Problems for Fourier and Wavelet Analysis

This is a colletion of problems collected from different sources: from the recommenced books to the course, from previous exam papers, from the lecture notes, etc. A note of the form "(BRx.y)" means that the problem is problem nr y from chapter x in Bracewell[1], or a minor modification of that problem. Some problems are taken from Frazier[2].

1. Show that $\mathcal{F}\mathcal{F}\mathcal{F}\mathcal{F}f = f$.
2. (a) Prove that all functions $f : \mathbb{R} \mapsto \mathbb{C}$ can be written as the sum of an even and an odd function. Think of possible *different* generaizations of this to functions $f : \mathbb{R}^2 \mapsto \mathbb{C}$
   (b) Prove that the fourier transform of a real, odd function is odd.
3. When splitting a function $f : \mathbb{R} \mapsto \mathbb{C}$ into an odd and even part, $f = f_e + f_o$, the result depends on the choice of origin. Show that $\int_{\mathbb{R}}(f_o^2 + f_e^2)\,dx$ does not depend on the choice of origin. Say something about the physical interpretation of this statement.
4. Show that if f and g are continuous functions, such that $f = g$ when interpreted as distributions, then $f(x) = g(x)$ for all $x \in \mathbb{R}$.
5. Let $f_n(x) = \sqrt{n}\exp(-n\pi x^2)$. Show that if $\varphi \in \mathcal{S}$, then $\lim_{n\to\infty}\langle f_n, \varphi\rangle = \varphi(0)$. A tempting way of expressing this is to say that $f_n \to_{n\to\infty} \delta$. In which sense could that be correct?
6. Does $\exp -x^2 \cos(\exp(x^2))$ belong to $\mathcal{S}$?
7. Let T be a tempered distribution. Recall that differentiation of T is *defined* by $\langle DT, \varphi\rangle = -\langle T, D\varphi\rangle$. Give a detailed proof that DT is a tempered distribution.
8. The *autocorrelation function* of a function $f : \mathbb{R} \mapsto \mathbb{C}$ is defined as

$$C(x) \;=\; \frac{f \star f(x)}{\|f\|_2^2} \;\equiv\; \frac{\int_{\mathbb{R}} \bar{f}(y) f(x+y)\,dy}{\int_{\mathbb{R}} |f(y)|^2\,dy}$$

   Prove that the autocorrelation function is Hermitian.
9. Translate f so that $f * f(x)$ attains its maximum at $x = 0$. Is it correct to state that the new origin is an "axis of maximal symmetry"?
10. Let $f(x) = 1 - x$ when $0 < x < 1$, and $f(x) = 0$ elsewhere, and let $g(x) = 1 - x/2$ when $0 < x < 2$, and $g8x) = 0$ elsewhere. Compute $f * g$, as much as possible by looking at the graphical representation of the functions. Give the answer as a graph.
11. **Optical sound track**. the optical sound track on old motin-picture films has a bredth b, and it is scanned by a slit of width w. With appropriate normailization, we may say that the scanning introduces convolution by a rectangle function of unit height and width w. Normally the slit should be oriented perpendicularly against the direction of the film. What is the effect of the sound quality if the orientation is perturbed by a small angle ε? How is the function by wich the convolution is done modified?
12. Prove that the map $T : \varphi \mapsto \sum_{n\in\mathbb{Z}} \varphi(n)$ is a tempered distribution.
13. Let φ be a function in the class $\mathcal{S}$. Show that under the given assumptions the following functions are also in $\mathcal{S}$, and check why the assumptions are necessary:

[1] Bracewell: The Fourier Transform and its Applications
[2] M. W. Frazier: An introduction to wavelets through linear algebra, Springer

(a) Assuming $\varphi(0) = 0$, let

$$\psi(x) = \begin{cases} \phi(x)/x & \text{if} x \neq 0 \\ \phi'(0) & \text{if} x = 0 \end{cases}$$

(b) Assuming $\int_{-\infty}^{\infty} \varphi(x)dx = 0$, let

$$\psi(x) = \int_{-\infty}^{x} \varphi(y)\, dy$$

14. Let $\ell(x) = x \log|x| - x$, and define $p_{-k} \in \mathcal{S}$ by taking distributional derivatives: $p_{-k} = D^{k+1}\ell$. Prove that $p_{-1}(\varphi) = \lim_{L\to\infty} \int_{-L}^{L} \frac{\varphi(x)-\varphi(0)}{x}\, dx$

15. Consider the family of functions $\{f_\lambda(x) = \lambda^{-1}\cos(\pi x^2/4\lambda^2)\}_{\lambda>0}$. Investigate whether

$$\lim_{\lambda\to 0^+} \int_{-\infty}^{\infty} \frac{1}{\lambda} \cos\left(\frac{\pi x^2}{2\lambda^2}\right) \varphi(x)\, dx \quad = \quad \varphi(0).$$

16. The composition of two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is denoted $f \circ g$, meaning

$$f \circ g(x) \quad = \quad f(g(x))$$

In general, distributions cannot be composed, but it is possible to compose distributions with smooth functions. Let δ be the Dirac distribution, and let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with isolated zeros x_1, x_2,... Show that, if $f'(x_j) \neq 0$ at all the points x_j, then

$$\delta \circ f \quad = \quad \sum_j \frac{\tau_{x_j}\delta}{|f'(x_j)|},$$

where τ_{x_j} is the translation operator. What can you say if $f'(x_j) = 0$ but $f''(x_j) \neq 0$?

17. Verify the following Fourier transform pairs:

$$\frac{\sin(\cdot)}{(\cdot)} \supset \pi\Pi(\pi\cdot) \qquad \left(\frac{\sin(\cdot)}{(\cdot)}\right)^2 \supset \pi\Lambda(\pi\cdot)$$

$$\frac{\sin(A\cdot)}{(A\cdot)} \supset \frac{\pi}{A}\Pi\left(\frac{\pi\cdot}{A}\right) \qquad \left(\frac{\sin(A\cdot)}{(A\cdot)}\right)^2 \supset \frac{\pi}{A}\Lambda\left(\frac{\pi\cdot}{A}\right)$$

$$\delta(a\cdot) \supset \frac{1}{|a|} \qquad e^{ix} \supset \delta\left(\cdot - \frac{1}{2\pi}\right)$$

18. Check rigorously that

$$\int_{-\infty}^{\infty} e^{-\pi x^2} \varphi(x/n)\, dx \to \varphi(0) \qquad \text{when} \qquad n \to \infty\,.$$

19. Check that

$$\langle T, \varphi\rangle \equiv \lim_{\varepsilon\to 0} \left(-\frac{1}{2} x^{-3/2} \varphi(x)\, dx + \varepsilon^{-1/2}\phi(0) \right)$$

is a distribution; try to do this directly whithout showing that it is the distributional second derivative of $2\sqrt{(x)}H(x)$.

20. Let $T \in \mathcal{S}'$ and $g \in \mathcal{S}$. Show that $D(gT) = (Dg)T + gDT$.

21. For a fixed $s \in \mathbb{R}$, let $\varepsilon_s(x) = \exp(-2\pi i s x)$ The the Fourier transfrom of a function $f(x)$ can be written

$$\int_{\mathbb{R}} f(x)\varphi_s(x)\,dx = \langle f, \varepsilon_s \rangle \,.$$

For a distribution $T \in \mathcal{S}'$ it would be tempting to write

$$\hat{T} = \langle T, \varepsilon_s \rangle \,.$$

What is wrong with that? Could you make sense of such a definition for $T = \delta$?

22. Let T be defined by

$$T(\varphi) = \text{pv} \int_{-\infty}^{\infty} \frac{1}{x}\varphi(x)\,dx \equiv \lim_{\varepsilon \to 0} \left(\int_{\varepsilon}^{\infty} \frac{1}{x}\varphi(x)\,dx + \int_{-\infty}^{-\varepsilon} \frac{1}{x}\varphi(x)\,dx \right) \,.$$

Show that T is a tempered distribution and compute its distributional derivative.

23. Let $f(x) = \begin{cases} 1 & -1/2 < x < 1/2 \\ 0 & \text{otherwise} \end{cases}$. Compute $f * f$ and $f * f * f$. What can you say about the regularity of the resulting functions? What can you say about their supports?

24. For a function f, define the dilation operator S_a by

$$S_z f(x) \quad = \quad \frac{1}{a} f\left(\frac{x}{a}\right) \qquad (a > 0)\,.$$

For $T \in \mathcal{S}'$, define $S_a T$, and show that $S_a T \in \mathcal{S}'$

25. Use the addition theorem for Fourier transforms to compute the Fourier transform of the following functions: a) $1 + \cos(\pi x)$, b) $\text{sinc}(x) + \frac{1}{2}\text{sinc}^2(x/2)$.

26. Use the shift theorem to find the Fourier transform of the following functions: a) $\cos(\pi x)/\left(\pi(x - 1/2)\right)$, b) $\Pi(x)\text{sgn}(x)$.

27. Show that a modulated pulse described by $\Pi(x/X)(1 + M\cos(2\pi F x)\cos(2\pi f x)$ has a spectrum

$$\begin{aligned} &\frac{1}{2}X\left(\text{sinc}(X(\xi + f)) + \text{sinc}(X(\xi - f))\right) \\ &+\frac{1}{4}MX\left(\text{sinc}(X(\xi + f + F)) + \text{sinc}(X(s + f - F)) + \text{sinc}(X(\xi - f + F)) + \text{sinc}(X(s - f - F))\right) \end{aligned}$$

Graph the spectrum.

28. Let z^* denote the complex conjugate of z, and let F and G denote the Fourier transforms of f and$g$. Prove that

$$\int_{-\infty}^{\infty} f^*(u)g^*(x-u)\,du \quad = \quad F^*(-\xi)G^*(-\xi)$$

29. Show from the energy theorem, that

$$\begin{aligned} \int_{-\infty}^{\infty} \text{sinc}^2(x)\cos(\pi x)\,dx \quad &= \quad \frac{1}{2} \qquad \text{and} \\ \int_{-\infty}^{\infty} e^{-\pi x^2}\cos(2\pi a x)\,dx \quad &= \quad e^{-\pi a^2} \end{aligned}$$

30. *Bracewell: chapter 6: 32* Consider a signal $s(t) = e^{-\pi t^2/T^2} e^{i2\pi(f_0 t + \beta t^2)}$. Show that the power spectrum is centered at f_0, and has an equivalent width Δ given by $\Delta = 2^{-1/2}T^{-1}(1 + 4\beta^2 T^4)^{1/2}$.

31. Give a simple expression for $e^{-\cdot^2} * e^{-\cdot^2}(x)$

32. Investigate $(1 + \cdot^2/a^2)^{-1} * (1 + \cdot^2/b^2)^{-1}(x)$ and its width in terms of the widths of the convolved functions.

33. Let W_f denote the equivalent width of f. Show that

$$W_{f*g} = \frac{W_f W_g}{W_{fg}}.$$

34. Show that

$$e^{-\beta^2} \cos(ax) \supset \left(\frac{\pi}{\beta}\right)^{1/2} e^{-\frac{\alpha^2+4\pi^2\xi^2}{4\beta}} \cosh\left(\frac{\pi\alpha\xi}{\beta}\right)$$

35. Suppose $g(x)$ is the result from smoothing a function $f(x)$ by convolution with a rectangle function $\Pi(x)$. Investigate wether it is possible to find an inverse operator $\Pi(x)^{-1}$ such that $\Pi^{-1} * \Pi = \delta$, or whether you can find a Fourier transform for the function $(\mathrm{sinc}(\xi))^{-1}$.

36. Prove the *Packing theorem*, the *Downsampling and upsampling theorems*, and the *Convolution theorem* for the discrete Fourier transfrom.

37. (Br9:2) "A sound track on film is fed into a high-fidelity reproducing system at twice the correct speed. It is physically obvious, and the similarity theorem confirms that the frequency of a sinusoidal input will be doubled. Ponder what the similarity theorem says about the amplitude until this is also obviouls physically".

38. (Br10:5) Assume that a signal $f(x)$ is approximately band limited, meaning that there is an a such that $\int_{|\xi|>a} |\hat{f}(\xi)|^2\, d\xi$ is small. Give an estimate on how great the difference between the original and reconstituted signal can be, if sampling has been carried out as if the signal *were* band limited at a.

39. (Br10:30)

30. Aliasing. It is necessary to predict the sunspot number 6 months ahead for scheduling frequencies for overseas radio communication. Each day the sunspot number is determined, as it has been for the last two centuries; at the end of each month the list is published, together with the mean value for the month. The monthly means, when graphed, give a rather jagged curve, not suited to the prediction of trends, so for each month a 12-month weighted running mean R_0' is calculated from the formula

$$R_0' = \frac{R_{-6}}{24} + \frac{R_{-5}}{12} + \frac{R_{-4}}{12} + \frac{R_{-3}}{12} + \frac{R_{-2}}{12} + \frac{R_{-1}}{12} + \frac{R_0}{12} + \frac{R_1}{12} + \frac{R_2}{12} + \frac{R_3}{12} + \frac{R_4}{12} + \frac{R_5}{12} + \frac{R_6}{24}$$

Naturally, this quantity is only available after a delay. From Fourier analysis of R_0', it is found that although R_0' is reasonably smooth, it nevertheless tends to have wiggles in it with a period of about 8 months; in fact, the Fourier transform peaks up at a frequency of 1/8.4 cycle per month.

Natural periodicities connected with the sun include the 11-year cycle and the 27-day interval between times when the sun presents roughly the same face to the earth, so it is hard to see how 8-month wiggles would have a solar origin. Can you explain why the wiggles are there?

40. (Br10:31)

Theorem for band-limited functions. An article in a current technical journal was being summarized before a discussion group by Smith, who said, "The author refers to a property of band-limited signals according to which two consecutive maxima cannot be more closely spaced than αT, where T is the critical sampling interval. Does anyone know this theorem?" Lee said, "It's more or less obvious. Consecutive maxima of a sinc function range from a greatest separation equal to 2.49 T down to $2T$ in the limit. A superposition of such sinc functions, which after all is what a band-limited function is, cannot possess maxima closer than $2T$. Therefore, $\alpha = 2$." Yanko, who had been thinking, then pointed out that $\sin 2t - \exp\left[-(t/1000)^2\right]$ would be adequately sampled at intervals $T = 1$ and made a sketch to show that the function possessed two rather closely spaced zeros at A and B. "I am pretty sure," he said, "that the integral of a band-limited function is also band-limited. Therefore, the maxima occurring at A and B in the integral of my function are spaced more closely than $2T$. What's more, I can make them as close as I please by adjusting the amplitude of the sine wave." Confirm or disprove the various statements. What is your value of α?

41. (BR11:7) Consider the sequence $f = \{f_0, ..., f_{15}\}$ with DFT $F = \{F_0, ..., F_{15}\}$. Let g be the 32-element sequence $\{f_0, ..., f_{15}, 0, ..., 0\}$, and let G be the corresponding DFT. By the packing theorem, we find that $G(0) = 0.5F(0)$, $G(2) = 0.5F(1)$, etc. Show that the odd-indexed elements of G can be obtained by

$$G(\nu) = \frac{1}{2} \sum_{k=-\infty}^{\infty} F\left(\frac{\nu - 1}{2} - k\right) \operatorname{sinc}(k - 1/2)$$

42. (Br 11:19) A function of a discrete variable τ, $\tau = 0, ..., 31$ is defined by $f(\tau) = \exp(-\tau^2/4)$ for $\tau = 1, ..., 31$, and $f(0) = 1/2$. Compute the DFT and compare with the Fourier transform of $v(t) = \exp(-t^2/4)H(t)$; here $H(t)$ is the Hevyside function.

43. (Br 13:2) Let $\mathcal{H}f$ be the Hilbert transform of a function f. Show that

$$\mathcal{H}(f * g) \quad = \quad \mathcal{H}f * g = f * \mathcal{H}$$

44. (Br 13:3) Explain why a function and its Hilbert transform have the same autocorrelation function

45. (an extension of Br13:19) Let $f : \mathbb{R}^2 \to \mathbb{R}$ have Fourier transform $\hat{f}$, *i.e.*

$$\hat{f}(\zeta) \quad = \quad \int_{\mathbb{R}^2} e^{-2\pi i(z \cdot \eta)} f(z)\, dx dy,$$

where $z = (x, y)^{tr}$, $\eta = (\xi, \eta)^{tr}$. Let A be an invertible 2×2 matrix, and let $b = (b_1, b_2)^{tr} \in \mathbb{R}^2$. Compute the Fourier transform of $g = f(A \cdot +b)$. What happens if A is not invertible?

46. (Br13:20)

Instantaneous musical pitch. A sound wave producing an air-pressure variation at the ear of 0.01 cos $[2\pi(500t + 50t^2)]$ newtons/meter^2 could be described as a moderately loud pure tone (54 decibels above reference level of 0.1 newton/meter^2) with a rising pitch. At $t = 0$ the frequency would be 500 hertz and rising at a rate of 100 hertz per second and after 2 or 3 minutes the sound would fade out as it rose through the limit of audibility.

(*a*) How long would it take for the pitch to rise 1 octave (1) starting from 500 hertz; and (2) starting from 6000 hertz?

(*b*) Calculate the rate of change of frequency in semitones per second.

(*c*) Construct a waveform that would be perceived by the ear as a pure tone having a uniformly rising pitch. Arrange that the tone rises through 500 hertz at a rate of 100 hertz per second.

47. (Br13:21) Two-dimensional diffraction. A plowed field of 300 hectares is three times longer than its width and the furrows are three to the meter. The long axis of the field is oriented 20° east of north. On a (u, v)- plane, with the u axis running east, sketch and dimension the principal features of the two-dimensional Fourier transform of $h(x, y)$, the height of the surface of the field. Conceive of a situation where the Fourier transform of a plowed field might arise.

48. (Br13:23)

X-ray diffraction. The atoms in a certain plane of a crystal lie on a square lattice of spacing 0.34 nanometer and give rise to a square pattern of diffraction spots when illuminated by an x-ray beam, as would be expected since the two-dimensional Fourier transform of $^2\mathrm{III}(x,y)$ is $^2\mathrm{III}(u,v)$. The crystal is now traversed by a microwave acoustic wave that places the plane mentioned in periodic shear. Opinions given by colleagues include the following.

(*a*) The diffraction spots will be displaced and the direction of displacement will depend on the wave direction in the plane.

(*b*) The spots will be widened in one direction only.

(*c*) The spots will be enlarged.

(*d*) The spot structure may be destroyed because the crystal strain could be large enough to prevent the constructive interference of the extremely short x-ray wavelengths on which the spots depend.

(*e*) Any effect would be so small compared with the spot size as to be undetectable.

Comment on these opinions.

49. (Br13:29) Let $\mathcal{H}f$ denote the Hankel transform of a function f. Show that

$$f(r) \quad = \quad \mathcal{H}\left(\frac{1}{q}\frac{d}{dq}\mathcal{H}\left(\frac{1}{r}\frac{d}{dr}f(r)\right)\right)$$

50. Recall that $\ell^2(\mathbb{Z})$ is the set of sequences $\{x_k\}_{k=-\infty}^{\infty}$ such that $\sum |x_k|^2 < \infty$. We also denote by $\ell^2(\mathbb{Z}_N)$ (periodic) sequences of length N.

Let $z \in \ell^2(\mathbb{Z}_{512})$ be defined by

$$z(k) \quad = \quad 3\sin(2\pi i 7k/512) - 4\cos(2\pi i 8k/512)\,.$$

Compute the DFT of$z$.

51. Let $E^{(n)}(k) = \frac{1}{2}e^{2\pi ikn/4}$, $(n = 0, 1, 2, 3)$. Check that $E^(0)$, $E^(1)$, $E^(2)$ and $E^(3)$ form an orthogonal basis for $\ell^2(\mathbb{Z}_4)$.

52. Let N and k be positive integers with $k < N$, and such that k and N are relatively prime.. Let $\omega = e^{2\pi ik/N}$. Prove that $1, \omega, \omega^2, ..., \omega^{N-1}$ are distinct Nth roots of unity (recall that z is an Nth root of unity if $z^N = 1$).

53. Let $T : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$ be a translation invariant linear transformation. Prove that each element of the Fourier basis of $\ell^2(\mathbb{Z}_N)$ is an eigenvector of T.

54. Prove that convolution in $\ell^2(\mathbb{Z}_N)$ is associative, *i.e.*, that

$$(x * y) * z \quad = \quad x * (y * z) \, .$$

55. Define $T : \ell^2(\mathbb{Z}_4) \to \ell^2(\mathbb{Z}_4)$ by

$$(T(z))(k) \quad = \quad 3z(k-2) + iz(k) - (2+i)z(k+1) \, .$$

   (a) Write the matrix that represents T with respect to the standard basis of $\ell^2(\mathbb{Z}_4)$.

   (b) Show by direct computation that the vectors $E^{(0)}$, $E^{(1)}$, $E^{(2)}$, amd $E^{(3)}$, defined above, are eigenvectors of T.

56. Show that if V is a closed subspace of $L^2(\mathbb{R})$ (or more generally, of any Hilbert space H), and $f \in L^2(\mathbb{R})$, then there is a unique element w in V such that $\|f - w\| \leq \|f - v\|$ for all $v \in V$.

57.