TMA371 Partial Differential Equations TM, 1998-12-15. Solutions

1. Let $N_1 = (0,0)$, $N_2 = (1,1)$ be the nodes and ϕ_1 and ϕ_2 the piecewise linear basis functions on the triangulation in the fig. below, such that $supp\phi_1 = T_1 \cup T_5$, $\phi_1(N_1) = 1$, and $supp\phi_2 = \Omega$, $\phi_2(N_2) = 1$. Then

$$\begin{aligned}
\phi_1|_{T_1} &= 1-x, & \nabla\phi_1|_{T_1} &= (-1,0)\\
\phi_1|_{T_5} &= 1-y, & \nabla\phi_1|_{T_5} &= (0,-1)\\
\phi_2|_{T_1} &= y, & \nabla\phi_2|_{T_1} &= (0,1)\\
\phi_2|_{T_5} &= x, & \nabla\phi_2|_{T_5} &= (1,0)\\
\phi_2|_{T_2} &= 1-x+y, & \nabla\phi_2|_{T_2} &= (-1,1)\\
\phi_2|_{T_3} &= 3-x-y, & \nabla\phi_2|_{T_3} &= (-1,-1)\\
\phi_2|_{T_4} &= 1+x-y, & \nabla\phi_2|_{T_4} &= (1,-1).
\end{aligned}$$

Now we multiply the differential equation $-\Delta u = 1$ by a test function $v \in H^1(\Omega)$, $v = 0$, on Γ_D and ($\partial_n v = 0$, on Γ_N) and integrate over Ω to get

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega v,$$

taking $v = \phi_j$, $j = 1,2$ we obtain the linear system of equations $AU = b$, where $A = (a_{ij}) = ((\nabla\phi_i, \nabla\phi_j))$, $i,j = 1,2$ and $b = (b_1, b_2)$, $b_j = \int_\Omega \phi_j, j = 1,2$. Now using the computed gradients and the volume formula for a prisma we have

$$\begin{aligned}
&\int_\Omega \phi_1 = \frac{1}{3}(|T_1| + |T_5|) = 1/3,\\
&\int_\Omega \phi_2 = 5 \times \frac{1}{3}(|T_1|) = 5/6,\\
&(\nabla\phi_1, \nabla\phi_1) = \int_{T_1\cup T_5} \nabla\phi_1 \cdot \nabla\phi_1 = 1 \cdot |T_1| + 1 \cdot |T_5| = 1,\\
&(\nabla\phi_2, \nabla\phi_2) = \int_\Omega \nabla\phi_2 \cdot \nabla\phi_2 = 1 \cdot |T_1| + 2 \times (|T_2| + |T_3| + |T_4|) + 1 \cdot |T_5| = 4,\\
&(\nabla\phi_1, \nabla\phi_2) = \int_{T_1\cup T_5} \nabla\phi_1 \cdot \nabla\phi_2 = 0.
\end{aligned}$$

Thus $AU = b$ is equivalent to

$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 5/6 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = 1/24 \begin{pmatrix} 8 \\ 5 \end{pmatrix}.$$

2. See your class notes (exercise solutions: Problem 9.19).

3. See lecture notes (Chapter 8: obs! a parallel version).

4. a) Multiply the equation for u by u and integrate with respect to x. Using partial integration and boundary conditions we get

$$0 = \int_0^1 fu = \int_0^1 (\dot{u} - u'')u = \int_0^1 \dot{u}u + \int_0^1 u'u' = \frac{1}{2}\frac{d}{dt}\int_0^1 u^2 + \int_0^1 (u')^2,$$

which is the desired identity (E1).

Now (E1) together with Poincare inequality $\|u\| \le \|u'\|$ gives that

$$\frac{d}{dt}\|u\|^2 + 2\|u\|^2 \le 0, \Longleftrightarrow \frac{d}{dt}(\|u\|^2 e^{2t}) = (\frac{d}{dt}\|u\|^2 + 2\|u\|^2)e^{2t} \le 0.$$

Integrating with respect to time variable from 0 to t leads to

$$\|u\|^2 e^{2t} - \|u_0\|^2 \le 0, \text{ i.e., } \|u\|^2 \le e^{-2t}\|u_0\|^2,$$

which, taking the square root, gives the estimate (E2).

b) Let $w = u - u_s$, then w satisfies the differential equation

$$\dot{w} - w'' = \dot{u} - u'' + u_s'' = f - f = 0,$$

so that we can apply (E2) to w to get

$$\|u - u_s\| \le e^{-t}\|u_0 - u_s\| \to 0, \quad \text{as } t \to \infty.$$

5. We multiply the differential equation by a test function $v \in H_0^1(I)$, $I = (0,1)$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + 2xu'v + 2uv) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{1}$$

A *Finite Element Method* with $cG(1)$ reads as followa: Find $U \in V_h^0$ such that

$$\int_I (U'v' + 2xU'v + 2Uv) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{2}$$

where

$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$

Now let $e = u - U$, then (1)-(2) gives that

$$\int_I (e'v' + 2xe'v + 2ev) = 0, \quad \forall v \in V_h^0. \tag{3}$$

A posteriori error estimate: We note that using $e(0) = e(1) = 0$, we get

$$\int_I 2xe'e = \int_I x\frac{d}{dx}(e^2) = (xe^2)|_0^1 - \int_I e^2 = -\int_I e^2, \tag{4}$$

so that

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I (e'e' + 2xe'e + 2ee) \\
&= \int_I ((u-U)'e' + 2x(u-U)'e + 2(u-U)e) = \{v = e \text{ in}(1)\} \\
&= \int_I fe - \int_I (U'e' + 2xU'e + 2Ue) = \{v = \pi_h e \text{ in}(2)\} \\
&= \int_I f(e - \pi_h e) - \int_I \Big(U'(e-\pi_h e)' + 2xU'(e - \pi_h e) + 2U(e - \pi_h e)\Big) \\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e - \pi_h e),
\end{aligned} \tag{5}$$

where $\mathcal{R}(U) := f + U'' - 2xU' - 2U = f - 2xU' - 2U$, (for approximation with picewise linears, $U \equiv 0$, on each subinterval). Thus (5) implies that

$$\begin{aligned}\|e\|_{H^1}^2 &\le \|h\mathcal{R}(U)\|\|h^{-1}(e-\pi_h e)\|\\ &\le C_i\|h\mathcal{R}(U)\|\|e'\| \le C_i\|h\mathcal{R}(U)\|\|e\|_{H^1},\end{aligned}$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i\|h\mathcal{R}(U)\|.$$

A priori error estimate: We use (4) and write

$$\begin{aligned}\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I (e'e' + 2xe'e + 2ee)\\ &= \int_I \Big(e'(u-U)' + 2xe'(u-U) + 2e(u-U)\Big) = \{v = U - \pi_h u \text{ in}(3)\}\\ &= \int_I \Big(e'(u-\pi_h u)' + 2xe'(u-\pi_h u) + 2e(u-\pi_h u)\Big)\\ &\le \|(u-\pi_h u)'\|\|e'\| + 2\|u-\pi_h u\|\|e'\| + 2\|u-\pi_h u\|\|e\|\\ &\le \{\|(u-\pi_h u)'\| + 4\|u-\pi_h u\|\}\|e\|_{H^1}\\ &\le C_i\{\|hu''\| + \|h^2u''\|\}\|e\|_{H^1},\end{aligned}$$

this gives that

$$\|e\|_{H^1} \le C_i\{\|hu''\| + \|h^2u''\|\},$$

which is the a priori error estimate.

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