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McLachlan, N. W., and P. Humbert: "Formulaire pour le calcul symbolique," 2d ed., Gauthier-Villars, Paris, 1950.

Tables of the one-sided Laplace transformation (in which the lower limit of integration is zero) are necessarily sources of Fourier transforms only of functions that are zero for negative arguments. A table of the double-sided Laplace transform is given in the following work.

Van der Pol, B., and H. Bremmer: "Operational Calculus Based on the Two-sided Laplace Integral," 2d ed., Cambridge University Press, Cambridge, England, 1955.

PROBLEMS

1. What condition must $F(s)$ satisfy in order that $f(x) \to 0$ as $x \to \pm\infty$?

2. Prove that $|F(s)|^2$ is an even function if $f(x)$ is real.

3. The Fourier transform in the limit of sgn x is $(i\pi s)^{-1}$. What conditions for the existence of Fourier transforms are violated by these two functions? (sgn x equals 1 when x is positive and -1 when x is negative.)

4. Show that all periodic functions violate a condition for the existence of a Fourier transform.

5. Verify that the function $\cos x$ violates one of the conditions for existence of a Fourier transform. Prove that $\exp(-\alpha x^2)\cos x$ meets this condition for any positive value of α.

6. Give the odd and even parts of $H(x)$, e^{ix}, $e^{-x}H(x)$, where $H(x)$ is unity for positive x and zero for negative x.

7. Graph the odd and even parts of $[1 + (x - 1)^2]^{-1}$.

8. Show that the even part of the product of two functions is equal to the product of the odd parts plus the product of the even parts.

9. Investigate the relationship of $\mathcal{F}\mathcal{F}f$ to f when f is neither even nor odd.

10. Show that $\mathcal{F}\mathcal{F}\mathcal{F}\mathcal{F}f = f$.

11. It is asserted that the odd part of $\log x$ is a constant. Could this be correct?

12. Is an odd function of an odd function an odd function? What can be said about odd functions of even functions and even functions of odd functions?

13. Prove that the Fourier transform of a real odd function is imaginary and odd. Does it matter whether the transform is the plus-i or minus-i type?

14. An antihermitian function is one for which $f(x) = -f^*(-x)$. Prove that its real part is odd and its imaginary part even and thus that its Fourier transform is imaginary. ▷

15. Point out the fallacy in the following reasoning. "Let $f(x)$ be an odd function. Then the value of $f(-a)$ must be $-f(a)$; but this is not the same as $f(a)$. Therefore an odd function cannot be even."

16. Let the odd and even parts of a function $f(x)$ be $o(x)$ and $e(x)$. Show that, irrespective of shifts of the origin of x,

$$\int_{-\infty}^{\infty} |o(x)|^2\,dx + \int_{-\infty}^{\infty} |e(x)|^2\,dx = \text{const.}$$

17. Note that the odd and even parts into which a function is analyzed depend upon the choice of the origin of abscissas. Yet the sum of the integrals of the squares of the odd and even parts is a constant that is independent of the choice of origin. What is the constant?

18. Let axes of symmetry of a real function $f(x)$ be defined by values of a such that if o and e are the odd and even parts of $f(x - a)$, then

$$\frac{\int o^2\,dx - \int e^2\,dx}{\int o^2\,dx + \int e^2\,dx}$$

has a maximum or minimum with respect to variation of a. Show that all functions have at least one axis of symmetry. If there is more than one axis of symmetry, can there be arbitrary numbers of each of the two kinds of axis?

19. Note that $\cos x$ is fully even and has no odd part, and that shift of origin causes the even part to diminish and the odd part to grow until in due course the function becomes fully odd. In fact the even part of any periodic function will wax and wane relative to the odd part as the origin shifts. Consider means of assigning "abscissas of symmetry" and quantitative measures of "degree of symmetry" that will be independent of the origin of x. Test the reasonableness of your conclusions—for example, on the functions of period 2 which in the range $-1 < x < 1$ are given by $\Lambda(x)$, $\Lambda(x) - \frac{1}{2}$, $\Lambda(x) - \frac{1}{4}$. See Chapter 4 for triangle-function notation $\Lambda(x)$. ▷

20. The function $f(x)$ is equal to unity when x lies between $-\frac{1}{2}$ and $\frac{1}{2}$ and is zero outside. Draw accurate loci on the complex plane of $F(s)$ from which values of $F(0)$, $F(\frac{1}{2})$, $F(1)$, $F(1\frac{1}{2})$, and $F(2)$ can be measured.

21. The function $f(x)$ is equal to 100 when x differs by less than 0.01 from 1, 2, 3, 4, or 5 and is zero elsewhere. Draw a locus on the complex plane of $F(s)$ from which $F(0.05)$ can be measured in amplitude and phase.

22. **Self-transforming functions.** Long-published tables of Fourier transforms have included two functions that transform into themselves, namely $\exp(-\pi x^2)$ and sech x. In 1956, $\text{III}(x)$ joined this select group. When impulses are included, other examples, such as $1 + \delta(x)$, can be given. Propose a general construction for self-transforming functions. ▷

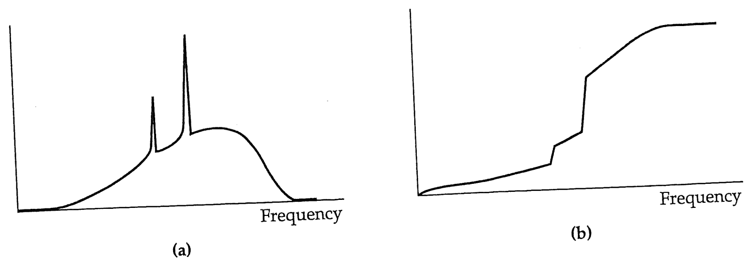

Fig. 3.12 **Spectra of X radiation from molybdenum: (a) power spectrum; (b) cumulative spectrum.**

line. Now we can consider a cumulative distribution function which gives the amount of energy in the range 0 to s:

$$\int_0^s |F(s)|^2 \, ds.$$

Any spectral *lines* would then appear as finite discontinuities in the cumulative energy spectrum as suggested by Fig. 3.12, and some mathematical convenience would be gained by using the cumulative spectrum in conjunction with Stieltjes integral notation. The convenience is especially marked when it is a question of using the theory of distributions for some question of rigor. However, the matter is purely one of notation, and in cases where we have to represent concentrations of energy within bands much narrower than can be resolved in the given context, we shall use the delta-symbol notation described later.

APPENDIX

We prove that the autocorrelation of the real nonzero function $f(x)$ is a maximum at the origin, that is,

$$\int_{-\infty}^{\infty} f(u)f(u+x)\,du \leqslant \int_{-\infty}^{\infty} [f(u)]^2\,du.$$

Let ϵ be a real number. Then, if $x \neq 0$,

$$\int_{-\infty}^{\infty} [f(u) + \epsilon f(u+x)]^2\,du > 0$$

and $$\int_{-\infty}^{\infty} [f(u)]^2\,du + 2\epsilon\int_{-\infty}^{\infty} f(u)f(u+x)\,du + \epsilon^2\int_{-\infty}^{\infty} [f(u+x)]^2\,du > 0;$$

that is, $$a\epsilon^2 + b\epsilon + c > 0,$$

where $$a = c = \int_{-\infty}^{\infty} [f(u)]^2\,du$$

$$b = 2\int_{-\infty}^{\infty} f(u)f(u+x)\,du.$$

Now, if the quadratic expression in ϵ may not be zero, that is, if it has no real root, then

$$b^2 - 4ac \leqslant 0.$$

Hence in this case $b/2 \leqslant a$, or

$$\frac{\int_{-\infty}^{\infty} f(u)f(u+x)\,du}{\int_{-\infty}^{\infty} [f(u)]^2\,du} \leqslant 1.$$

The equality is achieved at $x = 0$; consequently the autocorrelation function can nowhere exceed its value at the origin. The argument is the one used to establish the Schwarz inequality and readily generalizes to give the similar result for the complex autocorrelation function.

Exercise. Extend the argument with a view to showing that the equality cannot be achieved by any value of x save zero.

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PROBLEMS

1. Calculate the following serial products, checking the results by summation. Draw graphs to illustrate.
(*a*) {6 9 17 20 10 1} ∗ {3 8 11}
(*b*) {1 1 1 1 1} ∗ {1 1 1 1}
(*c*) {1 4 2 3 5 3 3 4 5 7 6 9} ∗ {1 1}
(*d*) {1 4 2 3 5 3 3 4 5 7 6 9} ∗ {$\frac{1}{2}$ $\frac{1}{2}$}

(e) $\{1\,4\,2\,3\,5\,3\,3\,4\,5\,7\,6\,9\} * \{1\,2\,1\}$
(f) $\{1\,4\,2\,3\,5\,3\,3\,4\,5\,7\,6\,9\} * \{\frac{1}{4}\,\frac{1}{2}\,\frac{1}{4}\}$
(g) $\{1\,2\,3\,7\,12\,19\,21\,22\,18\,13\,7\,5\,3\,2\,1\} * \{1\,-1\}$
(h) $\{1\,2\,3\,7\,12\,19\,21\,22\,21\,18\,13\,7\,5\,3\,2\,1\} * \{1\,1\,1\,1\ldots\}$
(i) $\{1\,1\} * \{1\,1\} * \{1\,1\}$
(j) $\{1\,1\,0\,0\,1\,0\,1\} * \{1\,1\,0\,0\,1\,0\,1\}$
(k) $\{1\,1\,0\,0\,1\,0\,1\} * \{1\,0\,1\,0\,0\,1\,1\}$
(l) $\{a\,b\,c\,d\,e\} * \{e\,d\,c\,b\,a\}$
(m) $\{1\,3\,1\} * \{1\,0\,0\,0\,0\,0\,\pm 1\}$
(n) $\{1\,3\,1\} * \{1\,2\,2\}$
(o) Multiply 131 by 122
(p) Multiply 10,301 by 10,202
(q) $\{1\,8\,1\} * \{1\,2\,2\}$
(r) Comment on the smoothness of your results in d and f relative to the longer of the two given sequences.
(s) Consider the result of i in conjunction with Pascal's pyramid of binomial coefficients.
(t) Seek longer sequences with the same property you discovered in k.
(u) Contemplate j, k, l, m, and n with a view to discerning what leads to serial products which are even.
(v) Master the implication of n, o, p, and q, and design a mechanical desk computer to perform serial multiplication.

2. Derive the following results, where $H(x)$ is the Heaviside unit step function (Chapter 4):

$$x^2H(x) * e^xH(x) = (2e^x - x^2 - 2x - 2)H(x)$$
$$[\sin x\, H(x)]^{*2} = \tfrac{1}{2}(\sin x - x\cos x)H(x)$$
$$[(1-x)H(x)] * [e^xH(x)] = xH(x)$$
$$H(x) * [e^xH(x)] = (e^x - 1)H(x)$$
$$[e^xH(x)]^{*2} = xe^xH(x)$$
$$[e^xH(x)]^{*3} = \tfrac{1}{2}x^2e^xH(x)$$

3. Prove the commutative property of convolution, that is, that $f * g \equiv g * f$.

4. Prove the associative rule $f * (g * h) \equiv (f * g) * h$.

5. Prove the distributive rule for addition $f * (g + h) \equiv f * g + f * h$.

6. The function f is the convolution of g and h. Show that the self-convolutions of f, g, and h are related in the same way as the original functions. ▷

7. If $f = g \star h$, show that $f \star f = (g \star g) \star (h \star h)$.

8. Show that if a is a constant, $a(f * g) = (af) * g = f * (ag)$.

9. Establish a theorem involving $f(g * h)$. ▷

10. Prove that the autocorrelation function is hermitian, that is, that $C(-u) = C^*(u)$, and hence that when the autocorrelation function is real it is even. Note that if the autocorrelation function is imaginary it is also odd; give some thought to devising a function with an odd autocorrelation function.

11. Prove that the sum and product of two autocorrelation functions are each hermitian. ▷

12. Alter the origin of $f(x)$ until $f * f|_0$ is a maximum. Investigate the assertion that the new origin defines an axis of maximum symmetry, making any necessary modification. Investigate the merits of the parameter

$$\frac{f * f|_0}{f \star f|_0}$$

to be considered a measure of "degree of evenness."

13. Show that if $f(x)$ is real,

$$\int_{-\infty}^{\infty} f(x)f(-x)\,dx = \int_{-\infty}^{\infty} [E(x)]^2\,dx - \int_{-\infty}^{\infty} [O(x)]^2\,dx,$$

and note that the left-hand side is the central value of the self-convolution of $f(x)$; that is, $f * f|_0$.

14. Find reciprocal sequences for $\{1\,3\,3\,1\}$ and $\{1\,4\,6\,4\,1\}$.

15. Find reciprocal sequences for $\{1\,1\}$ and $\{1\,1\,1\}$.

16. Establish a general procedure for finding the reciprocal of finite or semi-infinite sequences and test it on the following cases:

$$\{64\;32\;16\;8\;4\;2\;1\ldots\}$$
$$\{64\;64\;48\;32\;20\;12\;7\;4\ldots\}$$
$$\left\{1\quad e^{-1/10}\cos\left(\frac{\pi}{10}\right)\quad e^{-2/10}\cos\left(\frac{2\pi}{10}\right)\quad \ldots e^{-n/10}\cos\left(\frac{n\pi}{10}\right)\quad\ldots\right\}$$

17. Find approximate numerical values for a function $f(x)$ such that

$$f(x) * [e^{-x}H(x)]$$

is zero when evaluated numerically by serial multiplication of values taken at intervals of 0.2 in x, except at the origin. Normalize $f(x)$ so that its integral is approximately unity. ▷

18. The cross correlation $g \star h$ is to be normalized to unity at its maximum value. It is argued that

$$0 \leqslant \int [g(u) - h(u+x)]^2\,du = \int g^2\,du - 2\int g(u)h(u+x)\,du + \int h^2\,du,$$

and therefore that

$$\int g(u)h(u+x)\,du \leqslant \tfrac{1}{2}\int g^2\,du + \tfrac{1}{2}\int h^2\,du = M.$$

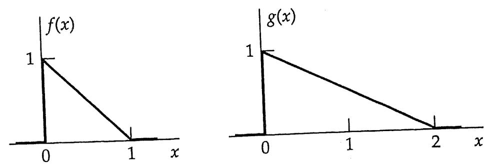
Consequently $(g \star h)/M$ is the desired quantity. Correct the fallacy in this argument.

19. Barker code. Calculate the autocorrelation sequence of

$$\{1\,1\,1\,1\,1\,-1\,-1\,1\,1\,-1\,1\,-1\,1\}.$$

This sequence is known as the Barker code of length 13 (Pettit, 1967). By coin tossing, establish a similar 13-element sequence at random and calculate its autocorrelation.

20. Convolution done analytically.
(*a*) Make a graph of the convolution $h(x)$ of the given functions $f(x)$ and $g(x)$, labeling both axes with numerical values.



(*b*) Label any interesting points of $h(x)$ with letters $A, B, C, \ldots$ and make a table of values such as the following:

Interesting point	x	$h(x)$
A		
B		
C		
...		

21. Self-convolution of sinc function. Let $f(x) = \text{sinc}\,(x+2) + \text{sinc}\,(x-2)$. What is the self-convolution of $f(x)$?

22. Convolution. In the integral

$$\int_{-\infty}^{\infty} f(x-u)g(u)\,du,$$

make the substitution

$$u = x - a,$$

where a is a constant. Then

$$\int_{-\infty}^{\infty} f(x-u)g(u)\,du = \int_{-\infty}^{\infty} f(a)g(x-a)\,dx = f(a)\int_{-\infty}^{\infty} g(x)\,dx.$$

Is this correct? If not, where is the fallacy in the derivation?

23. Optical sound track. The optical sound track on old motion-picture film has a breadth b, and it is scanned by a slit of width w. With appropriate normalization, we may say that the scanning introduces convolution by a rectangle function of unit height and width w. In a certain movie theater the projectionist clumsily dropped the whole projector on the floor and after that the slit was always inclined at a small angle ϵ to the striations on the sound track instead of making an angle of zero with them.
(*a*) What function now describes the convolution that takes place?
(*b*) Describe qualitatively the effect on the sound reproduction.

24. Numerical convolution compared with analytic. The exponential function e^{-x} may be represented discretely by the sequence

$$\{a\} = \{1 \quad 0.368 \quad 0.135 \quad 0.050 \quad \ldots\}$$

(*a*) Calculate the serial product or "autocorrelation sum"

$$\sum_{j=0,1,\ldots} a_j a_{i+j}$$

and make an accurate graph.
(*b*) Calculate the "autocorrelation function"

$$R(\tau) = \int_{-\infty}^{\infty} f(x)f(x+\tau)\,dx$$

when

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0, \end{cases}$$

and superimpose a graph of $R(\tau)$ on the previous graph.
(*c*) Naturally the continuous graph of $R(\tau)$ does not pass exactly through the points calculated from part *a*. Discuss the discrepancy in terms of round-off error, normalization, or any other effects which you think may account for the disagreement.

25. Two-dimensional convolution. The autocorrelation of two disks of unit diameter arises in the theory of optical instruments with circular apertures (and describes the two-dimensional transfer function). It is known as the Chinese hat function. Show that

$$\Pi(r) ** \Pi(r) = \text{chat}\, r = \tfrac{1}{2}[\cos^{-1} r - r(1-r^2)^{\frac{1}{2}}]\Pi\left(\frac{r}{2}\right). \triangleright$$

26. Two-dimensional autocorrelation. The two-dimensional autocorrelation function of $\Pi(x)\Pi(y)$ is $\Lambda(x)\Lambda(y)$. The central value is unity.
(*a*) Verify that the contour $\Lambda(x)\Lambda(y) = \epsilon$, where ϵ is small compared with unity, is approximately square.
(*b*) Verify that the contour $\Lambda(x)\Lambda(y) = 1 - \delta$, where δ is small compared with unity, is also approximately square.
(*c*) Choosing $\delta = \epsilon$, would you say that the two contours are equally square?

27. Autocorrelation of a convolution. Show that

$$(f * g) \star (f * g) = (f \star f) * (g \star g).$$

28. Deconvolution. Three sequences are related by convolution as follows:

$$\{a_0\, a_1\, a_2 \ldots\} * \{b_0\, b_1\, b_2 \ldots\} = \{c_0\, c_1\, c_2 \ldots\}.$$

If the sequences $\{b_k\}$ and $\{c_k\}$ are given, show that the rule for inverting the convolution to obtain $\{a_k\}$ is

$$a_k = b_0^{-1}\left(c_k - \sum_{j=0}^{k-1} a_j b_{k-j}\right).$$

Write a computer subprogram for inverting convolution.

29. Transmission line echoes. When a current impulse is injected into a transmission line that is short-circuited at the far end the voltage appearing at the input terminals is a sequence of equispaced impulses with coefficients $\{v\} = \{1\,2\,2\,2 \ldots\}$. The current response if an impulse of voltage is applied is $\{i\} = \{1\ -2\ 2\ -2\ 2 \ldots\}$. (a) Show that $\{v\} * \{i\} = \{1\}$. (b) Find two functions $v(t)$ and $i(t)$ such that $v(t)\text{III}(t)$ and $i(t)\text{III}(t)$ respectively evoke the impulse trains. ▷

30. Deconvolution. Let $\{b_k\}$ be an unknown sequence and let $\{a_k\} = \{1\,1\,1\,1\,1\,1\}$. Let $\{p_k\}$ be the periodic sequence of which one period is $\{0\,1\,1\,-1\,-1\,0\}$. Verify that $\{a_k\} * \{p_k\} = \{0\}$. As in the previous problem, let $\{c_k\} = \{a_k\} * \{b_k\}$ be given. Is it not true that convolving $\{a_k\}$ with $\{b_k\} + \{p_k\}$ will also yield $\{c_k\}$? If this is so, how can the deconvolution rule given above recover the unknown $\{b_k\}$? What, in fact, will the deconvolution algorithm produce? ▷

31. Speed of convolution. Specify a long data sequence f with N elements, smooth it by convolution with $g = \{1\,4\,6\,4\,1\}$, and obtain the elapsed time. We know that $f * g = g * f$, but does elapsed time depend on which sequence comes first? Does the time become proportional to N^2 as N increases? ▷

4

Notation for Some Useful Functions

Many useful functions in Fourier analysis have to be defined piecewise because of abrupt changes. For example, we may consider the function $f(x)$ such that

$$f(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leqslant x \leqslant 1 \\ 1 & x > 1. \end{cases}$$

This function, though simple in itself, is awkwardly expressed in comparison with a function such as, for example, $1 + x^2$, whose algebraic expression compactly states, over the infinite range of x, the arithmetical operations by which it is formed. For many mathematical purposes a function which is piecewise analytic is not simple to deal with, but for physical purposes, a "sloping step function," or "ramp-step function," may be at least as simple as a smoother function.

Fourier himself was concerned with the representation of functions given graphically, and according to English mathematician and historian E. W. Hobson "was the first fully to grasp the idea that a single function may consist of detached portions given arbitrarily by a graph."

To regain compactness and clarity of notation, we introduce a number of simple functions embodying various kinds of abrupt behavior. Also included here is a section dealing with sinc x, the important interpolating function, which is the transform of a discontinuous function, and some reference material on notations for the Gaussian function.

RECTANGLE FUNCTION OF UNIT HEIGHT AND BASE, $\Pi(x)$

The rectangle function of unit height and base, which is illustrated in Fig. 4.1, is defined by

PROBLEMS

1. Show that

$$H(ax+b) = \begin{cases} H\left(x+\frac{b}{a}\right) & a>0 \\ H\left(-x-\frac{b}{a}\right) & a<0, \end{cases}$$

and hence that

$$H(ax+b) = H\left(x+\frac{b}{a}\right)H(a) + H\left(-x-\frac{b}{a}\right)H(-a).$$

2. Discuss the function $\frac{1}{2}[1 + x/(x^2)^{\frac{1}{2}}]$ used by Cauchy.

3. Show that the operation $H(x)\,*$ is an integrating operation in the sense that

$$H(x) * [f(x)H(x)] = \int_0^x f(x)\,dx.$$

4. Calculate $(d/dx)\,[\Pi(x) * H(x)]$ and prove that $(d/dx)[f(x) * H(x)] = f(x)$.

5. By evaluating the integral, prove that $\operatorname{sinc} x * \operatorname{sinc} x = \operatorname{sinc} x$.

6. Prove that $\operatorname{sinc} x * J_0(\pi x) = J_0(\pi x)$.

7. Prove that $4 \operatorname{sinc} 4x * \sin x = \sin x$.▷

8. Show that

$$\begin{aligned} \Pi(x) &= H(x+\tfrac{1}{2}) - H(x-\tfrac{1}{2}) \\ &= H(\tfrac{1}{2}+x) + H(\tfrac{1}{2}-x) - 1 \\ &= H(\tfrac{1}{4}-x^2) \\ &= \tfrac{1}{2}[\operatorname{sgn}(x+\tfrac{1}{2}) - \operatorname{sgn}(x-\tfrac{1}{2})] \end{aligned}$$

and that $\Pi(x^2) = \Pi(x/2^{\frac{1}{2}})$.

9. Show that

$$\begin{aligned} \Lambda(x) &= \Pi(x) * \Pi(x) \\ &= \Pi(x) * H(x+\tfrac{1}{2}) - \Pi(x) * H(x-\tfrac{1}{2}). \end{aligned}$$

10. Experiment with the equation $f[f(x)] = f(x)$ and note that $f(x) = \operatorname{sgn} x$ is a solution. Find other solutions and attempt to write down the general solution compactly with the aid of step-function notation.

11. Show that $\operatorname{erf} x = 2\Phi(2^{\frac{1}{2}}\sigma x) - 1$, where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} e^{-u^2/2\sigma^2}\,du.$$

12. Show that the first derivative of $\Lambda(x)$ is given by

$$\Lambda'(x) = -\Pi\left(\frac{x}{2}\right)\operatorname{sgn} x$$

and calculate the second derivative. ▷

13. In abbreviated notation the relation of $\Lambda(x)$ to $\Pi(x)$ could be written $\Lambda = \Pi * \Pi$ or $\Lambda = \Pi^{*2}$. Show that

$$\Pi^{*2} = \Pi * \Lambda = \tfrac{1}{2}(x+1\tfrac{1}{2})^2\Pi(x+1) + (\tfrac{3}{4}-x^2)\Pi(x) + \tfrac{1}{2}(x-1\tfrac{1}{2})^2\Pi(x-1).$$

Show also that

$$\Pi^{*2} = \tfrac{1}{2}(x+1\tfrac{1}{2})^2H(x+1\tfrac{1}{2}) - \tfrac{3}{2}(x+\tfrac{1}{2})^2H(x+\tfrac{1}{2}) + \tfrac{3}{2}(x-\tfrac{1}{2})^2H(x-\tfrac{1}{2}) - \tfrac{1}{2}(x-1\tfrac{1}{2})^2H(x-1\tfrac{1}{2}).$$

14. Examine the derivatives of Π^{*3} at $x = \frac{1}{2}\pm$ and $x = 1\frac{1}{2}\pm$ and reach some conclusion about the continuity of slope and curvature.

15. Show that $(d/dx)|x| = \operatorname{sgn} x$ and that $(d/dx)\operatorname{sgn} x = 2\delta(x)$. Comment on the fact that

$$\frac{d^2|x|}{dx^2} = \frac{d^2}{dx^2}[2xH(x)] = 2\delta(x).\ ▷$$

16. Notation. Prove that

$$H(x) * H(x+1) - H(x) * H(x-1) = \Lambda(x),$$

or, if the RHS is not correct, derive the correct expression.

17. Notation. Prove that

$$\frac{d}{dx}\{[\Lambda(x)]^2\} = -\Lambda(x)\operatorname{sgn} x,$$

or, if the RHS is not correct, derive the correct expression.

18. Derivatives of the sinc function. Show that

$$\operatorname{sinc}'(x) = \frac{\cos \pi x}{x} - \frac{\sin \pi x}{\pi x^2},$$

$$\operatorname{sinc}''(x) = -\frac{2}{x^2}\cos \pi x + \frac{2x + \pi^2 x^3}{\pi x^4}\sin \pi x.$$

19. Integrals of the jinc function. When the jinc function was introduced it was mentioned that $\int_0^\infty \operatorname{jinc} r\, 2\pi r\, dr = 1$, i.e., regarded as a function of radius in two dimensions, the volume under the jinc function is unity. Confirm that $\int_{-\infty}^{\infty} \operatorname{jinc} x\, dx$ is also equal to unity. ▷

alized function $p(x)$:

$$\int_{-\infty}^{\infty} p'(x)F(x)\,dx = -\int_{-\infty}^{\infty} p(x)F'(x)\,dx.$$

Similarly,
$$\int_{-\infty}^{\infty} p^{(n)}(x)F(x)\,dx = (-1)^n \int_{-\infty}^{\infty} p(x)F^{(n)}(x)\,dx.$$

Since by definition $F^{(n)}(x)$ exists, however large n may be, it follows that we have an interpretation for the nth derivative of a generalized function, for any n.

Differentiation of ordinary functions. Generalized functions possess derivatives of all orders, and if an ordinary function could be regarded as a generalized function, then there would be a satisfactory basis for formulas such as

$$\frac{d}{dx}[H(x)] = \delta(x).$$

If $f(x)$ is an ordinary function and we form a sequence $f_\tau(x)$ such that

$$\lim_{\tau\to 0}\int_{-\infty}^{\infty} f_\tau(x)F(x)\,dx = \int_{-\infty}^{\infty} f(x)F(x)\,dx,$$

where $F(x)$ is any particularly well-behaved function, then the sequence defines a *generalized* function, which we may denote by the same symbol $f(x)$. The symbol $f(x)$ then has two meanings. We shall limit attention to functions $f(x)$ which as $|x|\to\infty$ behave as $|x|^{-N}$ for some value of N.

A suitable sequence $f_\tau(x)$ is given by

$$[\tau^{-1}e^{-\pi x^2/\tau^2}] * [f(x)e^{-\tau^2x^2}].$$

With this enlargement of the notion of generalized functions we can embrace the unit step function $H(x)$ as a generalized function and assign meaning to its derivative $H'(x)$. Thus

$$\begin{aligned}\int_{-\infty}^{\infty} H'(x)F(x)\,dx &= -\int_{-\infty}^{\infty} H(x)F'(x)\,dx\\ &= -\int_{0}^{\infty} F'(x)\,dx\\ &= \int_{\infty}^{0} F'(x)\,dx\\ &= F(0),\end{aligned}$$

but
$$\int_{-\infty}^{\infty} \delta(x)F(x)\,dx = F(0),$$

hence
$$H'(x) = \delta(x).$$

The generalized function $\delta(x)$ is thus the derivative of the generalized function $H(x)$, and this is how we interpret formulas such as $H'(x) = \delta(x)$; we take the symbol for an ordinary function such as $H(x)$ to stand for the corresponding generalized function.

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PROBLEMS

1. What is the even part of

$$\delta(x+3) + \delta(x+2) - \delta(x+1) + \tfrac{1}{2}\delta(x) + \delta(x-1) - \delta(x-2) - \delta(x-3)?$$

2. Attempting to clarify the meaning of $\delta(xy)$, a student gave the following explanation. "Where u is zero, $\delta(u)$ is infinite. Now xy is zero where $x = 0$ and where $y = 0$; therefore $\delta(xy)$ is infinite along the x and y axes. Hence $\delta(xy) = \delta(x) + \delta(y)$." Explain the fallacy in this argument, and show that

$$\delta(xy) = \frac{\delta(x) + \delta(y)}{(x^2 + y^2)^{\frac{1}{2}}}.$$

3. Show that

$$\mathrm{II}(x) = \delta(2x^2 - \tfrac{1}{2})$$

and that
$$\delta(x^2 - a^2) = \tfrac{1}{2}|a|^{-1}\{\delta(x-a) + \delta(x+a)\}. \triangleright$$

4. Show that

$$\int_{-\infty}^{\infty} e^{-i2\pi xs}\,ds = \delta(x)$$

and that

$$\int_{-\infty}^{\infty} \delta(x)e^{i2\pi sx}\,dx = 1.$$

5. Show that

$$\delta(ax+b) = \frac{1}{|a|}\,\delta\!\left(x + \frac{b}{a}\right), \qquad a \neq 0.$$

6. If $f(x) = 0$ has roots x_n, show that

$$\delta[f(x)] = \sum_n \frac{\delta(x - x_n)}{|f'(x_n)|}$$

wherever $f'(x_n)$ exists and is not zero. Consider the ideas suggested by $\delta(x^3)$ and $\delta(\operatorname{sgn} x)$.

7. Show that

$$\pi\delta(\sin \pi x) = \text{III}(x)$$

and

$$\delta(\sin x) = \pi^{-1}\,\text{III}\left(\frac{x}{\pi}\right). \triangleright$$

8. Show that

$$\text{III}(x) + \text{III}(x - \tfrac{1}{2}) = 2\,\text{III}(2x) = \text{III}(x) * 4\text{II}(2x - \tfrac{1}{2}).$$

9. Show that

$$\text{III}(x)\Pi\left(\frac{x}{8}\right) = \text{III}(x)\Pi\left(\frac{x}{7}\right) + \frac{\text{II}(x/8)}{8}$$

and also that

$$\text{III}(x)\left(\frac{x}{6\frac{1}{2}}\right) = \text{III}(x)\Pi\left(\frac{x}{7\frac{1}{2}}\right). \triangleright$$

10. Can the following equation be correct?

$$x\,\delta(x - y) = y\,\delta(x - y).$$

11. Show that $\Lambda(x) * \sum_{-\infty}^{\infty} a_n\,\delta(x - n)$ is the polygon through the points (n,a_n).

12. Prove that

$$\delta'(-x) = -\delta'(x)$$
$$x\,\delta'(x) = -\delta(x).$$

Show also that

$$f(x)\,\delta'(x) = f(0)\,\delta'(x) - f'(0)\,\delta(x),$$

for example, by differentiating $f(x)\,\delta(x)$. ▷

13. In attempting to show that $\delta'(x) = -\delta(x)/x$ a student presented the following argument. "A suitable sequence, as τ approaches zero, for defining $\delta(x)$ is $\tau/\pi(x^2 + \tau^2)$. Therefore a suitable sequence for $\delta'(x)$ is the derivative

$$\frac{d}{dx}\frac{\tau}{\pi(x^2 + \tau^2)} = \frac{-2\tau x}{\pi(x^2 + \tau^2)^2}$$
$$= \frac{-2x}{x^2 + \tau^2}\frac{\tau}{\pi(x^2 + \tau^2)}.$$

The second factor is the sequence for $\delta(x)$, and the first factor goes to $-2/x$ in the limit as τ approaches zero. Therefore $\delta'(x) = -2\delta(x)/x$." Explain the fallacy in this argument.

14. Show that

$$x^n\delta^{(n)}(x) = (-1)^n n!\,\delta(x)$$

and hence that

$$x^2\,\delta''(x) = 2\delta(x)$$

and

$$x^3\,\delta''(x) = 0.$$

15. The function $[x]$ is here defined as the mean of the greatest integer less than x and the greatest integer less than or equal to x. Show that

$$[x]' = \text{III}(x)$$

and also that

$$\frac{d}{dx}\{[x]H(x)\} = \text{III}(x)H(x) - \tfrac{1}{2}\delta(x).$$

(The common definition of $[x]$ as the greatest integer less than x is not fully suitable for the needs of this exercise; the two definitions differ by the null function which is equal to $\frac{1}{2}$ for integral values of x and is zero elsewhere.)

16. The sawtooth function $Sa(x)$ is defined by $Sa(x) = [x] - x + \frac{1}{2}$. Show that

$$Sa'(x) = \text{III}(x) - 1$$

and that

$$\frac{d}{dx}[Sa(x)H(x)] = [\text{III}(x) - 1]H(x).$$

17. Show that $\operatorname{sgn}^2 x = 1 - \delta^0(x)$.

18. The Kronecker delta is defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Show that it may be expressed as a null function of $i - j$ as follows:

$$\delta_{ij} = \delta^0(i - j).$$

19. We wish to consider the suitability of a sequence of asymmetrical profiles, such as $\tau^{-1}\{\Lambda(x/\tau) + \frac{1}{2}\Lambda[(x - \tau)/\tau]\}$, for representing the impulse symbol. Discuss the sifting property that leads to a result of the form

$$\delta_a * f = \mu\,\delta_+ * f + \nu\,\delta_- * f,$$

where δ_a is a symbol based on the asymmetrical sequence, δ_+ is based on the sequence $\tau^{-1}\Pi[(x - \frac{1}{2}\tau)/\tau]$, and δ_- is based on the sequence $\tau^{-1}\Pi[(x + \frac{1}{2}\tau)/\tau]$ (τ positive). ▷

20. Prove the relation $^2\delta(x,y) = \delta(r)/\pi|r|$.

21. Illustrate on an isometric projection the meaning you would assign to $\text{III}[(x^2 + y^2)^{\frac{1}{2}}]$. How would you express something which on this diagram would have the appearance of equally spaced concentric rings of equal height?

22. The function $f_r(x)$ is formed from $f(x)$ by reversing it; that is, $f_r(x) = f(-x)$. Show that the operation of forming f_r from f can be expressed with the aid of the impulse symbol by

$$f \star \delta$$

and hence that

$$(f \star \delta) \star \delta = f.$$

23. Under what conditions could we say that $(f \star \delta) \star \delta = f \star (\delta \star \delta)$?

24. All the sequences $f(x,\tau)$ given on page 76 have the property that $f(0,\tau)$ increases without limit as $\tau \to 0$. Show that $\frac{1}{2}\tau^{-1}\Lambda[(x/\tau) - 1] + \frac{1}{2}\tau^{-1}\Lambda[(x/\tau) + 1]$ is an equivalent sequence which, however, possesses a limit of zero, as $\tau \to 0$, for all x. Show that $f(0,\tau)$, far from needing to approach ∞ as $\tau \to 0$, may indeed approach $-\infty$. ▷

25. Show that

$$f(x)\,\delta''(x) = f(0)\,\delta''(x) - 2f'(0)\,\delta'(x) + f''(0)\,\delta(x)$$

and that in general

$$f(x) + \cdot\delta^{(n)}(x) = f(0)\,\delta^{(n)}(x) - \binom{n}{1} f'(0)\,\delta^{(n-1)}(x) + \dots$$

$$- \binom{n}{n-1} f^{(n-1)}(0)\,\delta'(x) + f^{(n)}(0)\,\delta(x). ▷$$

26. **Impulses and sequences.** The delta function possesses a "sifting property"

$$L = \lim_{\tau\to 0} \int_{\infty}^{\infty} f(x)h(x,\tau)\,dx = f(0),$$

which arises from physical situations where the function $h(x,\tau)$ can be thought of as a sequence of functions of x generated as a parameter τ ranges through a series of diminishing constant values.

(*a*) Give an example of a function $h(x,\tau)$ for which the sifting property does not hold.

(*b*) What conditions are sufficient for $h(x,\tau)$ to meet in order for the sifting property to be true?

(*c*) 1. Write down a particular case of $h(x,\tau)$ that meets your conditions.
2. Using your $h(x,\tau)$, take

$$f(x) = \begin{cases} -1 & x \ge 0 \\ 1 & x < 0. \end{cases}$$

Does the sifting property hold true for this example?

(*d*) A particular $h(x,\tau)$ has the property that, if attention is fixed on a given value of x, then $\lim_{\tau\to 0} h(x,\tau) = 0$, and the same applies for *all* values of x. Could such a function $h(x,\tau)$ satisfy the sifting property?

(*e*) Give an $h(x,\tau)$ such that L becomes $f(2)$.

(*f*) Give an $h(x,\tau)$ such that L becomes $f'(0)$.

(*g*) Give an $h(x,\tau)$ such that $L = \int_0^\infty f(x)\,dx$.

(h) Give an $h(x,\tau)$ such that $L = \sum_{n=0}^{\infty} f(n)$ (n integral).

27. **Limits.** A linear time-invariant system having an impulse response $I(t)$ is excited by an input voltage $V_1(t,\tau)$, where

$$V_1(t,\tau) = [1 + \tau^{-2}(1 - 2\tau)|x|]\Pi\left(\frac{x}{2\tau}\right).$$

and produces a response $V_2(t,\tau)$.

(*a*) Is it true that $\lim_{\tau\to 0} V_2(t,\tau) = I(t)$?

(*b*) Is it true that $\lim_{\tau\to 0} V_1(t,\tau) = 0$?

(*c*) Is it true that $\lim_{\tau\to 0} \int_{-\infty}^{\infty} V_1(t,\tau)\,dt = 0$?

(*d*) Is it true that $\int_{-\infty}^{\infty} \lim_{\tau\to 0} V_1(t,\tau)\,dt = 0$?

28. **Sequence defining $\delta(x)$.** Construct a sequence of particularly well-behaved functions $f(x,\tau)$ that defines $\delta(x)$ but has the property that $\lim_{\tau\to 0} f(x,\tau) = 0$ for all x.

29. **Generalized functions.** Consider the sequence of functions $\tau^{-1}\cos(\pi x^2/4\tau^2)$ generated as $\tau \to 0$. For *all* values of x, the function value diverges in an oscillatory manner without limit. Could such a sequence exhibit the sifting property of an impulse at $x = 0$; that is, could it be true that

$$\lim_{\tau\to 0} \int_{-\infty}^{\infty} \tau^{-1} \cos\left(\frac{\pi x^2}{4\tau^2}\right) F(x)\,dx = F(0)?$$

30. **Asymmetrical impulse.** Show that $\delta_+(x)$ introduced in Problem 5.19 differs from $\delta(x)$ by the derivative of a null function (p. 87)

$$\delta_+(x) = \delta(x) - \frac{1}{2}\frac{d}{dx}\delta^0(x).$$

31. **Energy of voltage impulse.** A voltage $V(t) = A\delta(t)$ is applied to a resistance R.

(*a*) How much charge is passed through the resistor?

(*b*) How much energy is dissipated in the resistor?

32. **Delta notation.** A voltage $V(t)$ is applied to a resistance R for a finite length of time during which 2 joules of energy are transferred to the resistor. If the experiment is repeated with a stronger voltage for a shorter time, too short a time, in fact, to be of interest, how could $V(t)$ be written in δ notation? ▷

33. **Product of delta symbols.** No definition is given to a product of impulses in the one-dimensional theory, but in two dimensions products arise naturally and are readily interpretable. Consider $\delta(x)\,\delta(y)$, each factor being regarded as a function of two variables and describing straight blades of unit height on the (x,y) plane (p. 335). (Unit height means unit line density or unit double integral per unit arc length.) Evaluate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,dx\,dy$ by (1) substituting $\tau^{-1}\Pi(x/\tau)$ for $\delta(x)$ and $\tau^{-1}\Pi(y/\tau)$ for $\delta(y)$ in accordance with the rule of p. 76, (2) performing the integration, and (3) proceeding to the limit as $\tau \to 0$. Also show that when two blades intersect at an angle θ, the double integral is increased by a factor $1/\sin\theta$. ▷

34. Autocorrelation of ring impulse. A circular ring impulse of total strength $2\pi a$, described by $\delta(r-a)$, arises in optics in dealing with annular slits and also comes up in other fields. Show that

$$\delta(r-a) \star\star\, \delta(r-a) = \left(\frac{r}{2a}\right)^{-1}\left[1-\left(\frac{r}{2a}\right)^2\right]^{-\frac{1}{2}} \Pi\left(\frac{r}{4a}\right).$$

Graph this function of r and explain the principal features. For example, why does the autocorrelation become infinite at $r = 0$ and $r = a$; why are these singularities unequal; why is the value exactly 2 at $r/2a = 2^{-\frac{1}{2}}$? Investigate the autocorrelation of $\Pi(r/21) - \Pi(r/19)$. What are the values at $r = 0$; $r = 20$; what is the minimum value; and at what value of r does it occur? (In this problem $\star\star$ stands for two-dimensional autocorrelation.)

35. Delta notation. As an exercise in delta function notation, evaluate the following integrals.

$$\int_{-\infty}^{\infty} \delta(\sin x)\Pi(x-\tfrac{1}{2})\,dx, \quad \int_{-\infty}^{\infty} \delta(\cos x)\Pi\left(\frac{x}{4}\right)dx, \quad \int_{-\infty}^{\infty} \delta(\sin 2x)\Pi\left(\frac{x}{4}\right)dx. \triangleright$$

36. Integral of impulse. If $\int_{-\infty}^{\infty} \delta(x)\,dx$ is agreed to be unity, is there any objection to $\int_0^{\infty} \delta(x)\,dx = \frac{1}{2}$? ▷

37. Two variables. Use the method for interpreting expressions containing delta functions to arrive at the meaning of delta (x, y). ▷

38. Checking analysis by computer. It has been suggested that $\text{III}(x)\operatorname{sgn} x$ has Fourier transform $-i\cot\pi s$. The innermost pair of impulses $-\delta(x+1)+\delta(x-1)$ has FT $-2i\sin 2\pi s$; then transforming pair by pair suggests that

$$\sum_{k=1}^{\infty} -2i\sin 2\pi ks = -i\cot\pi s.$$

To check for a sign error, a missing factor of 2, or more serious errors, compute both sides for a single value of s; for example, does $\sin 1° + \sin 2° + \sin 3° \ldots$ add up to $\frac{1}{2}\cot\frac{1}{2}°$? Graph the sum of N terms versus N and discuss your finding. ▷

6

The Basic Theorems

A small number of theorems play a basic role in thinking with Fourier transforms. Most of them are familiar in one form or another, but here we collect them as simple mathematical properties of the Fourier transformation. Most of their derivations are quite simple, and their applicability to impulsive functions can readily be verified by consideration of sequences of rectangular or other suitable pulses. As a matter of interest, proofs based on the algebra of generalized functions as given in Chapter 5 are gathered for illustration at the end of this chapter.

The emphasis in this chapter, however, is on illustrating the *meaning* of the theorems and gaining familiarity with them. For this purpose a stock-in-trade of particular transform pairs is first provided so that the meaning of each theorem may be shown as it is encountered.

A FEW TRANSFORMS FOR ILLUSTRATION

Six transform pairs for reference are listed below. They are all well known, and the integrals are evaluated in Chapter 7; we content ourselves at this point with asserting that the following integrals may be verified.

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-i2\pi xs}\,dx = e^{-\pi s^2} \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-\pi s^2} e^{+i2\pi sx}\,ds = e^{-\pi x^2}$$

$$\int_{-\infty}^{\infty} \operatorname{sinc} x\, e^{-i2\pi xs}\,dx = \Pi(s) \quad \text{and} \quad \int_{-\infty}^{\infty} \Pi(s) e^{+i2\pi sx}\,ds = \operatorname{sinc} x$$

$$\int_{-\infty}^{\infty} \operatorname{sinc}^2 x\, e^{-i2\pi xs}\,dx = \Lambda(s) \quad \text{and} \quad \int_{-\infty}^{\infty} \Lambda(s) e^{+i2\pi sx}\,ds = \operatorname{sinc}^2 x$$

Thus the transform of the Gaussian function is the same Gaussian function, the transform of the sinc function is the unit rectangle function, and the transform of the sinc^2 function is the triangle function of unit height and area.

TABLE 6.1
Theorems for the Fourier transform

Theorem	$f(x)$	$F(s)$
Similarity	$f(ax)$	$\frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$
Addition	$f(x)+g(x)$	$F(s)+G(s)$
Shift	$f(x-a)$	$e^{-i2\pi as}F(s)$
Modulation	$f(x)\cos\omega x$	$\frac{1}{2}F\left(s-\frac{\omega}{2\pi}\right)+\frac{1}{2}F\left(s+\frac{\omega}{2\pi}\right)$
Convolution	$f(x)*g(x)$	$F(s)G(s)$
Autocorrelation	$f(x)*f^*(-x)$	$\lvert F(s)\rvert^2$
Derivative	$f'(x)$	$i2\pi sF(s)$
Derivative of convolution	$\frac{d}{dx}[f(x)*g(x)]=f'(x)*g(x)=f(x)*g'(x)$	
Rayleigh	$\int_{-\infty}^{\infty}\lvert f(x)\rvert^2\,dx=\int_{-\infty}^{\infty}\lvert F(s)\rvert^2\,ds$	
Power	$\int_{-\infty}^{\infty}f(x)g^*(x)\,dx=\int_{-\infty}^{\infty}F(s)G^*(s)\,ds$	
(f and g real)	$\int_{-\infty}^{\infty}f(x)g(-x)\,dx=\int_{-\infty}^{\infty}F(s)G(s)\,ds$	

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PROBLEMS

1. Using the transform pairs given for reference, deduce the further pairs listed below by application of the appropriate theorem. Assume that A and σ are positive.

$$\frac{\sin x}{x} \supset \pi\Pi(\pi s) \qquad \left(\frac{\sin x}{x}\right)^2 \supset \pi\Lambda(\pi s)$$

$$\frac{\sin Ax}{Ax} \supset \frac{\pi}{A}\Pi\left(\frac{\pi s}{A}\right) \qquad \left(\frac{\sin Ax}{Ax}\right)^2 \supset \frac{\pi}{A}\Lambda\left(\frac{\pi s}{A}\right)$$

$$e^{-x^2} \supset \pi^{\frac{1}{2}}e^{-\pi^2 s^2} \qquad \delta(ax) \supset \frac{1}{\lvert a\rvert}$$

$$e^{-Ax^2} \supset \left(\frac{\pi}{A}\right)^{\frac{1}{2}}e^{-\pi^2 s^2/A} \qquad \delta(ax+b) \supset \frac{1}{\lvert a\rvert}e^{i2\pi bs/a}$$

$$e^{-x^2/2\sigma^2} \supset (2\pi)^{\frac{1}{2}}\sigma e^{-2\pi^2\sigma^2 s^2} \qquad e^{ix} \supset \delta\left(s-\frac{1}{2\pi}\right)$$

2. Show that the following transform pairs follow from the addition theorem, and make graphs.

$$1+\cos\pi x \supset \delta(s)+\mathrm{II}(s)$$
$$1+\sin\pi x \supset \delta(s)+i\,{}_{\mathrm{I}}\mathrm{I}(s)$$
$$\operatorname{sinc} x+\tfrac{1}{2}\operatorname{sinc}^2\tfrac{1}{2}x \supset \Pi(s)+\Lambda(2s)$$
$$A^{-\frac{1}{2}}e^{-\pi x^2/A}+A^{\frac{1}{2}}e^{-\pi Ax^2} \supset e^{-\pi s^2/A}+e^{-\pi As^2}$$
$$4\cos^2\pi x+4\cos^2\tfrac{1}{2}\pi x-3 \supset \delta(s+\tfrac{1}{2})+\delta(s-\tfrac{1}{2})+\delta(s)+\tfrac{1}{2}\delta(s-1)+\tfrac{1}{2}\delta(s+1).$$

3. Deduce the following transform pairs, using the shift theorem.

$$\frac{\cos\pi x}{\pi(x-\frac{1}{2})} \supset -e^{-i\pi s}\Pi(s)$$
$$\frac{\sin\pi x}{\pi(x-1)} \supset -e^{-i2\pi s}\Pi(s)$$
$$\Lambda(x-1) \supset e^{-i2\pi s}\operatorname{sinc}^2 s$$
$$\Pi(x-\tfrac{1}{2}) \supset e^{-i\pi s}\operatorname{sinc} s$$
$$\Pi(x)\operatorname{sgn} x \supset -i\sin\tfrac{1}{2}\pi s\operatorname{sinc}\tfrac{1}{2}s$$
$$\Pi\left(\frac{x-\frac{1}{2}a}{a}\right) \supset \lvert a\rvert e^{-i\pi as}\operatorname{sinc} as.$$

4. Use the convolution theorem to find and graph the transforms of the following functions: $\operatorname{sinc} x \operatorname{sinc} 2x$, $(\operatorname{sinc} x\cos 10x)^2$.

5. Let $f(x)$ be a periodic function with period a, that is, $f(x+a)=f(x)$ for all x. Since the Fourier transform of $f(x+a)$ is, by the shift theorem, equal to $\exp(i2\pi as)F(s)$, which must be equal to $F(s)$, what can be deduced about the transform of a periodic function?

6. Graph the transform of $f(x)\sin\omega x$ for large and small values of ω, and explain graphically how, for small values of ω, the transform of $f(x)\sin\omega x$ is proportional to the derivative of the transform of $f(x)$. ▷

7. Graph the transform of $\exp(-x)H(x)\cos\omega x$. Is it an even function of s?

8. Show that a pulse signal described by $\Pi(x/X)\cos 2\pi fx$ has a spectrum

$$\tfrac{1}{2}X\{\operatorname{sinc}[X(s+f)]+\operatorname{sinc}[X(s-f)]\}$$

9. Show that a modulated pulse described by $\Pi(x/X)(1 + M\cos 2\pi Fx)\cos 2\pi fx$ has a spectrum

$$\tfrac{1}{2}X\{\text{sinc}\,[X(s+f)] + \text{sinc}\,[X(s-f)]\} + \tfrac{1}{4}MX\{\text{sinc}\,[X(s+f+F)] + \text{sinc}\,[X(s+f-F)] + \text{sinc}\,[X(s-f+F)] + \text{sinc}\,[X(s-f-F)]\}.$$

Graph the spectrum to a suitably exaggerated scale for a case where there are 100 modulation cycles and 100,000 radio-frequency cycles in one pulse and the modulation coefficient M is 0.6. Show by dimensioning how the factors 100, 100,000, and 0.6 enter into the shape of the spectrum.

10. A function $f(x)$ is defined by

$$f(x) = \begin{cases} 0 & |x| > 2 \\ 2 - |x| & 1 < |x| < 2 \\ 1 & |x| < 1; \end{cases}$$

show that

$$f(x) = 2\Lambda\left(\frac{x}{2}\right) - \Lambda(x) = \Lambda(x) * [\delta(x+1) + \delta(x) + \delta(x-1)]$$

and hence that

$$F(s) = 4\,\text{sinc}^2\,2s - \text{sinc}^2\,s = \text{sinc}^2\,s(1 + 2\cos 2\pi s).$$

11. Prove that $f * g * h \supset FGH$ and hence that $f^{*n} \supset F^n$.

12. The notation f^{*n} meaning $f(x)$ convolved with itself $n-1$ times, where $n = 2, 3, 4, \dots,$ suggests the idea of fractional-order self-convolution. Show that such a generalization of convolution is readily made and that, for example, one reasonable expression for $f(x)$ convolved with itself half a time would be

$$f^{*1\frac{1}{2}} \equiv \int e^{i2\pi sx}\left[\int e^{-i2\pi su} f(u)\,du\right]^{1\frac{1}{2}} ds. \triangleright$$

13. Prove that

$$(f * g)(h * j) \supset (FG) * (HJ)$$

and that

$$(f + g) * (h + j) \supset FH + FJ + GH + GJ.$$

14. Use the convolution theorem to obtain an expression for

$$e^{-ax^2} * e^{-bx^2}. \triangleright$$

15. Prove that

$$\int_{-\infty}^{\infty} f^*(u) g^*(x-u)\,du \supset F^*(-s)G^*(-s).$$

16. Prove that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(u) g^*(u-x) e^{-i2\pi xs}\,du\,dx = F^*(-s)G^*(s).$$

17. Show by Rayleigh's theorem that

$$\int_{-\infty}^{\infty} \text{sinc}^2\,x\,dx = 1$$

$$\int_{-\infty}^{\infty} \text{sinc}^4\,x\,dx = \int_{-\infty}^{\infty} [\Lambda(x)]^2\,dx = \tfrac{2}{3}$$

$$\int_{-\infty}^{\infty} [J_0(x)]^2 dx = \infty$$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$$

18. Complete the following schemata for reference, including thumbnail sketches of the functions.

function	⊃ transform
autocorrelation $f \star f$	⊃ power spectrum $\|F(s)\|^2$

$\Pi(x)$	

$\Lambda(x)$	

$\gamma(x)$	

sinc x	

$\Pi(x)\cos 2\pi fx$	

19. Show the fallacy in the following reasoning. "The Fourier transform of $\int_{-\infty}^{x} f(x)\,dx$ must be $F(s)/i2\pi s$ because the derivative of $\int_{-\infty}^{\infty} f(x)\,dx$ is $f(x)$, and hence by the derivative theorem the transform of $f(x)$ would be $F(s)$, which is true."

20. Establish an integral theorem for the Fourier transform of the indefinite integral of a function. ▷

21. Use the derivative theorem to find the Fourier transform of $xe^{-\pi x^2}$.

22. Show that $2\pi x\Pi(x) \supset i\,\text{sinc}'\,s$.

23. The following brief derivation appears to show that the area under a derivative is zero. Thus

$$\int_{-\infty}^{\infty} f'(x)\,dx = \overline{f'(x)}\Big|_0 = i2\pi sF(s)\Big|_0 = 0.$$

Confirm that this is so, or find the error in reasoning.

24. Show that

$$f(ax-b) \supset \frac{1}{|a|} e^{-i2\pi bs/a} F\left(\frac{s}{a}\right).$$

25. Show from the energy theorem that

$$\int_{-\infty}^{\infty} e^{-\pi x^2} \cos 2\pi a x \, dx = e^{-\pi a^2}.$$

26. Show from the energy theorem that

$$\int_{-\infty}^{\infty} \operatorname{sinc}^2 x \cos \pi x \, dx = \tfrac{1}{2}. \triangleright$$

27. Show that the function whose Fourier transform is $|\operatorname{sinc} s|$ has a triangular autocorrelation function.

28. As a rule, the autocorrelation function tends to be more spread out than the function it comes from. But show that

$$\frac{1}{\pi x} * \frac{-1}{\pi x} = \delta(x).$$

Show that $(\pi x)^{-1}$ must have a flat energy spectrum, and from that deduce and investigate other functions whose autocorrelation is impulsive. ▷

29. The Maclaurin series for $F(s)$ is

$$F(0) + sF'(0) + \frac{s^2}{2!}F''(0) + \dots .$$

Consider the case of $F(s) = \exp(-\pi s^2)$, where the series is known to converge and to converge to $F(s)$. Thus, in this particular case,

$$F(s) = \sum_{n=0}^{\infty} \frac{s^n}{n!} F^{(n)}(0).$$

If $F(s)$ is the transform of $F(x)$, then transforming this equation we obtain

$$f(x) = \delta(x)\int_{-\infty}^{\infty} f(x)\,dx - \delta'(x)\int_{-\infty}^{\infty} x f(x)\,dx + \delta''(x)\int_{-\infty}^{\infty} \frac{x^2}{2!} f(x)\,dx + \dots .$$

How do you explain this result? ▷

30. Fourier pairs. Derive the following Fourier transform pairs:
(*a*) $e^{i\pi x^2} \supset e^{i\pi/4} e^{-i\pi s^2}$
(*b*) $\cos(\pi x^2) \supset 2^{-\frac{1}{2}}[\cos(\pi s^2) + \sin(\pi s^2)]$

(*c*) $\sin(\pi x^2) \supset 2^{-\frac{1}{2}}[\cos(\pi s^2) - \sin(\pi s^2)]$
(*d*) $e^{-\pi a x^2} \begin{matrix}\cos\\ \sin\end{matrix} \pi\beta x^2 \supset (\alpha^2+\beta^2)^{-\frac{1}{4}} \exp\left(-\frac{\pi\alpha s^2}{\alpha^2+\beta^2}\right) \begin{matrix}\cos\\ \sin\end{matrix} \left(\arctan\left(\frac{\beta}{\alpha}\right) - \frac{\pi\beta s^2}{\alpha^2+\beta^2}\right)$
(*e*) $e^{-\pi(\alpha+i\beta)x^2} \supset (\alpha+i\beta)^{-\frac{1}{2}} e^{-\pi s^2/(\alpha+i\beta)}$

31. Frequency analysis. A volcano on the floor of the Pacific Ocean erupted near an inhabited island, causing the sea surface to rise and fall, reaching a maximum height of about 10 meters. The height was recorded by the captain of a vessel standing offshore, using a sextant to determine the distance from the water to the top of a cliff. Later examination showed that the height $h(t)$ could be represented approximately by

$$h(t) = 11 \sin(45° - 72°t) \exp(-t^2/5),$$

where h is in meters and t is in minutes. The volcano erupts from time to time, often causing damage to the docks and shipping in the lagoon, but this is the first time a waveform has become available and it is to be the basis of a redesign of the port.
(*a*) Paying particular attention to the correctness of numerical values, but not necessarily carrying out all the arithmetic, obtain the Fourier transform $\overline{h}(f)$ of $h(t)$.
(*b*) At what frequency, in cycles per minute, will the excitation be at a maximum?

32. Chirp signal. A chirp is a signal that sweeps in frequency and is used in radar by bats and humans to facilitate the sorting out of the emitted signal from the echo under conditions where the first echoes will be returning while the emission is still continuing. An example is

$$s(t) = e^{-\pi t^2/T^2} e^{i2\pi(f_0 t + \beta t^2)}.$$

This chirped pulse has an equivalent duration T, a frequency f_0 at midpulse, and a frequency sweep rate 2β. Show that the power spectrum is centered at f_0 and has an equivalent width Δ given by $\Delta = 2^{-\frac{1}{2}}T^{-1}(1 + 4\beta^2T^4)^{\frac{1}{2}}$.

33. Voigt profiles. Spectral lines often have a profile of the form $[1 + 4(f-f_0)^2/B^2]^{-1}$, which arises, for example, from absorption by a resonator. This is a shifted Cauchy profile. Other spectral lines may have a Gaussian profile, as in the case of a gas in a state of turbulence where Doppler shifts greatly exceed the natural linewidth B. Intermediate profiles of the form Gaussian-convolved-with-Cauchy are known as Voigt profiles. They have interesting properties. Show that the convolution of two Voigt profiles is also a Voigt profile. ▷

34. Inverse theorems. Show that the inverse derivative and inverse shift theorems are

$$-i2\pi x f(x) \supset F'(s)$$
$$e^{i2\pi s_0 x} f(x) \supset F(s - s_0).$$

Are there any other theorems where the direct and inverse forms are not the same?

same curvature to each side of the origin. Therefore any discontinuity must be confined to the imaginary part of $F''(s)$. But the imaginary part of $F(s)$ is zero at $s = 0$; therefore any effect of a discontinuity in $F''(s)$ at $s = 0$ will die out in the limit. It is therefore sufficient to require $f(x)$ to have finite area, finite mean, and finite variance.

In the event of nonidentical functions being convolved, a finite absolute third moment is required plus a more elaborate condition due to Lyapunov to ensure that the third moments are not too strong.

Exercise. Consider the behavior of $(\mathrm{sinc}\, x)^{*n}$, $(\mathrm{sinc}^2\, x)^{*n}$, $[(1 + x^2)^{-1}]^{*n}$, $[x\Pi(x)]^{*n}$, $[\Pi(x) \sin x]^{*n}$, $[e^{-ax} \sin \beta x H(x)]^{*n}$.

If one of the transforms falls to zero for some finite value $s = s_1$ the product of all the transforms will have a zero too, and so their product cannot be Gaussian. Therefore none of the convolving functions of x may be a rectangle function. This may be a moot point because the transform values of the full convolution may be so close to Gaussian for $s > s_1$ that a zero value is indistinguishable from the exact Gaussian value for some intents and purposes. This is the case with $[\Pi(x)]^{*n}$.

SUMMARY OF CORRESPONDENCES IN THE TWO DOMAINS

The results of the preceding discussion are tabulated in Table 8.5 for reference.

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PROBLEMS

1. Deduce a simple expression for $\exp(-x^2) * \exp(-x^2)$. ▷

2. Show that the self-convolution of $(1 + x^2)^{-1}$ is identical with itself, except for scale factors. Show that the self-convolution is twice as wide as the original function, that is, that the width is additive under convolution in this case, and reconcile this with the known general fact that variance is additive.

3. Investigate the functional form of $(1 + x^2/a^2)^{-1} * (1 + x^2/b^2)^{-1}$ and its width in terms of the widths of the convolved functions.

4. Show by direct integration that

$$\Lambda(x)H(x) \supset \frac{1}{i2\pi s} - \frac{e^{-i\pi s}\,\mathrm{sinc}\, s}{i2\pi s};$$

verify the result by applying first the addition and shift theorems to find the Fourier transform of $(d/dx)[\Lambda(x)H(x)]$, and then the derivative theorem in reverse.

5. By separation of the preceding transform into real and imaginary parts, show that

$$\Lambda(x)H(x) \supset \tfrac{1}{2}\,\mathrm{sinc}^2 s + \frac{1 - \cos \pi s\, \mathrm{sinc}\, s}{i2\pi s};$$

as checks on the algebra, verify that the central ordinate and slope are connected in the appropriate way with the area and first moment of $\Lambda(x)H(x)$, respectively, and note whether the transform is indeed hermitian. Break $\Lambda(x)H(x)$ into its even and odd parts, and obtain the transform of each separately.

6. Because of the irregular variation in the number of sunspots, the sequence of daily sunspot numbers is smoothed by taking five-day running totals; that is, for each day we add the sunspot number for the preceding and following two days. Here is a sequence of five-day running totals beginning Jan. 1, 1900:

45, 35, 25, 15, 5, 0, 0, 0, 0, 15, 50, 80, 100, 125, 125, 100, 80, 70, 45, 30, 30, 30, 35, 60, 80, 90, 95, 100, 90, 85, 75. ▷

From this smoothed sequence, what can be deduced about the actual daily values?

7. Show that

$$W_{f\star g} = \frac{W_f W_g}{W_{fg}}$$

where W_f is the equivalent width of $f(x)$.

8. Show that squares of equivalent widths are additive under convolution of Gaussian functions.

9. Show that the equivalent width of $4\,\mathrm{sinc}^2 2s - \mathrm{sinc}^2 s$ is $\frac{1}{3}$, and calculate the equivalent width of $\mathrm{sinc}\, s + \mathrm{sinc}^2 2s$.

10. Investigate the properties of the Jones bandwidth of $f(x)$, which is defined as

$$\frac{\int_0^\infty FF^*\,ds}{F_{max}}. \triangleright$$

11. Show that the autocorrelation width of $f(x)H(x)$ is twice that of $f(x)$.

12. Verify the following autocorrelation widths.

Function:	$x\Pi(x-\frac{1}{2})$	$\Pi(2x+\frac{3}{4}) + \Pi(2x-\frac{3}{4})$	$e^{-x^2/2\sigma^2}$
Width:	$\frac{3}{4}$	1	$2\pi^{\frac{1}{2}}\sigma$

13. The abscissa x of a function $f(x)$ is divided into a finite number of finite segments (plus two semi-infinite end segments). The finite segments are rearranged without overlapping, thus defining a new function which we may describe as derivable from $f(x)$ by shuffling. For example, the string of 11 pulses

$$\sum_{n=-5}^{n=+5} \Pi(11x - n)$$

and the function $|x|\Pi(x/2)$ are derivable respectively from $\Pi(x)$ and $\Lambda(x)$ by shuffling, and vice versa. Show that the equivalent width is unaffected by shuffling if the segment containing $x = 0$ is not dislodged in the shuffle, and that the autocorrelation width is not affected in any case. Consider the effect of shuffling on the total energy of a waveform, and mention several parameters of its power spectrum which are invariant under shuffling. $\triangleright$

14. An even function $g(x)$ is derived from a function $f(x)$ by the process of symmetrization illustrated in Fig. 8.21, where $AB = CD$, $EF = GH + IJ$, and so on. This process, known

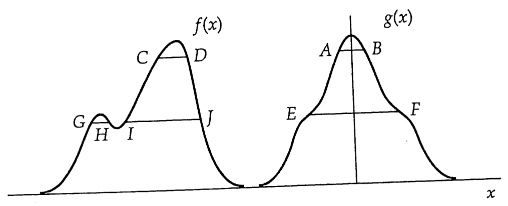

Fig. 8.21 **Steiner symmetrization.**

as Steiner symmetrization, was used by Jacob Steiner to prove that the circle is the figure of minimum perimeter for a given area. Show that the autocorrelation functions of $f(x)$ and $g(x)$ have the same equivalent width.

15. State the relationship between the functions $\exp(-x)H(x)$ and $\exp(-|x|)$ in the light of symmetrization, and say how their respective power spectra are related. $\triangleright$

16. Show that the convolution of two odd functions is even.

17. Establish the following relations between functions and their autocorrelation functions.

Function	Autocorrelation
$\Pi(x)$	$\Lambda(x)$
$e^{-\pi x^2}$	$2^{-\frac{1}{2}}e^{-\frac{1}{2}\pi x^2}$
$\delta(x)$	$\delta(x)$
$e^{-x}H(x)$	$\frac{1}{2}e^{-\lvert x\rvert}$
$e^{-\lvert x\rvert}$	$e^{-\lvert x\rvert}(1+\lvert x\rvert)$

18. Show by a simple argument in the Fourier transform domain that the autocorrelation function of $\exp(-\pi x^2)\cos \omega x$ is $2^{-3/2}\exp(-\frac{1}{2}\pi x^2)\cos \omega x$ when ω is large. $\triangleright$

19. Some difficulty arises over the autocorrelation of $\cos x$. Show that there is a sense in which the autocorrelation of a cosine function is also a cosine function.

20. Show that the product of the autocorrelation widths of a function and its transform is given by

$$W_{f\star f^*}W_{F\star F^*} = \frac{(\int f\,dx)^2(\int F\,ds)^2}{(\int f^2\,dx)^2}$$

and that the product does not have a nonzero lower limit.

21. Show that

$$|f * g| \leqslant \int_{-\infty}^{\infty} |FG|\,ds.$$

22. **Self-reciprocal transform.** We know that there is at least one function $f(\cdot)$ that is its own Fourier transform $F(\cdot)$, because

$$e^{-\pi x^2} \supset e^{-\pi s^2}.$$

It is reported that

$$f(x) = e^{-\pi(x/5)^2} + 5e^{-\pi(5x)^2}$$

is another function that is its own Fourier transform, that is, that $F(s) = f(s)$. Can you prove or disprove this claim?

23. Wavepacket spectrum. Show that

$$\exp(-\beta x^2)\cos\alpha x \supset \left(\frac{\pi}{\beta}\right)^{\frac{1}{2}} \exp\left[-\left(\frac{\alpha^2+4\pi^2 s^2}{4\beta}\right)\right]\cosh\left(\frac{\pi\alpha s}{\beta}\right).$$

24. Surfing. There is a tradition in surfing communities that every seventh wave is bigger. What would be the corresponding feature in the wave-spectrum domain?

25. Hermite polynomial pairs. Derive the following Fourier transform pairs

$$xe^{-\pi x^2} \supset -ise^{-\pi s^2}$$
$$(4\pi x^2-1)e^{-\pi x^2} \supset -(4\pi s^2-1)e^{-\pi s^2}$$
$$(4\pi x^3-3x)e^{-\pi x^2} \supset i(4\pi s^3-3s)e^{-\pi x^2}$$
$$(16\pi^2 x^4-24\pi x^2+3)e^{-\pi x^2} \supset (16\pi^2 s^4-24\pi s^2+3)e^{-\pi s^2}$$

and, in general,

$$H_n(\sqrt{2\pi}\,x)e^{-\pi x^2} \supset (-i)^n H_n(\sqrt{2\pi}\,s)e^{-\pi s^2},$$

where $H_n(x)$ are the Hermite polynomials $1,\ 2x,\ 4x^2-2,\ 8x^3-12x,\ 16x^4-48x^2+12,\ \ldots,\ n=0,1,2,\ldots$.

26. Computed music. Describe how it might be possible to generate a sound of perpetually rising pitch that never rises beyond the range of audibility. ▷

27. Acoustic perception. A musician with his back to a violinist is able to tell whether the player is slowly moving away or is playing diminuendo. On being questioned about his ability, the musician explained, "The violinist may play a single tone loudly or softly and maintain steady tone quality, but as he moves away while sustaining a single note, there is a clear change in the color of the note. It sounds purer—the overtones less prominent." An acoustics engineer said, "When you hear a click it is followed by reverberant energy that arrives after reflection from the walls. The same click farther away sounds fainter but the amount of reverberant energy entering the ear is about the same. Therefore, the subjective impression is not the same as for a faint nearby click. Since the impulse response is different, naturally the violin note sounds different." Explain how this explanation, if it is correct, is consistent with the musician's explanation.

28. Central slope of transform phase. Let $\phi(s)$ be the phase of the Fourier transform $F(s)$ of a given real function $f(x)$. Show that, if $\phi(s)$ passes through the origin, it does so with a slope

$$\phi'(0) = -2\pi \int_{-\infty}^{\infty} xf(x)\,dx \Big/ \int_{-\infty}^{\infty} f(x)\,dx.$$

Verify the result for the case where $f(x) = xe^{-x}H(x)$.

29. Restoration for running means. Suppose that we are given $g(x)$, a function that results from smoothing $f(x)$ by convolution with a rectangle function $\Pi(x)$. Thus $g(x) = \Pi(x) * f(x)$. We wish to find $f(x)$. See whether you can find an inverse operator $\Pi^{-1}(x)$ such that $\Pi^{-1}(x) * \Pi(x) = \delta(x)$ or whether you can find a Fourier transform for $(\text{sinc}\, s)^{-1}$. ▷

30. Restoration for weighted running mean. Attempt to find an inverse operator for running means taken with triangular weighting. Thus if

$$g(x) = \Lambda(x) * f(x)$$

seek $\Lambda^{-1}(x)$ with the property that, if $g(x)$ is given, $f(x)$ may be found by performing the inverse operation: $f(x) = \Lambda^{-1}(x) * g(x)$. If this approach works, it will be necessary that $\Lambda^{-1}(x) * \Lambda(x) = \delta(x)$. It has been proposed that

$$f(x) = g''(x-1) + 2g''(x-2) + 3g''(x-3) + \ldots.$$

Examine this formula for some special case by graphing the first two or three terms and attempt to derive it.

31. Moments. Determine constants a, b, and c such that $f(x) = a + b\cos 2\pi cx$ is a good fit to sinc x over the range $-1 < x < 1$. In what way could $F(s)$ be said to resemble $\Pi(s)$?

32. Sinc function properties. From p. 75 of Abramovitz and Stegun (1964) we discover that

$$\text{sinc}\, x = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right).$$

We may think of each parabolic factor progressively forcing the product through $x = 1, 2, \ldots, n$. Show that the coefficients in the expansion of $(1-x^2)(1-x^2/4)(1-x^2/9)$ do not agree with the coefficients of the Taylor series expansion of sinc x. Why is that? Why does the central limit theorem not seem to apply? ▷

33. Dual lines of reasoning. Problems that are stated in the time domain and require answers in the frequency domain can be reasoned out in either of two ways: Translate the statement into the frequency domain and find the answer to the new question, or solve in the time domain and translate the answer into the frequency domain. Consider the following problem. "A short pulse has a more or less flat spectrum up to a roll-off frequency that is some fraction of (pulse duration)$^{-1}$, but is it true that the spectrum of a waveform consisting of two identical pulses in succession, far from being approximately uniform, is such that there are frequencies where there is no content at all?"

A line of reasoning leading to this result is as follows: "The pulse pair is expressible as a convolution between the waveform of a single pulse and an impulse pair $\frac{1}{2}\delta(t) + \frac{1}{2}\delta(t-T)$, where T is the interval between pulses. Consequently, its spectrum contains a factor $\cos \pi Tf$, which produces zeros at certain frequencies." The problem did not require the full spectrum to be calculated and did not ask for the values of frequency at the zeros, but as with many important questions, was merely concerned with whether a certain phenomenon existed. Consequently, the reasoning presented can be brief. Now give the other line of reasoning leading to the same result.

34. Variance of wavepacket abscissa. Derive the variance $\langle x^2 \rangle$ of the wavepacket $f(x) = \exp[-\pi(x/W)^2]\cos 2\pi\nu x$. ▷

35. A function consisting of an asymmetrical triangular peak, with its reflection, is defined by

$$f(x) = \begin{cases} k|x|, & 0 \leqslant |x| < a \\ 1 - |x|, & a \leqslant |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

where k and a lie between 0 and 1. Find its Fourier transform $F(s)$. ▷

36. Radar pulse generator. In order to deliver a megawatt of radio frequency power output to an antenna in a pulse of $\Delta = 0.1$ μs duration, a generator has to be excited at its input terminals by a voltage $15{,}000\Pi(t - \frac{1}{2}\Delta)$ volts. A way of doing this would be to spend a millisecond or so charging one conductor of a transmission line to a steady voltage of 30 kV. The characteristic impedance Z_0 of the transmission line would be made equal to the input resistance of the generator. At $t = 0$, a switch would connect the charged transmission line to the generator, applying 15 kV to the generator input. Electric charge would pour from the transmission line at a constant rate for 0.1 μs until the total charge was expended, whereupon the generator excitation would drop to zero and the r.f. pulse going to the transmitting antenna would terminate. (*a*) What would the length of the transmission line have to be (in meters)? (*b*) As seen from the generator, what would be the input impedance to the transmission line segment and the voltage transfer function, as a function of frequency? (*c*) What would the impulse response of the transmission line segment be? ▷

37. Functions to be expressed as finite differences. Do the exercise relating to Fig. 8.17. ▷

38. Keeping up with periodicals. Scan recent issues of journals in the current field of study, find a paper that interests you, and submit a synopsis in a form that would equip your instructor to give a 10-minute talk to the class about the work without having to refer to the original paper.

39. Composing a homework problem. Compose a homework problem suitable for your class and hand it in together with a solution suitable in final form for distribution. Bear in mind the distinction between a problem and an exercise.

40. Schrödinger's equation. A basic principle of quantum mechanics, applied to the harmonic oscillator composed of a mass m constrained by a spring of stiffness k, says that the spatial wave function $\psi(x)$ (whose squared modulus $\psi\psi^*$ gives the relative probability of finding the mass in position $x \pm \frac{1}{2}dx$), obeys the second-order, linear differential equation

$$\frac{\hbar}{2m}\frac{d^2\psi}{dx^2} + E\psi - \tfrac{1}{2}kx^2\psi = 0.$$

Solutions, for the various allowed energies $E_n = \hbar\omega(n + \frac{1}{2})$, $n = 0, 1, 2, \ldots$ are

$$\psi_n(x) = \sqrt{\alpha/\sqrt{\pi}2^n n!}H_n(\alpha x)e^{-\alpha x^2/2},$$

where $\omega^2 = k/m$ and $\alpha = (mk/\hbar)^{1/4}$. The H_n are the Hermite polynomials listed above in Problem 8.25, where the Hermite-Gauss functions $H_n(x)\exp(-x^2/2)$ can be seen to be their own Fourier transforms. That seems to imply that if one takes the Fourier transform of this Schrödinger equation term by term, the resulting differential equation for the transform Ψ must still be a Schrödinger equation. Is that so? ▷