Exam for the course "Financial derivatives and PDE's" (CTH[*tma285*], GU[*mma711*]) March 13th$^{\text{th}}$, 2018

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Remark: (1) No aids permitted

1. Assume that the price $S(t)$ of a stock follows a geometric Brownian motion with instantaneous volatility $\sigma > 0$ and that the risk-free interest rate r is a positive constant.

   (a) Give and justify the definition for the fair price $\widetilde{\Pi}(0)$ at time $t = 0$ of the perpetual American put on the stock with strike K (max. 1 point).

   (b) Prove that $\widetilde{\Pi}(0) = v_{L_*}(S(0))$, where

   $$v_{L_*}(x) = \begin{cases} K - x, & 0 \leq x \leq L_* \\ (K - L_*)\left(\frac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}}, & x > L_* \end{cases}, \qquad L_* = \frac{2r}{2r + \sigma^2} K.$$

   (max. 4 points).

2. Assume that the price $S(t)$ of a stock follows a geometric Brownian motion with instantaneous volatility $\sigma > 0$ and that the risk-free interest rate r is a positive constant. The Asian call with geometric average is the European style derivative with pay-off

   $$Y = \left(\exp\left(\frac{1}{T}\int_0^T \log S(t)\, dt\right) - K\right)_+,$$

   where $T > 0$ and $K > 0$ are respectively the maturity and strike of the call. Derive an exact formula for the Black-Scholes price of this option at time $t = 0$ (max. 4 points) and the corresponding put-call parity (max. 1 point). HINT: $\int_0^t W(s)\, ds$ is normally distributed, for all $t > 0$.

3. Assume that the price $S(t)$ of a stock follows a generalized geometric Brownian motion with instantaneous volatility $\{\sigma(t)\}_{t\geq 0}$ given by the Heston model $d\sigma^2(t) = a(b - \sigma^2(t))\, dt + c\sigma(t)\, d\widetilde{W}(t)$, where $\{\widetilde{W}(t)\}_{t\geq 0}$ is a Brownian motion in the risk-neutral

probability measure and a, b, c are constants such that $2ab \geq c^2 > 0$. A volatility call option with strike K and maturity T is a financial derivative with pay-off

$$Y = N\left(\sqrt{\frac{\kappa}{T}\int_0^T \sigma^2(t)\,dt} - K\right)_+,$$

where κ is the number of trading days in one year and N is the notional amount of the option. Find the partial differential equation and the terminal value satisfied by the pricing function of this derivative (max. 5 points).

Solution of exercise 2. We have

$$S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma\widetilde{W}(t)}.$$

Hence

$$Y = \left(S(0)e^{(r-\sigma^2/2)T/2+\sigma X(T)/T} - K\right)_+, \quad X(T) = \int_0^T W(t)\,dt.$$

According to the hint, $X(T)$ is normally distributed. Moreover

$$\mathbb{E}[X(T)] = \int_0^T \mathbb{E}[W(t)]\,dt = 0$$

and

$$\mathrm{Var}[X(T)] = \mathbb{E}[X(T)^2] = \mathbb{E}[\int_0^T\int_0^T W(t)W(s)\,dt\,ds] = \int_0^T\int_0^T \mathbb{E}[W(t)W(s)]\,dt\,ds$$

If $s < t$, then $\mathbb{E}[W(t)W(s)] = \mathbb{E}[(W(t) - W(s))W(s)] + \mathbb{E}[W(s)^2] = \mathbb{E}[W(s)^2] = s$ and similarly $\mathbb{E}[W(t)W(s)] = t$ if $s > t$. Hence

$$\mathrm{Var}[X(T)] = \int_0^T\int_0^T \min(s,t)\,dt\,ds = 2\int_0^T\int_0^s t\,dt\,ds = \frac{T^3}{3}.$$

Thus $X(T) \in N(0, T^3/3)$ and so $G := X(T)/\sqrt{T^3/3} \in N(0,1)$. The pay-off of the derivative can be rewritten as

$$Y = \left(S(0)e^{(r-\sigma^2/2)T/2+\sigma\sqrt{T/3}\,G} - K\right)_+ = \left(S(0)e^{(\hat{r}-\hat{\sigma}^2/2)T+\hat{\sigma}\sqrt{T}G} - K\right)_+$$

where

$$\hat{\sigma} = \frac{\sigma}{\sqrt{3}}, \quad \hat{r} = \frac{r}{2} - \frac{\sigma^2}{12}$$

It follows that the Asian call option with geometric mean is equivalent to a standard call option in a market with parameters $\hat{r}$, $\hat{\sigma}$. Hence, by the Black-Scholes formula,

$$\Pi_Y(0) = S(0)\Phi(a) - K\,e^{-\hat{r}T}\Phi(b)$$

where

$$a = \frac{\log \frac{S(0)}{K} + (\hat{r} - \hat{\sigma}^2/2)T}{\hat{\sigma}\sqrt{T}}, \quad b = a + \hat{\sigma}\sqrt{T}$$

and Φ is the standard normal distribution. This concludes the first part of the exercise (4 points). Now let $Q = \left(K - \exp\left(\frac{1}{T}\int_0^T \log S(t)\, dt\right)\right)_+$ be the pay-off of the Asian put with geometric mean. Using the put-call parity for the standard call/put options we have

$$\Pi_Y(0) - \Pi_Q(0) = S(0) - Ke^{-\hat{r}T}$$

Note that the put call parity depends on the volatility of the asset. This concludes the second part of the exercise (1 point)