TMA371 Partial Differential Equations TM, 1998–08–29. Solutions.

1.

$$
\begin{aligned}
\|\nabla(v-\pi_h v)\|_{L_p} &\le C_i \|hD^2 v\|_{L_p}\\
\|v-\pi_h v\|_{L_p} &\le C_i \|h^2 D^2 v\|_{L_p}
\end{aligned}
$$

One proof in the case $p=\infty$, $d=1$ is in the book. Here is another one: let $I_i=(x_{i-1},x_i)$, $h_i=x_i-x_{i-1}$ and $e(x)=v(x)-(\pi_h v)(x)$. Then $e(x_{i-1})=e(x_i)=0$ and by Rolle's theorem there is $\tilde{x}\in I_i$ such that $e'(\tilde{x})=0$ and hence

$$e'(x)=\int_{\tilde{x}}^{x} e''(y)\,dy.$$

This proves

$$\|e'\|_{L_\infty(I_i)} \le h_i\|e''\|_{L_\infty(I_i)} = \|hv''\|_{L_\infty(I_i)},$$

since $e''(x)=v''(x)$ and $h_i=h(x)$ in I_i. Taylor's formula gives

$$e(x_i)=e(\tilde{x})+(x-\tilde{x})e'(x_i)+\tfrac{1}{2}(x-\tilde{x})^2 e''(\xi).$$

Here $e(x_i)=0$, $e'(\tilde{x})=0$ and the maximum of $|e(x)|$ is attained at $\tilde{x}$. Hence

$$\|e\|_{L_\infty(I_i)} \le \tfrac{1}{2}h_i^2\|e''\|_{L_\infty(I_i)} = \tfrac{1}{2}\|h^2 v''\|_{L_\infty(I_i)}.$$

This proves

$$\|e'\|_{L_\infty(0,1)} \le \|hv''\|_{L_\infty(0,1)}, \quad \|e\|_{L_\infty(0,1)} \le \tfrac{1}{2}\|h^2 v''\|_{L_\infty(0,1)}.$$

2. See the book.

3. Let $V=\{v\in H^1(0,1): v(0)=v(1)=0\}$ with $\|v\|_V=\|v\|_{H^1}=\sqrt{\|v\|^2+\|v'\|^2}$. Multiplication of the differential equation by $v\in V$, integration by parts taking the boundary conditions into account, leads to the variational formulation

$$
\begin{aligned}
&u\in V,\\
&a(u,v)=L(v) \quad \forall v\in V,
\end{aligned}
$$

with

$$
\begin{aligned}
a(u,v)&=(u',v')+\alpha(u,v),\\
L(v)&=(f,v).
\end{aligned}
$$

If $\alpha>0$ then we have

$$
\begin{aligned}
a(u,u)&=\|u'\|^2+\alpha\|u\|^2 \ge \min(1,\alpha)\|u\|_{H^1}^2,\\
a(u,v)&\le \|u'\|\,\|v'\|+\alpha\|u\|\,\|v\| \le (1+\alpha)\|u'\|_{H^1}\,\|v'\|_{H^1},\\
|L(v)|&\le \|f\|\,\|v\| \le \|f\|\,\|v\|_{H^1}.
\end{aligned}
$$

4.

$$
\begin{aligned}
&\dot{u}-\Delta u=f, \quad x\in\Omega,\ t>0,\\
&u=0, \quad x\in\partial\Omega,\ t>0,\\
&u(x,0)=u_0(x), \quad x\in\Omega.
\end{aligned}
$$

(a) Prove (with $\|u\|=(\int_\Omega u^2\,dx)^{1/2}$)

$$
\begin{aligned}
\|u(t)\|^2+\int_0^t\|\nabla u(s)\|^2\,ds &\le \|u_0\|^2+C\int_0^t\|f(s)\|^2\,ds,\\
\|\nabla u(t)\|^2+\int_0^t\|\Delta u(s)\|^2\,ds &\le \|\nabla u_0\|^2+C\int_0^t\|f(s)\|^2\,ds.
\end{aligned}
$$

Multiplication by u gives

$$(\dot{u},u)+\|\nabla u\|^2=(f,u)\le\|f\|\,\|u\|\le C\|f\|\,\|\nabla u\|\le\tfrac{1}{2}C\|f\|^2+\tfrac{1}{2}\|\nabla u\|^2.$$

Here $(\dot{u}, u) = \frac{1}{2}\frac{d}{dt}\|u\|^2$ and hence

$$\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 \le C\|f\|^2.$$

Integration $\int_0^t \cdot \, ds$ gives the first inequality. To get the second one we multiply by $-\Delta u$:

$$(\dot{u}, -\Delta u) + \|\Delta u\|^2 \le \|f\|\,\|\Delta u\| \le \tfrac{1}{2}\|f\|^2 + \tfrac{1}{2}\|\Delta u\|^2.$$

Here $(\dot{u}, -\Delta u) = \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2$ and hence

$$\frac{d}{dt}\|\nabla u\|^2 + \|\Delta u\|^2 \le \|f\|^2.$$

(b) See the book.

5. Multiplication by u gives

$$\epsilon\|u'\|^2 + \int_0^1 au'u\,dx + \|u\|^2 = (f,u) \le \|f\|\,\|u\| \le \tfrac{1}{2}\|f\|^2 + \tfrac{1}{2}\|u\|^2.$$

Here

$$\begin{aligned} \int_0^1 au'u\,dx &= \tfrac{1}{2}\int_0^1 a\frac{d}{dx}u^2\,dx \\ &= \tfrac{1}{2}a(1)u(1)^2 - \tfrac{1}{2}\int_0^1 a'u^2\,dx \ge 0, \end{aligned} \tag{1}$$

and hence

$$\epsilon\|u'\|^2 + \tfrac{1}{2}\|u\|^2 \le \tfrac{1}{2}\|f\|^2.$$

This proves

$$\sqrt{\epsilon}\|u'\| \le \|f\|, \quad \|u\| \le \|f\|. \tag{2}$$

Multiplication by au' gives

$$-\epsilon\int_0^1 u''au'\,dx + \|au'\|^2 + \int_0^1 au'u\,dx \le \tfrac{1}{2}\|f\|^2 + \tfrac{1}{2}\|au'\|^2.$$

Hence by (1)

$$\begin{aligned} \|au'\|^2 &\le \|f\|^2 + \epsilon\int_0^1 a\frac{d}{dx}(u')^2\,dx \\ &= \|f\|^2 - \epsilon a(0)u'(0)^2 - \epsilon\int_0^1 a'(u')^2\,dx \\ &\le \|f\|^2 + \|a'\|\,\epsilon\|u'\|^2 \\ &\le \|f\|^2 + C\epsilon\|u'\|^2. \end{aligned}$$

Using also (2) we conclude

$$\|au'\| \le C\|f\|. \tag{3}$$

Finally, by the differential equation and (2), (3),

$$\epsilon\|u''\| = \|f - au' - u\| \le \|f\| + \|au'\| + \|u\| \le C\|f\|.$$

/stig