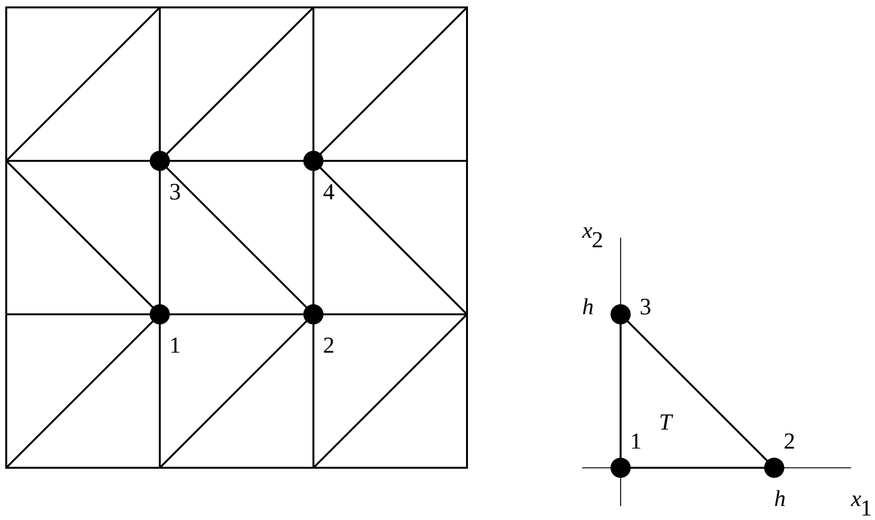
TMA371 Partial Differential Equations TM, 1998–04–14. Solutions.

1. Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega$. The cG(1) method is

$$\begin{aligned} &U \in V_h, \\ &(\nabla U, \nabla v) + (U, v) = (f, v) \quad \forall v \in V_h. \end{aligned}$$



Making the "Ansatz" $U(x) = \sum_{i=1}^{4} \xi_i \varphi_i(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{i=1}^{4} \xi_i \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx + \int_\Omega \varphi_i \varphi_j \, dx \Big) = \int_\Omega f \varphi_j \, dx, \quad j = 1, \dots 4,$$

or, in matrix form,

$$(K + M)\xi = F,$$

where $K_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_j = (f, \varphi_j)$ is the load vector.

We first compute the mass and stiffness matrices for the reference triangle T. The local basis functions are

$$\begin{aligned} \phi_1(x_1, x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, & \nabla\phi_1(x_1, x_2) &= -\frac{1}{h}\begin{bmatrix}1\\1\end{bmatrix}, \\ \phi_2(x_1, x_2) &= \frac{x_1}{h}, & \nabla\phi_2(x_1, x_2) &= \frac{1}{h}\begin{bmatrix}1\\0\end{bmatrix}, \\ \phi_3(x_1, x_2) &= \frac{x_2}{h}, & \nabla\phi_3(x_1, x_2) &= \frac{1}{h}\begin{bmatrix}0\\1\end{bmatrix}. \end{aligned}$$

Hence, with $|T| = \int_T dx = h^2/2$,

$$\begin{aligned} m_{11} &= (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2 \, dx_1 \, dx_2 = \frac{h^2}{12}, \\ k_{11} &= (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2 \, dx = \frac{2}{h^2}|T| = 1. \end{aligned}$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = \frac{|T|}{3} \sum_{j=1}^{3} \phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0 + \frac{1}{4} + \frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Continuing like this, we easily compute

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad k = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M, K from the local matrices m, k:

$$\begin{aligned}
&M_{ii} = 4m_{22} + 2m_{11} = \frac{1}{2}h^2, && K_{ii} = 4k_{22} + 2k_{11} = 4, \\
&M_{12} = M_{13} = M_{24} = M_{34} = 2m_{12} = \frac{1}{12}h^2, && K_{12} = K_{13} = K_{24} = K_{34} = 2k_{12} = -1, \\
&M_{14} = 0, && K_{14} = 0, \\
&M_{23} = 2m_{23} = \frac{1}{12}h^2, && K_{23} = 2k_{23} = 0.
\end{aligned}$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $K_{ij} = K_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 6 & 1 & 1 & 0 \\ 1 & 6 & 1 & 1 \\ 1 & 1 & 6 & 1 \\ 0 & 1 & 1 & 6 \end{bmatrix}, \quad K = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}.$$

2. The problem can be written

$$u \in V; \quad a(u, v) = (f, v) \quad \forall v \in V,$$

and the cG(1) problem is

$$U \in V_h; \quad a(U, v) = (f, v) \quad \forall v \in V_h,$$

where $V = H_0^1(0,1)$ and $a(u,v) = (\nabla u, \nabla v) + (u, v)$. Hence, we have the orthogonality

$$a(u - U, v) = 0 \quad \forall v \in V_h.$$

We prove an a priori error estimate in the energy norm. The energy norm is defined by $\|v\|_E^2 = \|\nabla v\|^2 + \|v\|^2 = a(v,v)$. Hence, with $e = u - U$ and the standard interpolant $\pi_h u \in V_h$,

$$\begin{aligned}
\|e\|_E^2 &= a(e, e) \\
&= a(e, u - \pi_h u) + a(e, \pi_h u - U) \\
&= a(e, u - \pi_h u) \quad \text{(orthogonality)} \\
&\leq \|e\|_E \, \|u - \pi_h u\|_E, \quad \text{(Cauchy's inequality)}
\end{aligned}$$

so that

$$\|e\|_E \leq \|u - \pi_h u\|_E.$$

The interpolation error estimates give

$$\begin{aligned}
\|\nabla(u - \pi_h u)\| &\leq C_1 \|hD^2 u\| \\
\|u - \pi_h u\| &\leq C_2 \|h^2 D^2 u\| \leq C_2 h_{\max} \|hD^2 u\| \leq C_3 \|hD^2 u\|,
\end{aligned}$$

so that $\|u - \pi_h u\|_E \leq \sqrt{C_1^2 + C_3^2}\|hD^2 u\| = C\|hD^2 u\|$ and $\|e\|_E \leq C\|hD^2 u\|$.

3. (Göteborg University). See the book.

3. (Chalmers). (a) Multiply the equation by $\dot{u}$:

$$\begin{aligned}
&(\ddot{u}, \dot{u}) - (\Delta u, \dot{u}) = 0,\\
&(\ddot{u}, \dot{u}) + (\nabla u, \nabla \dot{u}) = 0,\\
&\frac{1}{2}\frac{d}{dt}\big(\|\dot{u}\|^2 + \|\nabla u\|^2\big) = 0,\\
&\frac{1}{2}\big(\|\dot{u}(t)\|^2 + \|\nabla u(t)\|^2\big) = \frac{1}{2}\big(\|u_1\|^2 + \|\nabla u_0\|^2\big).
\end{aligned}$$

This means that the energy $E(t) = \frac{1}{2}\big(\|\dot{u}(t)\|^2 + \|\nabla u(t)\|^2\big)$ is conserved.

(b) Set $v_1 = \dot{u}$, $v_2 = u$. Then

$$\begin{aligned}
\dot{v}_1 - \Delta v_2 &= 0,\\
-\Delta \dot{v}_2 + \Delta v_1 &= 0.
\end{aligned}$$

The cG(1)-method is to find $U = (U_1, U_2) \in W_k^{(1)}$ such that

$$\begin{aligned}
&\int_{I_n} \big((\dot{U}_1, w_1) + (\nabla U_2, \nabla w_1)\big)\, dt = 0,\\
&\int_{I_n} \big((\nabla \dot{U}_2, \nabla w_2) - (\nabla U_1, \nabla w_2)\big)\, dt = 0,\\
&U_{1,n-1}^+ = P_n U_{1,n-1}^-, \quad U_{2,n-1}^+ = \pi_n U_{2,n-1}^-,
\end{aligned}$$

for all $(w_1, w_2) \in W_{kn}^{(0)}$ and

$$U_j(x,t)|_{S_n} = U_{j,n}^-(x)\frac{t - t_{n-1}}{k_n} + U_{j,n-1}^+(x)\frac{t_n - t}{k_n}, \quad j = 1, 2.$$

Taking $w_j = k_n^{-1}\int_{I_n} U_j\, dt$ (which is constant) and adding the resulting equations we get

$$\begin{aligned}
&\int_{I_n} \big((\dot{U}_1, U_1) + (\nabla \dot{U}_2, \nabla U_2)\big)\, dt = 0,\\
&\frac{1}{2}\frac{d}{dt}\int_{I_n} \big(\|U_1\|^2 + \|\nabla U_2\|^2\big)\, dt = 0,\\
&\|U_{1,n}^-\|^2 + \|\nabla U_{2,n}^-\|^2 = \|U_{1,n-1}^+\|^2 + \|\nabla U_{2,n-1}^+\|^2\\
&\qquad = \|P_n U_{1,n-1}^-\|^2 + \|\nabla \pi_n U_{2,n-1}^-\|^2\\
&\qquad = \|U_{1,n-1}^-\|^2 + \|\nabla U_{2,n-1}^-\|^2,
\end{aligned}$$

where in the last step we used the assumption $V_{n-1} \subset V_n$. Without this assumption we only get

$$\|U_{1,n}^-\|^2 + \|\nabla U_{2,n}^-\|^2 \le \|U_{1,n-1}^-\|^2 + \|\nabla U_{2,n-1}^-\|^2.$$

4. See the book, pp. 257–258. The quantity in (c) is a stability factor that measures the accumulation of error in the a posteriori error estimates.

5. (a) We have $\Gamma_- = \{x \in \Gamma : \beta(x)\cdot n(x) < 0\}$ and $\Gamma_+ = \{x \in \Gamma : \beta(x)\cdot n(x) \ge 0\}$. Multiply the equation by u:

$$(\dot{u}, u) + (\beta\cdot\nabla u, u) + (\alpha u, u) = (f, u). \tag{1}$$

Green's formula gives

$$\begin{aligned}(\beta\cdot\nabla u,u)&=\int_\Gamma \beta\cdot n\,u^2\,ds-\int_\Omega u\nabla\cdot(\beta u)\,dx\\&=\int_\Gamma \beta\cdot n\,u^2\,ds-\int_\Omega u\big(\nabla\cdot\beta u+\beta\cdot\nabla u\big)\,dx\\&=\int_{\Gamma_+}\beta\cdot n\,u^2\,ds+\int_{\Gamma_-}\beta\cdot n\,u^2\,ds-(\nabla\cdot\beta u,u)-(\beta\cdot\nabla u,u),\end{aligned}$$

so that, since $\beta\cdot n<0$ and $u=g$ on Γ_-,

$$2(\beta\cdot\nabla u,u)=\int_{\Gamma_+}\beta\cdot n\,u^2\,ds-\int_{\Gamma_-}|\beta\cdot n|\,g^2\,ds-(\nabla\cdot\beta u,u).$$

We also have

$$2(\dot u,u)=\frac{d}{dt}\|u\|^2.$$

We insert this into (1):

$$\text{(2)}\quad \frac{d}{dt}\|u\|^2+\int_{\Gamma_+}\beta\cdot n\,u^2\,ds+2\big((\alpha-\tfrac12\nabla\cdot\beta)u,u\big)=2(f,u)+\int_{\Gamma_-}|\beta\cdot n|\,g^2\,ds.$$

Here we have $\alpha-\frac12\nabla\cdot\beta\ge c>0$ and

$$2(f,u)\le 2\|f\|\,\|u\|=2\big(c^{-1/2}\|f\|\big)\big(c^{1/2}\|u\|\big)\le\frac1c\|f\|^2+c\|u\|^2,$$

so that (2) becomes

$$\frac{d}{dt}\|u\|^2+\int_{\Gamma_+}\beta\cdot n\,u^2\,ds+2c\|u\|^2\le\frac1c\|f\|^2+c\|u\|^2+\int_{\Gamma_-}|\beta\cdot n|\,g^2\,ds.$$

This implies

$$\begin{aligned}&\|u(\cdot,T)\|^2+c\int_0^T\|u(\cdot,t)\|^2\,dt+\int_0^T\int_{\Gamma_+}n\cdot\beta\,u^2\,ds\,dt\\&\qquad\le\|u_0\|^2+\frac1c\int_0^T\|f(\cdot,t)\|^2\,dt+\int_0^T\int_{\Gamma_-}|n\cdot\beta|\,g^2\,ds\,dt.\end{aligned}$$

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