A

A note on distributions in $\mathbb{R}^n$

This works exactly in the same way

except we need to define convergence in $S(\mathbb{R}^n)$.

We say $\varphi_n \to 0$ in $S(\mathbb{R}^n)$ if

for all $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^n$

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha \frac{\partial^\beta}{\partial x^\beta} \varphi_n \right| \to 0 \quad \text{when } n \to \infty$$

A tempered distribution on $\mathbb{R}^n$ is a

(complex-valued) linear functional T

$T : S(\mathbb{R}^n) \longrightarrow \mathbb{C}$ such that

$T(\varphi_n) \to 0$ for all sequences $\varphi_n \to 0$ in $S(\mathbb{R}^n)$

<u>Ex</u> $\delta \otimes \delta$:

$$\langle \delta \otimes \delta, \varphi \rangle = \varphi(0,0)$$

<u>Note</u> This is a common notation:

Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$

Then

$$h(x,y) = f(x)\, g(y)$$

is often denoted

$$h = f \otimes g.$$

<u>Note</u> In this way, $\otimes$ is <u>not</u> commutative.

Multi dimensional Fourier transform

Notation $x = (x_1, x_2, \dots, x_n)$

$s = (s_1, \dots\dots s_n)$

If $k = (k_1, k_2, \dots, k_n) \in \in \mathbb{N}^n$, let $|k| = k_1 + \dots + k_n$

We may DEFINE $x^k = x_1^{k_1} \cdots x_n^{k_n}$

$$\frac{\partial^{|k|}}{\partial x^k} = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$$

Def If $f : \mathbb{R}^n \to \mathbb{C}$

we define

$$\hat{f}(s) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot s} f(x)\,dx$$

and the inverse

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot s} \hat{f}(s)\,ds$$

satisfies almost all rules from 1-D F-transform:

Examples $f(x,y) \supset \hat{f}(u,v)$

$f(ax, by) \supset \frac{1}{|ab|} \hat{f}\left(\frac{u}{a}, \frac{v}{b}\right)$

Note if a, b complex... need to be careful.

$f(x-a, y-b) \supset e^{-2\pi i(au+bv)} \hat{f}(u,v)$

$\left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial y}\right)^n f(x,y) \supset (2\pi i u)^m (2\pi i v)^n \hat{f}(u,v)$

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y) \supset -4\pi^2(u^2+v^2) \hat{f}(u,v)$

One can in general define $T_1 \otimes T_2$

by
$$\langle T_1 \otimes T_2 , \varphi_1 \otimes \varphi_2 \rangle = T_1(\varphi_1)\, T_2(\varphi_2)$$

Problem: Is it enough to define

$T_1 \otimes T_2$ on the subset of $S(\mathbb{R}^2)$

given by $\{ \varphi \in S(\mathbb{R}^2) : \varphi(x,y) = \varphi_1(x)\, \varphi_2(y) ,\ \varphi_1, \varphi_2 \in S(\mathbb{R}) \}$?

Solution One can approximate $\varphi(x,y)$

as $$\varphi(x,y) = \sum_{k=1}^{N} \varphi_{1k}(x)\, \varphi_{2k}(y)$$

Then $$\langle T_1 \otimes T_2 , \varphi \rangle = \sum_{k=1}^{N} \langle T_1 \otimes T_2 , \varphi_{1k} \otimes \varphi_{2k} \rangle + \dots$$
$$= \sum_{k=1}^{N} T_1(\varphi_{1k})\, T_2(\varphi_{2k}) \quad \dots\dots$$

Fourier transform of tensor products:

Let $\varphi(x,y) = \varphi_1(x)\, \varphi_2(y)$

Then $$\hat{\varphi}(\xi, \eta) = \iint e^{-2\pi i (x\xi + y\eta)} \varphi_1(x)\, \varphi_2(y)\, dx\,dy$$

$$= \hat{\varphi}_1(\xi)\, \hat{\varphi}_2(\eta)$$

So $\mathcal{F}(\varphi_1 \otimes \varphi_2) = \hat{\varphi}_1\, \hat{\varphi}_2$

This makes it easy to compute
the Fourier transform of some
tempered distributions on $\mathbb{R}^2$ (or any $\mathbb{R}^n$)