Mathematics Chalmers & GU

TMA462/MMA410: Fourier and Wavelet Analysis, 2012–12–20; kl 14:00-18:00.

Telephone: Richard Lärkäng: 0703-088304
Course Books: Bergh et al AND Bracewell, Lecture Notes and Calculator are allowed.
Each problem gives max 5p. Breakings: **3**: 12-15p, **4**: 16-19p och **5**: 20p-
For GU **G** students :12p, **VG**: 18p- (if applicable)
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma462/1213/index.html

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1. Show that $\{\Lambda(t-n)\}_{n=-\infty}^{\infty}$ are linearly independent.

2. Determine the periodic autocorrelation of the function $f(x) = \cos 2\pi x$.

3. Let $a > 0$ and derive the Fourier transform of $\displaystyle\sum_{n=-\infty}^{\infty} \delta_{an}$.

4. Assume that φ is the scaling function in an orthogonal multi-resolution anlysis (OG-MRA). Show that the support of $\hat{\varphi}$ measures at least 2π.

5. Assume that φ and ψ are the scaling function and the wavelet of an MRA, respectively. Assume further that φ has vanishing moments of order $\leq L$, i.e.

$$\int t^k \varphi(t)\, dt = 0, \qquad 1 \leq k \leq L.$$

Show that ψ has vanishing moments of order $\leq L/2$:

$$\int t^k \psi(t)\, dt = 0, \qquad 1 \leq k \leq L/2.$$

Note that zero moments are not included.

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TMA462/MMA410: Fourier and Wavelet Analysis, 2012–12–20; kl 14:00-18:00.. Lösningar/Solutions.

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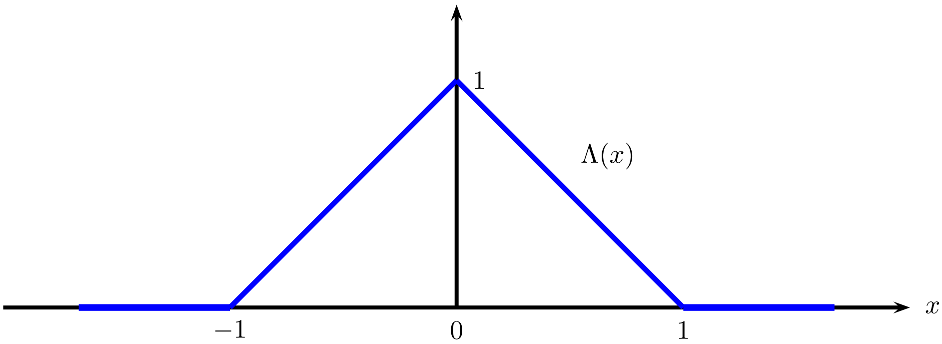
1. Show that $\{\Lambda(t-n)\}_{n=-\infty}^{\infty}$ are linearly independent.
Solution: We have that

$$\Lambda(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$



$$\sum_{n=M}^{N} C_n \Lambda(t-n) = 0, \quad \Longrightarrow \quad t = k, \quad \text{gives} \quad C_k = 0, \quad -M \le k \le N.$$

In other words $\{\Lambda(t-n)\}_{n=-M}^{N}$ is linearly independent for all M and N. Hence, $\{\Lambda(t-n)\}_{n=-\infty}^{\infty}$ is linearly independent.

2. Determine the periodic autocorrelation of the function $f(x) = \cos 2\pi x$.
Solution: The function $f(x) = \cos 2\pi x$ is periodic with period 1. Its periodic autocorrelation is then defined as

$$\begin{aligned}
f \star f(x) &= \int_0^1 f(u)f(u-x)\,du = [u-x=y] \\
&= \int_{-x}^{1} -x f(x+y)f(y)\,dy = [\text{the integrand is 1-periodic}] \\
&= \int_0^1 f(u)f(u-x)\,du = \int_0^1 = \cos 2\pi u \cdot \cos 2\pi(u-x)\,du \\
&= \int_0^1 \frac{1}{2}[\cos(2\pi u - 2\pi(u-x)) + \cos(2\pi u + 2\pi(u-x))]\,du \\
&= \frac{1}{2}\int_0^1 [\cos 2\pi x + \cos(2\pi u + 2\pi(2u-x))]\,du = \frac{1}{2}\cos 2\pi x + \frac{1}{2}\frac{1}{4\pi}[\sin 2\pi(2u-x)]_0^1 \\
&= \frac{1}{2}\cos 2\pi x + \frac{1}{8\pi}(\sin 2\pi(1-x) + \sin 2\pi x) = \frac{1}{2}\cos 2\pi x.
\end{aligned}$$

3. Let $a > 0$ and derive the Fourier transform of $\sum_{n=-\infty}^{\infty} \delta_{an}$.

Solution: We have that

$$\langle \delta_{an}, \varphi \rangle = \langle \delta, \varphi_{-an} \rangle = \varphi(an), \qquad \varphi \in S.$$

Hence

$$\langle \mathcal{F}\delta_{an}, \varphi \rangle = \langle \delta_{an}, \hat{\varphi} \rangle = \hat{\varphi}(an) = \frac{1}{a}\mathcal{F}[\varphi(\frac{x}{a})](n) = \langle \delta_n, \frac{1}{a}\mathcal{F}[\varphi(\frac{x}{a})] \rangle = \frac{1}{a}\langle \mathcal{F}\delta_n, \varphi(\frac{x}{a}) \rangle.$$

Using this identity we can write

$$\begin{aligned}
\langle \mathcal{F}[\sum_{n=-\infty}^{\infty} \delta_{an}], \varphi \rangle &= \frac{1}{a}\sum_{n=-\infty}^{\infty} \langle \mathcal{F}\delta_n, \varphi(\frac{x}{a}) \rangle = \frac{1}{a}\langle \sum_{n=-\infty}^{\infty} \mathcal{F}\delta_n, \varphi(\frac{x}{a}) \rangle \\
&= \frac{1}{a}\langle \mathcal{F}\sum_{n=-\infty}^{\infty} \delta_n, \varphi(\frac{x}{a}) \rangle = \frac{1}{a}\langle \sum_{n=-\infty}^{\infty} \delta_n, \varphi(\frac{x}{a}) \rangle = \frac{1}{a}\sum_{n=-\infty}^{\infty} \langle \delta_n, \varphi(\frac{x}{a}) \rangle \\
&= \frac{1}{a}\sum_{n=-\infty}^{\infty} \varphi(\frac{n}{a}) = \frac{1}{a}\sum_{n=-\infty}^{\infty} \langle \delta_{n/a}, \varphi \rangle = \langle \frac{1}{a}\sum_{n=-\infty}^{\infty} \delta_{n/a}, \varphi \rangle.
\end{aligned}$$

Thus

$$\mathcal{F}[\sum_{n=-\infty}^{\infty} \delta_{an}] = \frac{1}{a}\sum_{n=-\infty}^{\infty} \delta_{n/a}.$$

4. Assume that φ is the scaling function in an orthogonal multi-resolution anlysis (OG-MRA). Show that the support of $\hat{\varphi}$ measures at least 2π.

Solution: We have using Plancherel relation that

$$\begin{aligned}
\delta_k &= \int_{-\infty}^{\infty} \varphi(t)\varphi(t-k)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\omega)\overline{e^{-ik\omega}\hat{\varphi}(\omega)}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\varphi}(\omega)|^2 e^{ik\omega}\,d\omega = \frac{1}{2\pi}\sum_{-\infty}^{\infty}\int_{2n\pi}^{2(n+1)\pi} |\hat{\varphi}(\omega)|^2 e^{ik\omega}\,d\omega = [\omega - 2\pi n \to \omega] \\
&= \frac{1}{2\pi}\sum_{-\infty}^{\infty}\int_{0}^{2\pi} |\hat{\varphi}(\omega+2n\pi)|^2 e^{ik(\omega+2n\pi)}\,d\omega = \frac{1}{2\pi}\int_{0}^{2\pi}\sum_{-\infty}^{\infty} |\hat{\varphi}(\omega+2n\pi)|^2 e^{ik\omega}\,d\omega.
\end{aligned}$$

This yields

$$\sum_{-\infty}^{\infty} |\hat{\varphi}(\omega+2n\pi)|^2 = 1, \qquad \forall\ \omega \quad \text{(or for a.e.)}.$$

Let now $S = supp\hat{\varphi} = \overline{\{\hat{\varphi}(\omega) \neq 0\}}$. Then $|\hat{\varphi}(\omega)|^2 \leq 1$ for all ω gives the measure of

$$|S| = \int_S d\omega \geq \int_S |\hat{\varphi}(\omega)|^2\,d\omega = \int_{-\infty}^{\infty} |\hat{\varphi}(\omega)|^2\,d\omega = 2\pi\int_{-\infty}^{\infty} |\varphi(x)|^2\,dx = 2\pi,$$

where we have assumed that $||\varphi||_{L^2} = 1$.

5. Assume that φ and ψ are the scaling function and the wavelet of an MRA, respectively. Assume further that φ has vanishing moments of order $\leq L$, i.e.

$$\int t^k \varphi(t)\, dt = 0, \qquad 1 \leq k \leq L.$$

Show that ψ has vanishing moments of order $\leq L/2$:

$$\int t^k \psi(t)\, dt = 0, \qquad 1 \leq k \leq L/2.$$

Note that zero moments are not included.

Solution: We have that

$$\mathcal{F}[t^k \varphi(t)](\omega) = i^k D^k \hat{\varphi}(\omega).$$

The assumption

$$\int t^k \varphi(t)\, dt = i^k D^k \hat{\varphi}(0) = 0, \quad k = 1, \dots, L,$$

yields

$$D^k \hat{\varphi}(0) = 0,$$

which together with

$$\hat{\varphi}(2\omega) = H(\omega)\hat{\varphi}(\omega), \quad \hat{\varphi}(0) = 1, \quad H(0) = 1,$$

and the trivial expansion

$$2^k D^k \hat{\varphi}(2\omega) = \sum_{\alpha=0}^{k} \binom{k}{\alpha} D^{k-\alpha} H(\omega) D^{\alpha} \hat{\varphi}(\omega), \quad \text{implies that} \quad 2^k D^k \hat{\varphi}(0) = D^k H(0) \cdot 1$$

Thus

$$D^k H(0) = 0, \quad 1 \leq k \leq L.$$

Further, for the orthogonal system we have that

$$|H(\omega)|^2 + |G(\omega)|^2 = 1,$$

which together with Binomial expansions and properties of H ($H(0) = 1,\ D^k H(0) = 0,\ 1 \leq k \leq L$) gives that

$$H(\omega) - 1 = \omega^{L+1} H_1(\Omega).$$

Thus for sufficiently small $|\omega|$ with $|H(\omega)| \leq 1$ we have that

$$\begin{aligned} |G(\omega)|^2 &= 1 - |H(\omega)|^2 = (1 - |H(\omega)|)(1 + |H(\omega)|) \\ &\leq 2(1 - |H(\omega)|) \leq 2|1 - H(\omega)| = 2|\omega|^{L+1} |H_1(\Omega)| \leq C_1 |\omega|^{L+1}. \end{aligned}$$

Hence

$$|G(\omega)| \leq C|\omega|^{\frac{L+1}{2}},$$

which implies that

$$D^k G(0) = 0, \quad k = 0, 1, \dots, [\frac{L}{2}].$$

Consequently, using

$$\int t^k \psi(t)\, dt = i^k D^k \hat{\psi}(0), \qquad \text{and} \qquad \hat{\psi}(2\omega) = G(\omega)\hat{\psi}(\omega)$$

we may use Binomisal polynomial again and write

$$2^k D^k \hat{\psi}(2\omega) = \sum_{\alpha=0}^{k} \binom{k}{\alpha} D^{k-\alpha} G(\omega) D^{\alpha} \hat{\varphi}(\omega),$$

where $D^k \hat{\varphi}(0) = 0$ for $k = 1, \dots, L$ implies that

$$2^k D^k \hat{\psi}(0) = D^k G(0) = 0, \quad \Longrightarrow \quad \int t^k \psi(t)\, dt = 0.$$

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