TMA371 Partial Differential Equations TM, 2000-04-25. Solutions

1. Consider the Dirichlet problem

$$-\nabla \cdot (a(x)\nabla u) = f(x), \quad x \in \Omega \subset \mathbb{R}^2, \qquad u = 0, \text{ for } x \in \partial\Omega.$$

Assume that c_0 and c_1 are constants such that $c_0 \le a(x) \le c_1$, $\forall x \in \Omega$ and let $U = \sum_{j=1}^{N} \alpha_j w_j(x)$ be a Galerkin approximation of u in a finite dimensional subspace M of $H_0^1(\Omega)$. Prove the a priori error estimate

$$||u - U||_{H_0^1(\Omega)} \le C \inf_{\chi \in M} ||u - \chi||_{H_0^1(\Omega)}.$$

Solution: We formulate the continuous and approximate weak formulations as:

$$(a\nabla u, \nabla v) = (f, v), \qquad \forall v \in H_0^1(\Omega), \tag{1}$$

and

$$(a\nabla U, \nabla v) = (f, v), \qquad \forall v \in M, \tag{2}$$

respectively, so that

$$(a\nabla (u - U), \nabla v) = 0, \qquad \forall v \in M. \tag{3}$$

We may write

$$u - U = u - \chi + \chi - U,$$

where χ is an arbitrary element of M, it follows that

$$\begin{aligned}(a\nabla(u-U), \nabla(u-U)) &= (a\nabla(u-U), \nabla(u-\chi)) \\ &\le ||a\nabla(u-U)|| \cdot ||u-\chi||_{H_0^1(\Omega)} \\ &\le c_1 ||u-U||_{H_0^1(\Omega)} ||u-\chi||_{H_0^1(\Omega)},\end{aligned} \tag{4}$$

on using (3), Schwarz's inequality and the boundedness of a. Also, from the boundedness condition on a, we have that

$$(a\nabla(u-U), \nabla(u-U)) \ge c_0 ||u-U||^2_{H_0^1(\Omega)}. \tag{5}$$

Combining (4) and (5) gives

$$||u - U||_{H_0^1(\Omega)} \le \frac{c_1}{c_0} ||u - \chi||_{H_0^1(\Omega)}.$$

Since χ is an arbitrary element of M, and the proof is complete.

2. See the proof of Theorem 14.1 in the book.

3. Consider the boundary value problem

$$\begin{cases} -\Delta u + u = f, & x \in \Omega \subset \mathbb{R}^d, \\ n \cdot \nabla u = g, & \text{on} \quad \Gamma := \partial\Omega, \end{cases}$$

where n is the outward unit normal to Γ.

(a) Show the following stability estimate: for some constant C,

$$||\nabla u||^2_{L_2(\Omega)} + ||u||^2_{L_2(\Omega)} \le C(||f||^2_{L_2(\Omega)} + ||g||^2_{L_2(\Gamma)}).$$

(b) Formulate a finite element method for the $1D$-case and derive the resulting system of equations for $\Omega = [0, 1]$, $f(x) = 1$, $g(0) = 3$ and $g(1) = 0$.

Solution: a) Multiplying the equation by u and performing partial integration we get

$$\int_\Omega \nabla u \cdot \nabla u + uu - \int_\Gamma n \cdot \nabla u u = \int_\Omega fu,$$

i.e.,

$$\|\nabla u\|^2 + \|u\|^2 = \int_\Omega fu + \int_\Gamma gu \le \|f\|\|u\| + \|g\|_\Gamma C_\Omega(\|\nabla u\| + \|u\|) \tag{6}$$

where $\|\cdot\| = \|\cdot\|_{L_2(\Omega)}$ and we have used the inequality $\|u\| \le C_\Omega(\|\nabla u\| + \|u\|)$. Further using the inequality $ab \le a^2 + b^2/4$ we have

$$\|\nabla u\|^2 + \|u\|^2 \le \|f\|^2 + \frac{1}{4}\|u\|^2 + C\|g\|_\Gamma^2 + \frac{1}{4}\|\nabla u\|^2 + \frac{1}{4}\|u\|^2$$

which gives the desired inequality.
b) Consider the variational formulation

$$\int_\Omega \nabla u \cdot \nabla v + uv = \int_\Omega fv + \int_\Gamma gv, \tag{7}$$

set $U(x) = \sum U_j \psi_j(x)$ and $v = \psi_i$ in (7) to obtain

$$\sum_{j=1}^N U_j \int_\Omega \nabla\psi_j \cdot \nabla\psi_i + \psi_j\psi_i = \int_\Omega f\psi_i + \int_\Gamma g\psi_i, \quad i = 1, \dots, N.$$

This gives $AU = b$ where $U = (U_1, \dots, U_N)^T,\ b = (b_i)$ with the elements

$$b_i = h,\ i = 2, \dots, N-1, \quad b(N) = h/2, \quad b(1) = h/2 + 3,$$

and $A = (a_{ij})$ with the elements

$$a_{ij} = \begin{cases} -1/h + h/6, & \text{for } i = j+1 \text{ and } i = j-1 \\ 2/h + 2h/3, & \text{for } i = j \text{ and } i = 2, \dots, N-1 \\ 0, & \text{else.} \end{cases}$$

4. Consider the initial-boundary value problem

$$\begin{cases} \dot{u} - \Delta u = 0, & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Show the stability estimates:

$$\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2\, ds \le \|u_0\|^2 + C\int_0^t \|f(s)\|^2\, ds,$$

$$\|\nabla u(t)\|^2 + \int_0^t \|\Delta u(s)\|^2\, ds \le \|\nabla u_0\|^2 + C\int_0^t \|f(s)\|^2\, ds.$$

Solution: Multiplication by u gives

$$(\dot{u}, u) + \|\nabla u\|^2 = (f, u) \le \|f\|\,\|u\| \le C\|f\|\,\|\nabla u\| \le \frac{1}{2}C\|f\|^2 + \frac{1}{2}\|\nabla u\|^2.$$

Here $(\dot{u}, u) = \frac{1}{2}\frac{d}{dt}\|u\|^2$ and hence

$$\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 \le C\|f\|^2.$$

Integrating $\int_0^t \cdot\, ds$ gives the first inequality. To get the second one we multiply by $-\Delta u$:

$$(\dot{u}, -\Delta u) + ||\Delta u||^2 \leq ||f||\,||\Delta u|| \leq \frac{1}{2}||f||^2 + \frac{1}{2}||\Delta u||^2.$$

Here $(\dot{u}, -\Delta u) = \frac{1}{2}\frac{d}{dt}||\nabla u||^2$ and hence

$$\frac{d}{dt}||\nabla u||^2 + ||\Delta u||^2 \leq ||f||^2.$$

5. Prove an a priori and an a posteriori error estimate for a finite element method for problem

$$-u'' + u' + u = f, \quad \text{in} \quad (0,1), \qquad u(0) = u(1) = 0.$$

Solution: Multiplication by $v \in H_0^1(I)$ and integration gives the variational formulation: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + u'v + uv) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{8}$$

The corresponding finite element method $cG(1)$ is: Find $U \in V_h^0$ such that

$$\int_I (U'v' + U'v + Uv) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{9}$$

where

$$V_h^0 = \{v : v \text{ piecewise linear, continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Let now $e = u - U$, then (8)-(9) gives that

$$\int_I (e'v' + e'v + ev) = 0, \quad \forall v \in V_h^0. \tag{10}$$

We define the energy inner product and norm viz;

$$(v,w)_E = \int_I (v'w' + vw), \qquad ||v||_E^2 = (v.v)_E = \int_I [(v')^2 + v^2].$$

To derive **a posteriori** error estimates first we note that

$$\int_I e'e = \frac{1}{2}\int_0^1 \frac{d}{dx}(e^2) = 0, \qquad \text{since } e(0) = e(1) = 0. \tag{11}$$

Thus

$$\begin{aligned}
||e||_E^2 &= \int_I (e'e' + ee) = \int_I (e'e' + e'e + ee) \\
&= \int_I ((u-U)'e' + (u-U)'e + (u-U)e) = \{v = e \text{ in } (8)\} \\
&= \int_I fe - \int_I (U'e' + U'e + Ue) = \{v = \pi_h e \text{ in } (9)\} \\
&= \int_I f(e - \pi_h e) - \int_I (U'(e-\pi_h e)' + U'(e - \pi_h e) + U(e - \pi_h e)) \\
&= \{\text{partial integration on each subinterval}\} = \int_I R(U)(e - \pi_h e),
\end{aligned}$$

where $R(U) := f + U'' - U' - U = f - U' - U$, in each subinterval. Let now $||\cdot|| = ||\cdot||_{L_2(I)}$, then

$$\begin{aligned} ||e||_E^2 &\leq ||hR(U)||\,||h^{-1}(e - \pi_h e)|| \\ &\leq C_i ||hR(U)||\,||e'|| \leq C_i ||hR(U)||\,||e||_E, \end{aligned}$$

and hence

$$||e||_E \leq C_i ||hR(U)||.$$

As for the **a priori** error estimate, a similar approach would result

$$\begin{aligned} ||e||_E^2 &= \int_I (e'e' + ee) = (11) = \int_I (e'e' + e'e + ee) \\ &= \int_I (e'(u-U)' + e'(u-U) + e(u-U)) = \{v = U - \pi_h u \text{ in } (10)\} \\ &= \int_I (e'(u-\pi_h u)' + e'(u-\pi_h u) + e(u-\pi_h u)) \\ &\leq ||e'||\,||(u-\pi_h u)'|| + ||e'||\,||u-\pi_h u|| + ||e||\,||u-\pi_h u|| \\ &\leq ||e||_E \{||(u-\pi_h u)'|| + 2||u-\pi_h u||\} \\ &\leq C_i ||e||_E \{||hu''|| + ||h^2 u''||\}, \end{aligned}$$

consequently

$$||e||_E \leq C_i \{||hu''|| + ||h^2 u''||\}.$$

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