Exam for the course “Financial derivatives and PDE’s” (CTH[*tma285*], GU[*mma711*]) June 9th$^{\text{th}}$, 2017

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Remark: (1) No aids permitted

1. Assume that the price $S(t)$ of a stock follows a geometric Brownian motion with instantaneous volatility $\sigma > 0$ and that the risk-free interest rate r is a positive constant.
   (a) Give and justify the definition for the fair price $\widetilde{\Pi}(0)$ at time $t = 0$ of the perpetual American put on the stock with strike K (max. 1 point).
   (b) Prove that $\widetilde{\Pi}(0) = v_{L_*}(S(0))$, where
   $$v_{L_*}(x) = \begin{cases} K - x, & 0 \leq x \leq L_* \\ (K - L_*)\left(\frac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}}, & x > L_* \end{cases}, \qquad L_* = \frac{2r}{2r + \sigma^2}K.$$
   (max. 4 points).

2. Assume that the price $S(t)$ of a stock follows a geometric Brownian motion with instantaneous volatility $\sigma > 0$ and that the risk-free interest rate r is a positive constant. The Asian call with geometric average is the European style derivative with pay-off
   $$Y = \left(\exp\left(\frac{1}{T}\int_0^T \log S(t)\, dt\right) - K\right)_+,$$
   where $T > 0$ and $K > 0$ are respectively the maturity and strike of the call. Derive an exact formula for the Black-Scholes price of this option (max. 3 points) and the corresponding put-call parity (max. 2 points).

3. Assume that the spot interest rate in the risk-neutral probability is given by the *Hull-White* model:
   $$dR(t) = (a - bR(t))\, dt + c d\widetilde{W}(t),$$
   where a, b, c are constants. Derive the partial differential equation and the terminal condition satisfied by the pricing function $f(t, x)$ of the zero-coupon bond with maturity T and face value 1 (max. 1 point). Find $f(t, x)$ (max. 1 point). HINT: Use the ansatz $f(t, x) = \exp(-xA(T - t) - G(T - t))$, where A, G are deterministic functions of time. Finally, use the HJM method to derive the dynamics of the instantaneous forward rate $F(t, T)$ in the physical probability (max 3 points).