).$$

Since the drift term is non-positive, then $\{e^{-rt}v_{L_*}(t)\}_{t\geq 0}$ is a supermartingale and thus by the optional sampling theorem, the process $\{e^{-r(t\wedge\tau)}v_{L_*}(S(t\wedge\tau)\}_{t\geq 0}$ is a also a supermartingale, for all $\tau \in Q$. Now, if $S(0) > L_*$, then, by continuity of the paths of the geometric Brownian motion, $S(u, \omega) > L_*$ as long as $u < \tau_*(\omega)$. Hence by stopping the process at τ_* the stock price will never fall below L_* and therefore the drift term vanishes, that is

$$e^{-r(t\wedge\tau_*)}v_{L_*}(S(t\wedge\tau_*)) = v_{L_*}(S(0)) + \int_0^{t\wedge\tau_*} e^{-ru}\sigma S(u)v'_{L_*}(S(u))d\widetilde{W}(u).$$

The Itô integral is a martingale and thus the Itô integral stopped at time τ_* is also a martingale by the optional sampling theorem. The claim follows.

Step 2: *The identity* (6.80) *holds.* The supermartingale property of the process $\{e^{-r(t\wedge\tau)}v_{L_*}(S(t\wedge\tau))\}_{t\geq 0}$ implies that its expectation is non-increasing, hence

$$\widetilde{\mathbb{E}}[e^{-r(t\wedge\tau)}v_{L_*}(S(t\wedge\tau))] \leq v_{L*}(S(0)).$$

As $v_{L_*}(x)$ is bounded and continuous, the limit $t \to +\infty$ gives

$$\widetilde{\mathbb{E}}[e^{-r\tau}v_{L_*}(S(\tau))] \leq v_{L_*}(S(0)).$$

As $v_{L_*}(x) \geq (K-x)_+$ we also have

$$\widetilde{\mathbb{E}}[e^{-r\tau}(K - S(\tau))_+] \leq v_{L_*}(S(0)).$$

Taking the maximum over all $\tau \in Q$ we obtain

$$\widetilde{\Pi}(0) = \max_{\tau\in Q}\widetilde{\mathbb{E}}[e^{-r\tau}(K - S(\tau))_+] \leq v_{L_*}(S(0)).$$

Now we prove $\widetilde{\Pi}(0) \geq v_{L_*}(S(0))$. This is obvious for $S(0) \leq L_*$. In fact, letting for instance $\widetilde{\tau} = \min\{t \geq 0 : S(t) \leq L_*\}$, we have $\widetilde{\tau} \equiv 0$ for $S(0) \leq L_*$ and so $\max_{\tau\in Q}\widetilde{\mathbb{E}}[e^{-r\tau}(K - S(\tau))_+] \geq \widetilde{\mathbb{E}}[e^{-r\widetilde{\tau}}(K - S(\widetilde{\tau}))_+] = (K - S(0))_+ = v_{L_*}(S(0))$, for $S(0) \leq L_*$. For $S(0) > L_*$ we use the martingale property of the stochastic process $\{e^{-r(t\wedge\tau_*)}v_{L_*}(S(t\wedge\tau_*))\}_{t\geq 0}$, which implies

$$\widetilde{\mathbb{E}}[e^{-r(t\wedge\tau_*)}v_{L_*}(S(t\wedge\tau_*))] = v_{L_*}(S(0)).$$

Hence in the limit $t \to +\infty$ we obtain

$$v_{L_*}(S(0)) = \widetilde{\mathbb{E}}[e^{-r\tau_*}v_{L_*}(S(\tau_*))].$$

Moreover $e^{-r\tau_*}v_{L_*}(S(\tau_*)) = e^{-r\tau_*}v_{L_*}(L_*) = e^{-r\tau_*}(K - S(\tau_*))_+$, hence

$$v_{L_*}(S(0)) = \widetilde{\mathbb{E}}[e^{-r\tau_*}(K - S(\tau_*))_+].$$

It follows that

$$\widetilde{\Pi}(0) = \max_{\tau\in Q}\widetilde{\mathbb{E}}[e^{-r\tau}(K - S(\tau))_+] \geq v_{L_*}(S(0)),$$

which completes the proof. □

Next we discuss the problem of hedging the perpetual American put with a portfolio invested in the underlying stock and the risk-free asset.

Definition 6.12. *A portfolio process $\{h_S(t), h_B(t)\}_{t\geq 0}$ is said to be replicating the perpetual American put if its value $\{V(t)\}_{t\geq 0}$ equals $\widetilde{\Pi}(t)$ for all $t \geq 0$.*

Thus by setting-up a replicating portfolio, the writer of the perpetual American put is sure to always be able to afford to pay-off the buyer. Note that in the European case a self-financing hedging portfolio is trivially replicating, as the price of European derivatives has been defined as the value of such portfolios. However in the American case a replicating portfolio need not be self-financing: if the buyer does not exercise at an optimal exercise time, the writer must withdraw cash from the portfolio in order to replicate the derivative. This leads to the definition of portfolio generating a cash flow.

Definition 6.13. *A portfolio* $\{h_S(t), h_B(t)\}_{t\geq 0}$ *with value* $\{V(t)\}_{t\geq 0}$ *is said to generate a* **cash flow** *with rate* $c(t)$ *if* $\{c(t)\}_{t\geq 0}$ *is adapted to* $\{\mathcal{F}_W(t)\}_{t\geq 0}$ *and*

$$dV(t) = h_S(t)dS(t) + h_B(t)dB(t) - c(t)dt \tag{6.81}$$

Remark 6.16. Note that the cash flow has been defined so that $c(t) > 0$ when the investor withdraws cash from the portfolio (causing a decrease of its value).

The following theorem is Corollary 8.3.7 in [21].

Theorem 6.19. *The portfolio given by*

$$h_S(t) = v'_{L_*}(S(t)), \quad h_B(t) = \frac{v_{L_*}(S(t)) - h_S(t)S(t)}{B(0)e^{rt}}$$

is replicating the perpetual American put while generating the case flow $c(t) = rK\mathbb{I}_{S(t)<L_*}$ *(i.e., cash is withdrawn at the rate* rK *whenever* $S(t) < L_*$*, provided of course the buyer does not exercise the derivative).*

Proof. By definition, $V(t) = h_S(t)S(t) + h_B(t)B(t) = v_{L_*}(S(t)) = \widetilde{\Pi}(t)$, hence the portfolio is replicating. Moreover

$$dV(t) = d(v_{L_*}(S(t))) = h_S(t)dS(t) + \frac{1}{2}v''_{L_*}(S(t))\sigma^2 S(t)^2 dt. \tag{6.82}$$

Now, a straightforward calculation shows that $v_{L_*}(x)$ satisfies

$$-rv_{L_*} + rxv'_{L_*} + \frac{1}{2}\sigma x^2 v''_{L_*} = -rK\mathbb{I}_{S(t)<L_*},$$

a relation which was already used in step 1 in the proof of Theorem 6.16. It follows that

$$\begin{aligned}\frac{1}{2}v''_{L_*}(S(t))\sigma^2 S(t)^2 dt &= r(v_{L_*}(S(t)) - S(t)h_S(t))dt - rK\mathbb{I}_{S(t)<L_*}dt \\ &= h_B(t)dB(t) - rK\mathbb{I}_{S(t)<L_*}dt.\end{aligned}$$

Hence (6.82) reduces to (6.81) with $c(t) = rK\mathbb{I}_{S(t)<L_*}$, and the proof is complete. □

6.9.2 American calls on a dividend-paying stock

Let $\widehat{c}_a(t, S(t), K, T)$ denote the Black-Scholes price at time t of the American call with strike K and maturity T assuming that the underlying stock pays the dividend $aS(t_0^-)$ at time $t_0 \in (0, T)$. We denote by $c_a(t, S(t), K, T)$ the Black-Scholes price of the corresponding European call. We omit the subscript a to denote prices in the absence of dividends. Moreover replacing the letter c with the letter P gives the price of the corresponding put option. We say that it is optimal to exercise the American call at time t if its Black-Scholes price at this time equals the intrinsic value of the call, i.e., $\widehat{c}_a(t, S(t), K, T) = (S(t) - K)_+$.

Theorem 6.20. *consider the American call with strike K and expiration date T and assume that the underlying stock pays the dividend $aS(t_0^-)$ at the time $t_0 \in (0, T)$. Then*

$$\widehat{c}_a(t, S(t), K, T) > (S(t) - K)_+, \quad \text{for } t \in [t_0, T),$$

i.e., it is not optimal to exercise the American call prior to maturity after the dividend is paid. Moreover, there exists $\delta > 0$ such that, if

$$S(t_0^-) > \max(\frac{\delta}{1-a}, K),$$

then the equality

$$\widehat{c}_a(t_0^-, S(t_0^-), K, T) = (S(t_0^-) - K)_+$$

holds, and so it is optimal to exercise the American call "just before" the dividend is to be paid.

Proof. For the first claim we can assume $(S(t) - K)_+ = S(t) - K$, otherwise the American call is out of the money and so it is clearly not optimal to exercise. By Theorem 6.7 we have

$$c_a(t, S(t), K, T) = c(t, S(t), K, T), \quad P_a(t, S(t), K, T) = P(t, S(t), K, T), \quad \text{for } t \geq t_0.$$

Hence, by Theorem 6.5, the put-call parity holds after the dividend is paid:

$$c_a(t, S(t), K, T) = P_a(t, S(t), K, T) + S(t) - Ke^{-r(T-t)}, \quad t \geq t_0.$$

Thus, for $t \in [t_0, T)$,

$$\widehat{c}_a(t, S(t), K, T) \geq c_a(t, S(t), K, T) > S(t) - K = (S(t) - K)_+,$$

where we used that $P(t, S(t), K, T) > 0$ and $r \geq 0$. This proves the first part of the theorem, i.e., the fact that it is not optimal to exercise the American call prior to expiration after the dividend has been paid. In particular

$$\widehat{c}_a(t, S(t), K, T) = c_a(t, S(t), K, T), \quad \text{for } t \geq t_0. \tag{6.83}$$

Next we show that it is optimal to exercise the American call "just before the dividend is paid", i.e., $\widehat{c}_a(t_0^-, S(t_0^-), K, T) = (S(t_0^-) - K)_+$, provided the price of the stock is sufficiently high. Of course it must be $S(t_0^-) > K$. Assume first that $\widehat{c}_a(t_0^-, S(t_0^-), K, T) > S(t_0^-) - K$; then, owing to (6.83), $\widehat{c}_a(t_0^-, S(t_0^-), K, T) = c_a(t_0^-, S(t_0^-), K, T)$ (buying the American call just before the dividend is paid is not better than buying the European call, since it is never optimal to exercise the derivative prior to expiration). By Theorem (6.7) we have $c_a(t_0^-, S(t_0^-), K, T) = c(t_0^-, (1-a)S(t_0^-), K, T) = c(t_0, (1-a)S(t_0^-), K, T)$, where for the latter equality we used the continuity in time of the Black-Scholes price function in the absence of dividends. Since $(1-a)S(t_0^-) = S(t_0)$, then

$$\widehat{c}_a(t_0^-, S(t_0^-), K, T) > S(t_0^-) - K \Rightarrow \widehat{c}_a(t_0^-, S(t_0^-), K, T) = c(t_0, S(t_0), K, T).$$

Hence

$$\widehat{c}_a(t_0^-, S(t_0^-), K, T) > S(t_0^-) - K \Rightarrow c(t_0, S(t_0), K, T) > S(t_0^-) - K = S(t_0) + (1-a)S(t_0^-) - K.$$

Therefore, taking the contrapositive statement,

$$c(t_0, S(t_0), K, T) \leq S(t_0) + (1-a)S(t_0^-) - K \Rightarrow \widehat{c}_a(t_0^-, S(t_0^-), K, T) = S(t_0^-) - K. \quad (6.84)$$

Next we remark that the function $x \to c(t, x, K, T) - x$ is decreasing (since $\Delta = \partial_x c = \Phi(d_1) < 1$, see Theorem 6.6), and

$$\lim_{x \to 0^+} c(t, x, K, T) - x = 0,$$

$$\lim_{x \to +\infty} c(t, x, K, T) - x = \lim_{x \to +\infty} P(t, x, K, T) - Ke^{-r(T-t)} = -Ke^{-r(T-t)},$$

see Exercise 6.7. Thus if $(1-a)S(t_0^-) - K > -Ke^{-r(T-t)}$, i.e., $S(t_0^-) > (1-a)^{-1}K(1 - e^{-r(T-t)}$, there exists ω such that if $S(t_0) > \omega$, i.e., $S(t_0^-) > \omega/(1-a)$, then the inequality $c(t_0, S(t_0), K, T) \leq S(t_0) + (1-a)S(t_0^-) - K$ holds. It follows by (6.84) that for such values of $S(t_0^-)$ it is optimal to exercise the call "at time t_0^-". Letting $\delta = \max(\omega, K(1 - e^{-r(T-t)})$ concludes the proof of the theorem. □

Exercise 6.18. *Prove that it is not optimal to exercise the American call at time $t \in [0, t_0)$ if $S(t) < \frac{K}{a}(1 - e^{-r(T-t)})$.*

6.A Appendix: Solutions to selected problems

Exercise 6.5. The pay-off function is $g(z) = k + z \log z$. Hence the Black-Scholes price of the derivative is $\Pi_Y(t) = v(t, S(t))$, where

$$\begin{aligned}
v(t,s) &= e^{-r\tau} \int_{\mathbb{R}} g\left(s e^{\left(r - \frac{\sigma^2}{2}\right)\tau - \sigma\sqrt{\tau}x} \right) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= e^{-r\tau} \int_{\mathbb{R}} \left(k + s e^{(r - \frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x} (\log s + (r - \frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x) \right) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= k e^{-r\tau} + s \log s \int_{\mathbb{R}} e^{-\frac{1}{2}(x + \sigma\sqrt{\tau})^2} \frac{dx}{\sqrt{2\pi}} \\
&\quad + s(r - \frac{\sigma^2}{2})\tau \int_{\mathbb{R}} e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2} \frac{dx}{\sqrt{2\pi}} - s\sigma\sqrt{\tau} \int_{\mathbb{R}} x e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2} \frac{dx}{\sqrt{2\pi}}
\end{aligned}$$

Using that

$$\int_{\mathbb{R}} e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2}\frac{dx}{\sqrt{2\pi}} = 1, \quad \int_{\mathbb{R}} x e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2}\frac{dx}{\sqrt{2\pi}} = -\sigma\sqrt{\tau},$$

we obtain

$$v(t,s) = ke^{-r\tau} + s\log s + s(r + \frac{\sigma^2}{2})\tau.$$

Hence

$$\Pi_Y(t) = ke^{-r\tau} + S(t)\log S(t) + S(t)(r + \frac{\sigma^2}{2})\tau.$$

This completes the first part of the exercise. The number of shares of the stock in the hedging portfolio is given by

$$h_S(t) = \Delta(t, S(t)),$$

where $\Delta(t,s) = \frac{\partial v}{\partial s} = \log s + 1 + (r + \frac{\sigma^2}{2})\tau$. Hence

$$h_S(t) = 1 + (r + \frac{\sigma^2}{2})\tau + \log S(t).$$

The number of shares of the bond is obtained by using that

$$\Pi_Y(t) = h_S(t)S(t) + B(t)h_B(t),$$

hence

$$\begin{aligned} h_B(t) &= \frac{1}{B(t)}(\Pi_Y(t) - h_S(t)S(t)) \\ &= e^{-rt}(ke^{-r\tau} + S(t)\log S(t) + S(t)(r + \frac{\sigma^2}{2})\tau - S(t) - S(t)(r + \frac{\sigma^2}{2})\tau - S(t)\log S(t)) \\ &= ke^{-rT} - S(t)e^{-rt}. \end{aligned}$$

This completes the second part of the exercise. To compute the probability that $Y > 0$, we first observe that the pay-off function $g(z)$ has a minimum at $z = e^{-1}$ and we have $g(e^{-1}) = k - e^{-1}$. Hence if $k \geq e^{-1}$, the derivative has probability 1 to expire in the money. If $k < e^{-1}$, there exist $a < b$ such that

$$g(z) > 0 \quad \text{if and only if} \quad 0 < z < a \text{ or } z > b.$$

Hence for $k < e^{-1}$ we have

$$\mathbb{P}(Y > 0) = \mathbb{P}(S(T) < a) + \mathbb{P}(S(T) > b).$$

Since $S(T) = S(0)e^{\alpha T - \sigma\sqrt{T}G}$, with $G \in N(0,1)$, then

$$S(T) < a \Leftrightarrow G > \frac{\log\frac{S(0)}{a} + \alpha T}{\sigma\sqrt{T}} := A, \quad S(t) > b \Leftrightarrow G < \frac{\log\frac{S(0)}{b} + \alpha T}{\sigma\sqrt{T}} := B.$$

Thus

$$\mathbb{P}(Y>0) = \mathbb{P}(G>A) + \mathbb{P}(G<B) = \int_A^{+\infty} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} + \int_{-\infty}^{B} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$
$$= 1 - \Phi(A) + \Phi(B).$$

This completes the solution of the third part of the exercise.

Exercise 6.14. Let

$$Q(t) = \int_0^t \sigma(s)^2\, ds.$$

We have $dQ(t) = \sigma(t)^2\, dt$. We compute

$$\begin{aligned} d(e^{-rt} f(t, \sigma^2(t), Q(t))) =& e^{-rt}[-rf\, dt + \partial_t f\, dt + \partial_x f\, d\sigma^2(t) + \frac{1}{2}\partial_x^2 f\, d\sigma^2(t) d\sigma^2(t) \\ &+ \partial_y f\, dQ(t) + \frac{1}{2}\partial_y^2 f\, dQ(t) dQ(t) + \partial_{xy}^2 f\, dQ(t) d\sigma^2(t)] \\ &= e^{-rt}[\partial_t f + a(b - \sigma^2(t))\partial_x f + \sigma^2(t)\partial_y f + \frac{c^2}{2}\sigma^2(t)\partial_x^2 f - rf]dt \\ &+ e^{-rt} c\sigma(t)\partial_x f d\widetilde{W}(t). \end{aligned}$$

where the function f and its derivatives are evaluated at $(t, \sigma^2(t), Q(t))$. As the discounted risk-neutral price must be a martingale in the risk-neutral probability measure, we need the drift term in the above equation to be zero. This is achieved by imposing that f satisfies the PDE

$$\partial_t f + a(b - x)\partial_x f + x\partial_y f + \frac{c^2}{2} x \partial_x^2 f = rf \tag{6.85}$$

Since $\Pi_Y(T) = Y = f(T, \sigma^2(T), Q(T))$, the terminal condition is

$$f(T, x, y) = N\left(\sqrt{\frac{\kappa}{T} y} - K\right)_+ .$$

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