MATEMATISKA VETENSKAPER, Chalmers & Göteborgs universitet
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MMA410, TMA 462 Fourier and Wavelet Analysis
Suggestions for solving the problems.

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1. Compute the Fourier transform of

$$f(x) = \begin{cases} 2x & 0 < x < a \\ 1-x & a < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

for $0 < a < 1$. (5p)

Solution: Perhaps easiest to compute directly:

$$\begin{aligned} \hat{f}(\xi) &= 2\int_0^a xe^{-2\pi i x\xi}\,dx + \int_a^1 (1-x)e^{-2\pi i x\xi}\,dx \\ &= -\frac{1-(4i\pi a\xi + 2\pi i\xi - 2)e^{-2\pi i a\xi} + e^{-2\pi i\xi}}{4\pi^2\xi^2} \end{aligned}$$

2. Prove that $|f * g| \le \int_{-\infty}^{\infty} |FG| d\xi$, if F and G are the Fourier transforms of F and G respectively. How does this relation change depending on the definition of the Fourier transform that is used? (5p)

Solution: With $F(\xi) = \int_{-\infty}^{\infty} \exp(-2\pi i x\xi) f(x)\,dx$, we have

- $\mathcal{F}(f * g) = FG$
- $f(x) = \int_{-\infty}^{\infty} \exp(2\pi i x\xi) F(\xi)\,dx$

so $|f * g(x)| = \left|\int_{-\infty}^{\infty} \exp(2\pi i x\xi) F(\xi)G(\xi)\,dx\right| \le \int_{-\infty}^{\infty} |F(\xi)G(\xi)|\,d\xi$.

Using e.g. $\hat{f} = \int_{-\infty}^{\infty} \exp(ix\xi) f(x)\,dx$ you must compensate with factors of 2π in the formula.

3. Write down a reasonable definition of an *even* tempered distribution, and prove that δ'' is even. (5p)

Solution: A function $\phi \in \mathcal{S}(\mathbb{R})$ is ***even*** if $\phi(x) = \phi(-x)$, and ***odd*** if $\phi(x) = -\phi(-x)$. One can say that a distribution $T \in \mathcal{S}'$ is even if $T(\phi) = 0$ for every *odd* function $\phi \in \mathcal{S}$. By definition $(D^2T)(\phi) = T(\phi'')$, and so $\delta''(\phi) = \phi''(0)$. But if ϕ is odd, so is ϕ'', and an odd function in $\mathcal{S}$ must be zero at $x = 0$.

4. Let $\{\psi_{j,k}\}$ be an orthonormal basis in $L^2(\mathbb{R})$. Put $\Psi(t) = 2^{1/2}\psi(2t)$, and define $\Psi_{j,k}$ by rescaling and translation in the same way as the $\psi_{j,k}$ are obtained from the wavelet ϕ. Prove that $\{\Psi_{j,k}\}$ is an orthonormal system that is not a Riesz basis. (5p)

Solution: $\Psi_{j,k}(t) = 2^{j/2}\Psi(2^j t - k) = 2^{(j+1)/2}\psi(2^{j+1}t - 2k) = \psi_{j+1,2k}(t)$. Hence the ON-condition is automatically satisfied. Morover we have, for all j, k, j', k' that $\langle \Psi_{j,k}, \psi_{j',2k'+1}\rangle = 0$, which contradicts the Riesz-condition, that there exist two constants $A < B$ so that for all $f \in L^2$, it must hold that $A\|f\| \le \sum \langle f, \psi_k\rangle^2 \le B\|f\|$.

5. Let $f(x,y) = \mathrm{sinc}(x)^2\mathrm{sinc}(2y)^2$, and let $R_\theta f(r)$ be the Radon transform of f. Show that for a given angle θ, it is enough to sample $R_\theta f(r)$ at discrete points $\{r_j\}$. How densely must it be sampled? (5p)

Solution: In general, $\mathcal{F}(R_\theta f)(\sigma) = \hat{f}(\sigma\theta) = \hat{f}(\sigma\cos(\theta), \sigma\sin(\theta))$ (recall that we identify θ with the unit vector with coordinates $(\cos(\theta), \sin(\theta))$. The Fourier transform of f is given by $\hat{f}(\xi, \eta) = \Lambda(\xi)\Lambda(\eta/2)/2$. It is zero outside the rectangle $\{(\xi, \eta) \,|\, |\xi| < 1, |\eta| < 2\}$. This means that if $|\tan(\theta)| < 2$, then $\mathcal{F}(R_\theta f)$ is band limited with bound $\sqrt{1 + \tan(\theta)^2}$, and if $|\tan(\theta)| > 2$ then $\mathcal{F}(R_\theta f)$ is band limited with bound $\sqrt{4 + 4\cot(\theta)^2}$. The sampling rate of $R_\theta f(r)$ should be chosen accordingly.

Good luck!
Bernt W.