Written exam TMA401 / MMA400 2015-01-05

Solutions (sketches)

Problem 1: See previous written exam

Problem 2: Set $u_1(x) = 1$, $u_2(x) = x$, $u_3(x) = x^2$, $x \in [0,1]$.

$H = L^2([0,1])$ with standard scalar product. Apply Gram-Schmidt

$e_1 = \frac{1}{\|u_1\|} u_1$ i.e. $e_1(x) = 1$, $x \in [0,1]$

$v_1(x) = u_2(x) - \langle u_2, e_1 \rangle e_1(x) = x - \frac{1}{2}$, $e_2 = \frac{1}{\|v_1\|} v_1$ i.e.

$e_2(x) = 2\sqrt{3}\,(x - \frac{1}{2})$, $x \in [0,1]$

$v_2(x) = u_3(x) - \langle u_3, e_1 \rangle e_1 - \langle u_3, e_2 \rangle e_2$, $e_3 = \frac{1}{\|v_2\|} v_2$ i.e.

$e_2(x) = \frac{2\sqrt{15}}{\sqrt{7}}(x^2 - x + \frac{1}{2})$

For an arbitrary $h \in L^2([0,1])$, the closest f in M,

$M = \{u_1, u_2, u_3\}^{\perp}$ is given by

$$f(x) = h(x) - \langle h, e_1 \rangle e_1 - \langle h, e_2 \rangle e_2 - \langle h, e_3 \rangle e_3 =$$

$$= h(x) - \int_0^1 h(t)\,dt - \int_0^1 h(t)\, 2\sqrt{3}(t - \tfrac{1}{2})\,dt \cdot 2\sqrt{3}(x - \tfrac{1}{2}) -$$

$$- \int_0^1 h(t) \frac{2\sqrt{15}}{\sqrt{7}}(t^2 - t + \tfrac{1}{2})\,dt \; \frac{2\sqrt{15}}{\sqrt{7}}(x^2 - x + \tfrac{1}{2}) =$$

$$= h(x) - \int_0^1 h(t)\,dt - \int_0^1 h(t)(t - \tfrac{1}{2})\,dt \cdot 12(x - \tfrac{1}{2}) -$$

$$- \int_0^1 h(t)(t^2 - t + \tfrac{1}{2})\,dt \; \frac{60}{7}(x^2 - x + \tfrac{1}{2}).$$

Answer: see above

Problem 3: Assume k continuous with $\int_0^\infty |k(y)|\,dy < \infty$

Set $Kf(x) = \int_0^\infty k(x+y) f(y)\,dy$, $x \in [0,\infty)$.

① K linear on $L^2([0,\infty))$: trivial

② K bounded:

$$\|Kf\|_{L^2}^2 = \int_0^\infty \left| \int_0^\infty k(x+y) f(y)\,dy \right|^2 dx \le$$

$$\le \int_0^\infty \left(\int_0^\infty |k(x+y)| \cdot |f(y)|\,dy \right)^2 dx \le \{\text{Hölder's } \le\} \le$$

$$\le \int_0^\infty \int_0^\infty |k(x+y)|\,dy \int_0^\infty |k(x+y)| \cdot |f(y)|^2\,dy\;dx \le$$

$$\le \int_0^\infty \int_0^\infty |k(y)|\,dy \cdot \int_0^\infty |k(x+y)| \cdot |f(y)|^2\,dy\;dx =$$

$$= \int_0^\infty |k(y)|\,dy \cdot \int_0^\infty \int_0^\infty |k(x+y)|\,dx\; |f(y)|^2\,dy \le$$

$$\le \left(\int_0^\infty |k(y)|\,dy \right)^2 \|f\|_{L^2}^2$$

Hence $\|Kf\|_{L^2} \leq \int_0^\infty |k(y)|\,dy \, \|f\|_{L^2}$ for $f \in L^2([0,\infty))$

So $\|K\|_{L^2\to L^2} \leq \int_0^\infty |k(y)|\,dy < \infty$

③ K compact

Set $h_n(x) = \begin{cases} 1 & x \in [0,n] \\ 1+n-x & x \in (n,n+1] \\ 0 & x \in (n+1,\infty) \end{cases}$

and $k_n(x) = k(x) h_n(x)$. Then $k_n \in C([0,\infty))$,

$k(x) = k_n(x)$ for $x \in [0,n]$ and $k_n(x) = 0$ for $x \in [n+1,\infty)$

and also $\int_0^\infty |k(y) - k_n(y)|\,dy \leq \int_n^\infty |k(y)|\,dy \to 0, \ n \to \infty$.

Set $K_n f(x) = \int_0^\infty k_n(x+y) f(y)\,dy$, $x \in [0,\infty)$

Note that $K_n f(x) = 0$ for $x \geq n+1$ and hence

$K_n f(x) = \int_0^{n+1} k_n(x+y) f(y)\,dy \quad x \in [0,n+1]$

It is well-known that K_n is a compact linear mapping on $L^2([0,n+1])$ and hence on $L^2([0,\infty))$ since $k_n \in C([0,\infty))$.

Since $\|K - K_n\|_{L^2\to L^2} \leq \int_0^\infty |k(y) - k_n(y)|\,dy \to 0, \ n \to \infty$

it follows that K is a compact linear mapping on $L^2([0,\infty))$.

Problem 4: See textbook

Problem 5: We first note that

$x \in N(T) \Leftrightarrow T(x) = 0 \Leftrightarrow \langle T(x), y\rangle = 0$ all $y \in H$

$\Leftrightarrow \langle x, T^*(y)\rangle = 0$ all $y \in H \Leftrightarrow x \in R(T^*)^\perp$

Hence $N(T)^\perp = R(T^*)^{\perp\perp}$

We know that $A^{\perp\perp} = \overline{\operatorname{Span} A}$ for any set $A \subset H$

Hence $R(T^*)^{\perp\perp} = \overline{\operatorname{Span} R(T^*)} = \overline{R(T^*)}$

Problem 6: H Hilbert space, $T \in \mathcal{K}(H)$ self-adjoint

$(e_n)_{n=1}^\infty$ complete ON-sequence of eigenvectors to T corresp. to the eigenvalues $(\lambda_n)_{n=1}^\infty$

$\lambda \notin \{0\} \cup \{\lambda_n : n = 1,2,\dots\}$

$\Rightarrow \lambda I - T$ invertible and

$$(\lambda I - T)^{-1}(x) = \frac{1}{\lambda}\left(x + \sum_{n=1}^\infty \frac{\lambda_n}{\lambda - \lambda_n}\langle x, e_n\rangle e_n\right)$$

Solution: Assume $(\lambda I - T)(y) = x$. Then using HS-thm

$$\lambda y = x + Ty = x + \sum_{n=1}^{\infty} \lambda_n \langle y, e_n \rangle e_n .$$

This gives

$$\lambda \langle y, e_k \rangle = \langle x, e_k \rangle + \lambda_k \langle y, e_k \rangle \qquad k = 1, 2, \ldots$$

and so

$$\langle y, e_k \rangle = \frac{1}{\lambda - \lambda_k} \langle x, e_k \rangle$$

We get

$$y = \frac{1}{\lambda}\left(x + \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda - \lambda_n} \langle x, e_n \rangle e_n \right) \quad \ldots (*).$$

Let y be defined by $(*)$. Note that, by Bessel's $\leq$, $\left(\frac{\lambda_n}{\lambda-\lambda_n} \langle x, e_n \rangle \right)_{n=1}^{\infty} \in \ell^2$ since $\lambda \neq 0$ and $\lambda \neq \lambda_n$ all n and $\lambda_n \to 0$ as $n \to \infty$. Moreover we note that

$$(\lambda I - T)\left(\frac{1}{\lambda}\left(x + \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda - \lambda_n} \langle x, e_n \rangle e_n\right)\right) =$$

$$= x + \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda-\lambda_n} \langle x, e_n \rangle e_n - \frac{1}{\lambda}\left(T(x) + \sum_{n=1}^{\infty} \frac{\lambda_n^2}{\lambda - \lambda_n} \langle x, e_n \rangle e_n\right) =$$

$$= x + \frac{1}{\lambda}\left[\sum_{n=1}^{\infty} \frac{\lambda \lambda_n}{\lambda - \lambda_n} \langle x, e_n \rangle e_n - \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n - \sum_{n=1}^{\infty} \frac{\lambda_n^2}{\lambda-\lambda_n} \langle x, e_n \rangle e_n \right.$$

$$= x + \frac{1}{\lambda}\left[\sum_{n=1}^{\infty} \underbrace{\left(\frac{\lambda\lambda_n}{\lambda-\lambda_n} - \lambda_n - \frac{\lambda_n^2}{\lambda-\lambda_n} \right)}_{=0} \langle x, e_n \rangle e_n \right] = x$$

We have shown that for each $x \in H$ there exists a unique $y \in H$ s.t. $(\lambda I - T)(y) = x$ and this y is given by $(*)$. Hence

$$(\lambda I - T)^{-1}(x) = \frac{1}{\lambda}(I + \tilde{K})(x)$$

where $\tilde{K}(x) = \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda - \lambda_n} \langle x, e_n \rangle e_n$.

Here $\tilde{K} \in B(H)$ since $\left(\frac{\lambda_n}{\lambda - \lambda_n} \right)_{n=1}^{\infty}$ bounded

Hence $(\lambda I - T)^{-1} \in B(H)$.