Convergence of Fourier Series

Theorem Suppose that f has period 1 and $f \in C^2$. Then, for all $x \in \mathbb{R}$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x} \qquad (*)$$

where

$$c_k = \int_0^1 e^{-2\pi i k x} f(x)\,dx$$

Proof

$$c_k = \int_0^1 e^{-2\pi i k x} f(x)\,dx = (\text{partial integration twice})$$

$$= \int_0^1 \frac{e^{-2\pi i k x}}{(-2\pi i k)^2} f''(x)\,dx = O\left(\frac{1}{k^2}\right)$$

so the series $(*)$ is absolutely convergent.

Hence $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x}$ is continuous, and belongs to S'. Therefore also

$$\hat{f} = \mathcal{F}\left(\sum c_k e^{2\pi i k \cdot}\right) \in S', \text{ and}$$

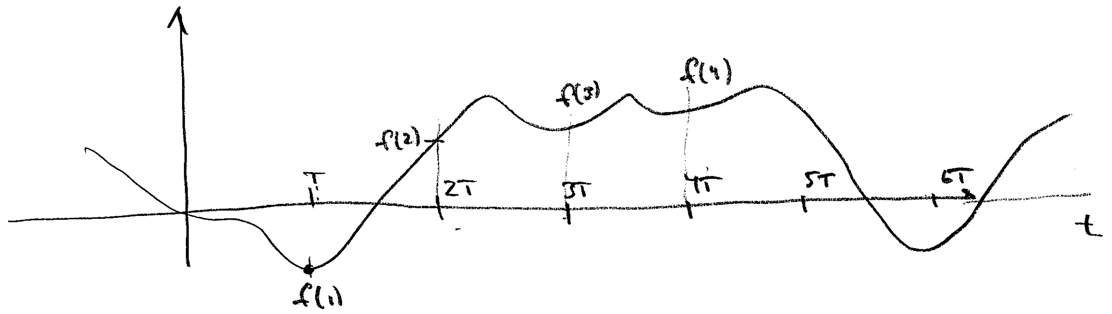
$$\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle = \int_{-\infty}^{\infty} f(x)\,\hat{\varphi}(x)\,dx = \int_0^1 f(x) \sum_{k=-\infty}^{\infty} \hat{\varphi}(x+k)\,dx$$

$$= \int_0^1 f(x) \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)\, e^{-2\pi i k x}\,dx = \sum_{k=-\infty}^{\infty} \varphi(k)\, c_k = \langle \sum c_k \delta_k, \varphi \rangle$$

The DFT and the FFT

[this part contains notation from Bracewell.
Hence T is a "sampling interval",
$\tau \in \{0,1,\dots,N-1\}$, and we write, for a function $V(t)$,

$$f(\tau) = V(t_0 + \tau T)$$]



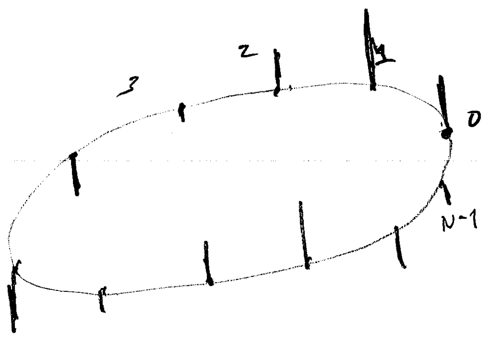
Definition $f(\cdot)$ has Discrete Fourier Transform

$$F(\nu) = \frac{1}{N} \sum_{\tau=0}^{N-1} f(\tau)\, e^{-2\pi i \nu \tau / N}$$

Prop $f(\tau)$ can be recovered by

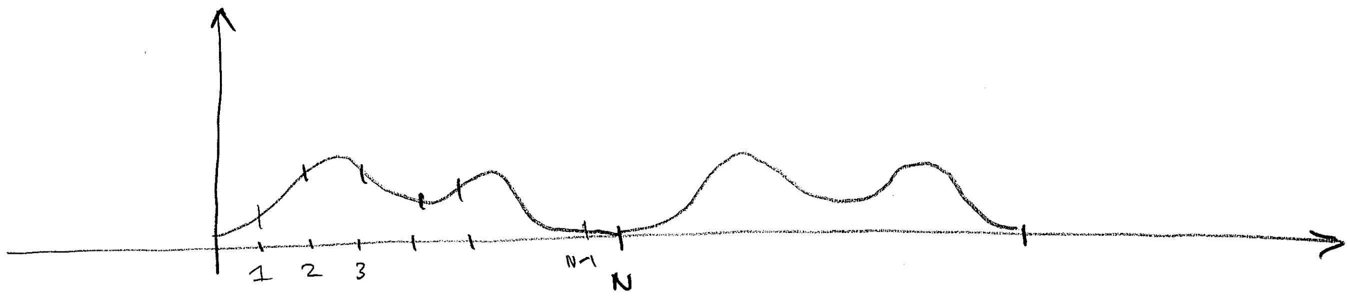
$$f(\tau) = \sum_{\nu=0}^{N-1} F(\nu)\, e^{2\pi i \nu \tau / N}$$

Proof By direct calculation.



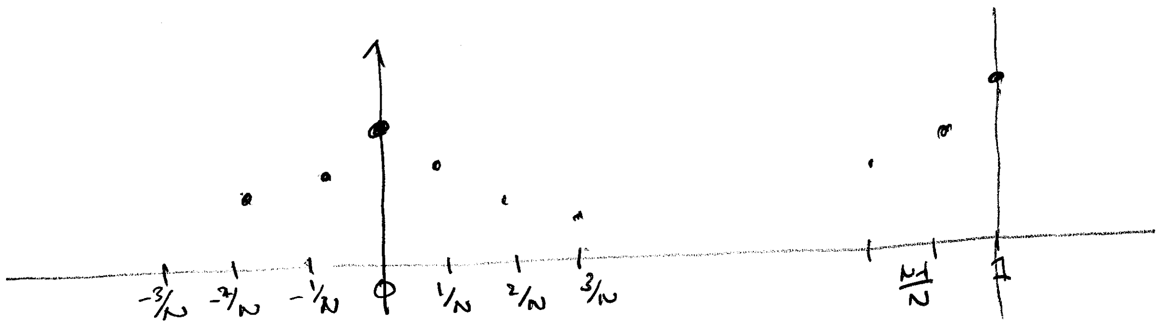
From here on, F and f are considered to be periodic with period N.

Interpretation



$f(t)$ periodic with period N, sampling frequency 1

$\hat{f}(\omega)$ has period 1 (and this implies that $\hat{f}(\omega)$ can only be represented up to $|\omega| < \frac{1}{2}$)



It is more natural to consider

$F_{\frac{N}{2}+1}, \ldots F_{N-1}$ as $F_{-\frac{N}{2}+1}, \ldots, F_{-1}$

Also, if $f(\tau)$ is real, then

$$F(N-\nu) = \frac{1}{N}\sum_{\tau=0}^{N-1} f(\tau)\, e^{-2\pi i (N-\nu)\tau/N}$$

$$= \frac{1}{N}\sum_{\tau=0}^{N-1} f(\tau)\, e^{-2\pi i (-\nu)\tau/N} = F(-\nu) = F(\nu)^*$$

so there are only N independent <u>real</u> coefficients.

Cyclic convolution

For f and g 2π-periodic, we define

$$f * g(\theta) = \int_0^{2\pi} f(x)\, g(\theta - x)\, dx$$

and for two N-periodic sequences:

$$f * g(\tau) = \sum_{\tau'=0}^{N-1} f(\tau')\, g(\tau - \tau')$$

Note if f and g are not considered periodic, but only defined on $0, 1, 2, \ldots, N-1$ one must write

$$f * g(\tau) = \sum_{\tau=0}^{N-1} f(\tau)\, g(\tau - \tau' + N\, H(\tau' - \tau))$$

where H is the Heaviside function.

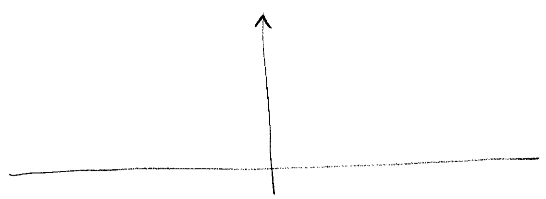
Computational rules

we write $f \supset F$

Note we may write $F(\nu) = \frac{1}{N} \sum_{\tau=-N/2}^{\frac{N}{2}-1} f(\tau)\, e^{-2\pi i \nu\tau/N}$

and $f(\tau) = \sum_{\nu=-N/2}^{\frac{N}{2}-1} F(\nu)\, e^{2\pi i \nu\tau/N}$

provided that N is even.



Rules for computation

$f(\tau) \supset F(\nu)$ (recall: $\tau, \nu \in \{0, \dots, N-1\}$)

$F(\nu) \supset \frac{1}{N} f(-\tau)$

$f^*(\nu) \supset F^*(-\nu)$ (for complex sequences)

$f_1 * f_2(\nu) \supset N F_1(\nu) F_2(\nu)$

[discrete convolution:

$$f_1 * f_2(\nu) = \sum_{k=0}^{N-1} f_1(\nu - k) f_2(k)$$

and here $\nu - k$ must be considered mod $N-1$, i.e. counting

$0 \; 1 \; 2 \; \dots \; N-1, \underset{\overset{\|}{0}}{N}, 1, 2, \dots$]

$$f_1(\tau) f_2(\tau) \supset \sum_{\nu'=0}^{N-1} F_1(\nu') F_2(\nu - \nu')$$

$$f(0) = \sum_{\nu=0}^{N-1} F(\nu)$$

$$\sum_{\tau=0}^{N-1} |f(\tau)|^2 = N \sum_{\nu=0}^{N-1} |F(\nu)|^2$$

The Packing theorem

The operator Pack_k is defined as

$$f = \{f(0), f(1), \ldots, f(N-1)\} \longmapsto$$

$$\text{Pack}_k f = \{\underbrace{\underbrace{f(0), f(1), \ldots, f(N-1)}_{N}, \underbrace{0, 0, \ldots, 0}_{N}, \ldots \underbrace{0, 0\,0\,0 \ldots 0}_{N}}_{k \text{ times}}\}.$$

We then have

$$g = \text{Pack}_k(f(\tau)) \supset G(\nu) \qquad \nu = 0, \ldots, Nk-1$$

where

$$G(\nu) = \begin{cases} \frac{1}{k} F\left(\frac{\nu}{k}\right) & \nu = 0, k, \ldots, kN-k \end{cases}$$

Proof:

$$G(\nu) = \frac{1}{kN} \sum_{\tau=0}^{kN-1} g(\nu)\, e^{-2\pi i \tau \nu / kN}$$

$$= \frac{1}{kN} \sum_{\tau=0}^{N-1} f(\nu)\, e^{-2\pi i \tau \nu / kN} =$$

$$= \frac{1}{k} F\left(\frac{\nu}{k}\right) \qquad \text{when} \quad \nu = 0, k, 2k, \ldots$$

Similarity theorem

$\text{Stretch}_K\ f$ (or upsampling) is defined as

$$g = \text{Stretch}_K\ f$$

$$g(\tau) = \begin{cases} f(\tau/K) & \tau = 0, K, \dots, (N-1)K \\ 0 & \text{elsewhere} \end{cases}$$

$$f = \{ f(0), \dots, f(N-1) \}$$

$$g = \{ \underbrace{f(0), 0, 0, 0, \dots 0}_{K}, \underbrace{f(1), 0 \dots 0}_{K} \dots \dots \underbrace{f(N-1), 0, 0 \dots 0}_{K} \}$$

Then $g \supset G$ with

$$G(\nu) = \begin{cases} \frac{1}{K} F(\nu) & \nu = 0, \dots, N-1 \\ \frac{1}{K} F(\nu - N) & \nu = N \dots 2N-1 \\ \frac{1}{K} F(\nu - 2N) \dots & \nu = 2N \dots 3N-1 \end{cases}$$

Example

$$\{ f(0), f(1) \dots f(3) \} \xrightarrow{\text{Stretch}_2} \{ f(0), 0, f(1), 0, f(2), 0, f(3), 0 \}$$

$$\{ F(0), F(1), F(2), F(3) \} \longrightarrow \frac{1}{2} \{ F(0), F(1), F(2), F(3), F(0), F(1), F(2), F(3) \}$$

$$\{ f(0), f(1), f(2), f(3) \} \xrightarrow{\text{Repeat}_2} \{ f(0), f(1), f(2), f(3), f(0), f(1), f(2), f(3) \}$$

$$\longrightarrow \{ F(0), 0, F(1), 0, F(2), 0, F(3), 0 \}$$

Note: What Bracewell calls stretch is more often called upsampling

The Summation Rule

Ex $\{f_0, f_1, \ldots, f_5\} \supset \{F_0, F_1, \ldots, F_5\}$

$\{f_0+f_3, f_1+f_4, f_2+f_5\} \supset 2\{F_0, F_2, F_4\}$

$\{f_0+f_2+f_4, f_1+f_3+f_5\} \supset 3\{F_0, F_3\}$

Theorem Let $g(\tau) = \sum_{l=0}^{k-1} f(\cancel{\tau + l}N)$ f(t-l N/k)

then $G(\nu) = k\,F(k\nu)$

The Decimation Rule (down sampling)

Ex $\{f_0, f_2, f_4\} \supset \{F_0+F_3, F_1+F_4, F_2+F_5\}$

Theorem Let $g(\tau) = f(\tau k)$.

Then $G(\nu) = \sum_{l=0}^{k-1} F(\nu - lk)$

F(v-l N/k)

Proof Both theorems are proven

by direct computation.

Examples

Let $f = \{f_0, f_1, f_2, f_3\} \supset F = \{F_0, F_1, F_2, F_3\}$

Shift Let $\tau_1 f = \{f_3, f_0, f_1, f_2\}$

Then $\tau_1 f \supset \mathcal{E}_1 f = \{F_0, \omega F_1, \omega^2 F_2, \omega^3 F_3\}$

where $\omega = e^{-2\pi i/N}$

(prove this by direct calculation!)

A diagram

$$
\begin{tikzcd}
\{f_0, f_1, f_2, f_3\} = f \arrow[r, "\mathcal{F}"] \arrow[d, "\tau_1"'] & F = \{F_0, F_1, F_2, F_3\} \arrow[d, "\mathcal{E}_1"] \\
\{f_3, f_0, f_1, f_2\} \arrow[r, "\mathcal{F}"] & \{F_0, \omega F_1, \omega^2 F_2, \omega^3 F_3\}
\end{tikzcd}
$$

Def For a periodic sequence $f = \{f_0, \dots, f_{N-1}\}$ we denote by $\tau_1 f$ the translated sequence $\{f_{N-1}, f_0, \dots, f_{N-2}\}$.

Def For a periodic sequence $F = \{F_0, \dots, F_{N-1}\}$ we write $\mathcal{E}_1 F$ for the sequence obtained by multiplication by powers of $\omega = e^{-2\pi i/N}$:

$$\mathcal{E}_1 F = \{F_0, \omega F_1, \omega^2 F_2, \dots, \omega^{N-1} F_{N-1}\}.$$

Stretch (upsampling) and Repeat.

For a sequence $f = \{f_0, \dots, f_{N-1}\}$ we denote by $Z_2 f$, the sequence

$$Z_2 f = \{f_0, 0, f_1, 0, \dots, f_{N-1}, 0\}$$

and similarly we write $Z_k f$ for the sequence $\{\underbrace{f_0, 0, \dots 0}_{k}, f_1, 0 \dots\dots, \underbrace{f_{N-1}, 0 \dots 0}_{k}\}$

For $F = \{F_0, \dots, F_{N-1}\}$ we write

$$R_k F = \{\underbrace{\underbrace{F_0, \dots, F_{N-1}}, \underbrace{F_0, \dots, F_{N-1}}, \dots, \underbrace{F_0, \dots, F_{N-1}}}_{k \text{ times}}\}.$$

A diagram:

$$\begin{tikzcd} f \arrow[r, "\mathscr{F}"] \arrow[d, "Z_2"'] & F \arrow[d, "\frac{1}{2} R_2"] \\ g \arrow[r, "\mathscr{F}"] & G \end{tikzcd}$$

This notation is useful for describing the FFT (Fast Fourier transform).

Note that the <u>cost</u> of computing the DFT is rather high:

$$F(\nu) = \frac{1}{N} \sum_{\tau=0}^{N-1} \omega^{\tau\nu} f(\tau)$$

$\Rightarrow$ in total $N \times N$ multiplications and $N(N-1)$ additions if all the $\omega^{\nu\tau}$ have been precomputed.

DFT in Matrix form

Let $f = \{f_0, f_1, \ldots, f_{N-1}\} = \begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix} \in \mathbb{R}^N$ (or $\mathbb{C}^N$)

i.e. we consider f to be a vector in $\mathbb{R}^N$ or $\mathbb{C}^N$.

We also let $F = \begin{bmatrix} F_0 \\ \vdots \\ F_{N-1} \end{bmatrix} \in \mathbb{C}^N$

Then we have

$$F = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & & \omega^{2(N-1)} \\ \vdots & & & & \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & & \omega^{(N-1)(N-1)} \end{bmatrix} f$$

where $\omega = e^{-2\pi i/N}$. The matrix can be written $\left\{ \omega^{(i-1)(j-1)} \right\}_{j=1\ i=1}^{N\ N}$.

Fast Fourier Transform

"decimation in time" (Bracewell's notation)

Recall:

$$\begin{tikzcd} f \arrow[r, "\mathcal{F}"] \arrow[d, "\tau_1"'] & F \arrow[d, "\mathcal{E}_1"] \\ g \arrow[r, "\mathcal{F}"] & G \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, "\mathcal{F}"] \arrow[d, "Z_2"'] & F \arrow[d, "\frac{1}{2}\mathcal{R}_2"] \\ g \arrow[r] & G \end{tikzcd}$$

One step in the FFT

(an even number)

(split even and odd)

$$\begin{tikzcd}[column sep=small]
 & \{f_0, f_1, \dots, f_{2N-1}\} \arrow[d] \arrow[dr] & & \\
\{\tilde{F}_0, \dots, \tilde{F}_{2N-2}\} \arrow[dd, "\frac{1}{2}\mathcal{R}_2"'] & \{f_0, f_2, \dots, f_{2N-2}\} \arrow[l, "\mathcal{F}"'] \arrow[dd, "Z_2"'] & \{f_1, f_3, \dots, f_{2N-1}\} \arrow[r, "\mathcal{F}"] \arrow[d, "Z_2"'] & \{\tilde{F}_1, \tilde{F}_3, \dots, \tilde{F}_{2N-1}\} \arrow[d, "\frac{1}{2}\mathcal{R}_2"] \\
 & & \{f_1, 0, f_3 \dots f_{2N-1}\} \arrow[r] \arrow[d, "\tau_1"'] & \{\ \} \arrow[d, "\mathcal{E}_1"] \\
\{\ \} \arrow[ddr] & \{f_0, 0, f_2, \dots, f_{2N-2}, 0\} \arrow[l] \arrow[d] & \{0, f_1, 0, f_3, \dots, f_{2N-1}\} \arrow[r] \arrow[dl, "\text{add}"] & \{\ \} \arrow[ddll, "\text{add}"] \\
 & \{f_0, f_1, f_2, \dots, f_{2N-1}\} \arrow[d] & & \\
 & \{F_0, F_1, \dots, F_{N-1}\} & &
\end{tikzcd}$$