

TMA521/MMA510
Optimization, project course
Lecture 2
**Decomposition and solution of a difficult
problem—facility location**

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Location of facilities which serve customers

Problem settings

- ▶ Potential depot sites: $\mathcal{J} = \{1, \dots, n\}$ (geographical locations)
 - ▶ Existing customers: $\mathcal{I} = \{1, \dots, m\}$ (geographical locations)
- f_j = fixed cost of opening depot (facility) $j \in \mathcal{J}$
- c_{ij} = transportation cost when customer i 's demand is fulfilled entirely from depot j ($i \in \mathcal{I}, j \in \mathcal{J}$)

Decision problem

- ▶ Which depots to **open**?
- ▶ Which **depots** to serve which **customers**, and **how much**?
- ▶ **Goal** minimize cost
- ▶ **Assumption**: depots have unlimited capacity (to be removed)

Uncapacitated facility location (UFL)

Variables

$$y_j = \begin{cases} 1, & \text{if depot } j \text{ is opened} \\ 0, & \text{otherwise} \end{cases}$$

$$x_{ij} = \begin{bmatrix} \text{proportion of customer } i\text{'s demand} \\ \text{to be delivered from depot } j \end{bmatrix}$$

Mathematical model

$$z_0^* = \min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j \quad (0)$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}} x_{ij} = 1, \quad i \in \mathcal{I} \quad (1)$$

$$x_{ij} - y_j \leq 0, \quad i \in \mathcal{I}, j \in \mathcal{J} \quad (2)$$

$$x_{ij} \in [0, 1], \quad i \in \mathcal{I}, j \in \mathcal{J} \quad (3)$$

$$y_j \in \{0, 1\}, \quad j \in \mathcal{J} \quad (4)$$

The mathematical model

$$z_0^* = \min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j \quad (0)$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}} x_{ij} = 1, \quad i \in \mathcal{I} \quad (1)$$

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$$x_{ij} \in [0, 1], \quad i \in \mathcal{I}, j \in \mathcal{J} \quad (3)$$

$$y_j \in \{0, 1\}, \quad j \in \mathcal{J} \quad (4)$$

- (0) Minimize cost
- (1) Deliver precisely the demand
- (2) Deliver from open depots only
- (3) x_{ij} is the *proportion* of the demand of customer i to be delivered from depot j
- (4) A depot may *not be partially* opened

Suppose that the depots have limited capacity

- ▶ d_i = demand of customer i ($D = \sum_{i \in \mathcal{I}} d_i$)
- ▶ b_j = capacity of depot j —if it is opened

Constraints:

$$\sum_{i \in \mathcal{I}} d_i x_{ij} \leq b_j y_j, \quad j \in \mathcal{J} \quad (5) \quad (\Rightarrow x_{ij} \leq y_j, \quad \forall i, j)$$

$\Rightarrow$ replace (2) (i.e., $x_{ij} \leq y_j$) by (5) $\Rightarrow$

$$z_0^* = \min \quad \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j \quad (0)$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}} x_{ij} = 1, \quad i \in \mathcal{I} \quad (1)$$

$$\sum_{i \in \mathcal{I}} d_i x_{ij} - b_j y_j \leq 0, \quad j \in \mathcal{J} \quad (5)$$

$$x_{ij} \in [0, 1], \quad i \in \mathcal{I}, j \in \mathcal{J} \quad (3)$$

$$y_j \in \{0, 1\}, \quad j \in \mathcal{J} \quad (4)$$

Capacitated facility location (CFL)

$$z^* = \min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j \quad (0)$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}} x_{ij} = 1, \quad i \in \mathcal{I} \quad (1)$$

$$\sum_{i \in \mathcal{I}} d_i x_{ij} - b_j y_j \leq 0, \quad j \in \mathcal{J} \quad (5)$$

$$x_{ij} \in [0, 1], \quad i \in \mathcal{I}, j \in \mathcal{J} \quad (3)$$

$$y_j \in \{0, 1\}, \quad j \in \mathcal{J} \quad (4)$$

Observation: The total capacity of open depots must cover the entire demand $\implies$ an additional (redundant) constraint:

$$(1), (5) \Rightarrow \overbrace{\sum_{j \in \mathcal{J}} b_j y_j}^{\text{capacity}} \geq \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} d_i x_{ij} = \sum_{i \in \mathcal{I}} d_i \sum_{j \in \mathcal{J}} x_{ij} = \sum_{i \in \mathcal{I}} d_i \cdot 1 = \overbrace{\phantom{\sum_{i \in \mathcal{I}} d_i}}^{\text{demand}} D \quad (6)$$

Add this constraint to the model

Resulting model

$$\begin{aligned} z^* = \min \quad & \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j \\ \text{s.t.} \quad & \sum_{j \in \mathcal{J}} x_{ij} = 1, \quad i \in \mathcal{I} \end{aligned} \quad (1)$$

$$\sum_{i \in \mathcal{I}} d_i x_{ij} - b_j y_j \leq 0, \quad j \in \mathcal{J} \quad (5)$$

$$\sum_{j \in \mathcal{J}} b_j y_j \geq D, \quad (6)$$

$$x_{ij} \in [0, 1], \quad i \in \mathcal{I}, j \in \mathcal{J} \quad (3)$$

$$y_j \in \{0, 1\}, \quad j \in \mathcal{J} \quad (4)$$

Trick: variable splitting

- ▶ Replace x_{ij} by w_{ij} in constraint (1) and in “half” the objective
- ▶ Let $0 \leq \alpha \leq 1$.
- ▶ Add the constraints $x_{ij} = w_{ij}$

$$\begin{aligned} z^* = \min \quad & \alpha \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} x_{ij} + (1 - \alpha) \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} w_{ij} + \sum_{j \in \mathcal{J}} f_j y_j \\ \text{s.t.} \quad & \sum_{j \in \mathcal{J}} w_{ij} = 1, \quad i \in \mathcal{I} \end{aligned} \quad (1)$$

$$\sum_{i \in \mathcal{I}} d_i x_{ij} - b_j y_j \leq 0, \quad j \in \mathcal{J} \quad (5)$$

$$\sum_{j \in \mathcal{J}} b_j y_j \geq D, \quad (6)$$

$$w_{ij} - x_{ij} = 0, \quad i \in \mathcal{I}, j \in \mathcal{J} \quad (7)$$

$$x_{ij} \in [0, 1], \quad i \in \mathcal{I}, j \in \mathcal{J} \quad (3)$$

$$w_{ij} \geq 0, \quad i \in \mathcal{I}, j \in \mathcal{J} \quad (8)$$

$$y_j \in \{0, 1\}, \quad j \in \mathcal{J} \quad (4)$$

Lagrangian relaxation

- ▶ The constraints (7) tie together the variables $(\mathbf{x}, \mathbf{y})$ and $\mathbf{w}$
- ▶ Lagrangian relax these with multipliers λ_{ij}

⇒ Lagrange function

$$\begin{aligned} L(\mathbf{x}, \mathbf{w}, \mathbf{y}, \boldsymbol{\lambda}) &= \\ &= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \left[\alpha c_{ij} x_{ij} + (1 - \alpha) c_{ij} w_{ij} + \overbrace{\lambda_{ij} (w_{ij} - x_{ij})}^{\text{penalty}} \right] + \sum_{j \in \mathcal{J}} f_j y_j \\ &= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} (\alpha c_{ij} - \lambda_{ij}) x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} [(1 - \alpha) c_{ij} + \lambda_{ij}] w_{ij} \end{aligned}$$

- ▶ For a fixed value of $\boldsymbol{\lambda}$:
Minimize the Lagrange function under the constraints (1), (5), (6), (3), (8) & (4)

The Lagrangian subproblem

The constraints (7) are “moved” to the objective, the rest are rearranged:

$$q(\lambda) = \min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} (\alpha c_{ij} - \lambda_{ij}) x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} [(1 - \alpha) c_{ij} + \lambda_{ij}] w_{ij}$$

$$\text{s.t.} \quad \sum_{i \in \mathcal{I}} d_i x_{ij} - b_j y_j \leq 0, \quad j \in \mathcal{J} \quad (5)$$

$$\sum_{j \in \mathcal{J}} b_j y_j \geq D, \quad (6)$$

$$x_{ij} \in [0, 1], \quad i \in \mathcal{I}, j \in \mathcal{J} \quad (3)$$

$$y_j \in \{0, 1\}, \quad j \in \mathcal{J} \quad (4)$$

$$\sum_{j \in \mathcal{J}} w_{ij} = 1, \quad i \in \mathcal{I} \quad (1)$$

$$w_{ij} \geq 0, \quad i \in \mathcal{I}, j \in \mathcal{J} \quad (8)$$

Separates into one problem in $(\mathbf{x}, \mathbf{y})$ and $|\mathcal{I}|$ problems in $\mathbf{w}$

The subproblem in x and y (for a fixed value of λ)

$$q_{xy}(\lambda) = \min_{x,y} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} [\alpha c_{ij} - \lambda_{ij}] x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j$$

s.t.

$$\sum_{j \in \mathcal{J}} b_j y_j \geq D, \quad (6)$$

$$\sum_{i \in \mathcal{I}} d_i x_{ij} \leq b_j y_j, \quad j \in \mathcal{J} \quad (5)$$

$$x_{ij} \in [0, 1], \quad i \in \mathcal{I}, j \in \mathcal{J} \quad (3)$$

$$y_j \in \{0, 1\}, \quad j \in \mathcal{J} \quad (4)$$

- ▶ This problem can be further decomposed through the following observation
- ▶ For every solution $\mathbf{y}$ (such that $\sum_{j \in \mathcal{J}} b_j y_j \geq D$ holds) we have the following:
 - ▶ If $y_j = 0$ then $x_{ij} = 0, i \in \mathcal{I}$ must hold
 - ▶ If $y_j = 1$ then $\sum_{i \in \mathcal{I}} d_i x_{ij} \leq b_j$ and $x_{ij} \in [0, 1]$ must hold

The value/cost of opening depot j , i.e., letting $y_j = 1$ (in the (x, y) -subproblem)

- ▶ $|\mathcal{J}|$ continuous knapsack problems (easy to solve)

$$\begin{aligned} \text{[CKSP}_j\text{]} \quad v_j(\lambda) &= f_j + \min_{\mathbf{x}} \sum_{i \in \mathcal{I}} [\alpha c_{ij} - \lambda_{ij}] x_{ij} \\ \text{s.t.} \quad &\sum_{i \in \mathcal{I}} d_i x_{ij} \leq b_j \\ &x_{ij} \in [0, 1], \quad i \in \mathcal{I} \end{aligned}$$

- ▶ Then, decide which depots to open (for a certain value of λ)
- ▶ Project onto the $\mathbf{y}$ space $\Rightarrow$ one 0/1 knapsack problem

$$\begin{aligned} \text{[0/1-KSP]} \quad q_{\mathbf{xy}}(\lambda) &= \min_{\mathbf{y}} \sum_{j \in \mathcal{J}} v_j(\lambda) \cdot y_j \\ \text{s.t.} \quad &\sum_{j \in \mathcal{J}} b_j y_j \geq D, \\ &y_j \in \{0, 1\}, \quad j \in \mathcal{J} \end{aligned}$$

Solving the continuous knapsack problems [CKSP_j]

Greedy algorithm

- ▶ Sort the values $\frac{\alpha c_{ij} - \lambda_{ij}}{d_i} < 0$, $i \in \mathcal{I}$, in increasing order
 $\Rightarrow$ indices $\{i_1, i_2, \dots, i_p\}$, where $p \leq m = |\mathcal{I}|$

- ▶ $\bar{x}_{ij} := 0$, $i \in \mathcal{I}$, $k := 0$

repeat

$$k := k + 1$$

$$\bar{x}_{i_k j} := \min \left\{ 1; \left(b_j - \sum_{s=1}^{k-1} d_{i_s} \bar{x}_{i_s j} \right) / d_{i_k} \right\}$$

until $\sum_{s=1}^k d_{i_s} \bar{x}_{i_s j} = b_j$ or $k = p$

- ▶ The solution fulfills $\sum_{i \in \mathcal{I}} d_i \bar{x}_{ij} \leq b_j$ and $\bar{x}_{ij} \in [0, 1]$, $i \in \mathcal{I}$

$\Rightarrow$ **The value/cost of opening depot j**

- ▶ $v_j(\lambda) = f_j + \sum_{k=1}^p \sum_{j \in \mathcal{J}} [\alpha c_{i_k j} - \lambda_{i_k j}] \bar{x}_{i_k j}$

Solving the 0/1 knapsack problems [0/1-KSP]

$$\begin{aligned} q_{xy}(\lambda) = \min_y \quad & \sum_{j \in \mathcal{J}} v_j(\lambda) \cdot y_j \\ \text{s.t.} \quad & \sum_{j \in \mathcal{J}} b_j y_j \geq D, \\ & y_j \in \{0, 1\}, \quad j \in \mathcal{J} \end{aligned}$$

where $v_j(\lambda) = f_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} [\alpha c_{ij} - \lambda_{ij}] \bar{x}_{ij}$
and $\bar{x}_{ij}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$, are computed by the greedy algorithm

- ▶ 0/1-KSP *cannot* be solved in polynomial time
- ▶ Solve using Dynamic Programming or Branch & Bound (CPLEX)

Summary of the solution of the $(\mathbf{x}, \mathbf{y})$ -problem

For a fixed value of the penalty λ

- ▶ Solve $|\mathcal{J}|$ continuous knapsack problems
- $\Rightarrow$ Solution $\bar{x}_{ij}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$,
- $\Rightarrow$ The value of opening depot j : $v_j(\lambda)$, $j \in \mathcal{J}$

- ▶ Solve a 0/1-knapsack problem
- $\Rightarrow y_j(\lambda) \in \{0, 1\}$, $j \in \mathcal{J}$
 - ▶ If $y_j(\lambda) = 0 \Rightarrow x_{ij}(\lambda) = 0$, $i \in \mathcal{I}$
 - ▶ If $y_j(\lambda) = 1 \Rightarrow x_{ij}(\lambda) = \bar{x}_{ij}$, $i \in \mathcal{I}$, (greedy)

- ▶ Solution $(\mathbf{x}(\lambda), \mathbf{y}(\lambda))$

The subproblem in w (for a fixed value of λ)

$|\mathcal{I}|$ semi-assignment problems (SAP)

$$q_w(\lambda) = \sum_{i \in \mathcal{I}} \left[\begin{array}{ll} \min_w & \sum_{j \in \mathcal{J}} [(1 - \alpha)c_{ij} + \lambda_{ij}] w_{ij} \\ \text{s.t.} & \sum_{j \in \mathcal{J}} w_{ij} = 1, \quad w_{ij} \geq 0, \quad j \in \mathcal{J} \end{array} \right]$$

Solving semi-assignment problem $i \in \mathcal{I}$

- ▶ $\ell_i := \arg \min_{j \in \mathcal{J}} \{(1 - \alpha)c_{ij} + \lambda_{ij}\}$
- ▶ $w_{i\ell_i}(\lambda) := 1, w_{ij}(\lambda) := 0, j \neq \ell_i$

The value of the relaxed problem for a fixed value of λ

$$q(\lambda) = \underbrace{q_{xy}(\lambda)}_{\text{difficult}} + \underbrace{q_w(\lambda)}_{\text{simple}}$$

- ▶ Can show that $q(\lambda) \leq z^*$ for all $\lambda \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{J}|}$ (weak duality)
- ▶ λ_{ij} is the penalty for violating the constraint $w_{ij} = x_{ij}$
- ▶ Find best *underestimate* of $z^* \iff$ find optimal values for the penalties λ_{ij}

- ▶ That is

$$q^* := \max_{\lambda \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{J}|}} q(\lambda) \leq z^*$$

- ▶ Most often

$$q^* < z^*$$

(not strong duality)

How to find better values for λ_{ij} ?

Penalty: $\min \dots + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \lambda_{ij} (w_{ij} - x_{ij})$

- ▶ If $w_{ij}(\lambda) > x_{ij}(\lambda) \Rightarrow$ Increase the value of λ_{ij} (higher penalty for violating the constraint)
- ▶ If $w_{ij}(\lambda) < x_{ij}(\lambda) \Rightarrow$ Decrease the value of λ_{ij} (higher penalty for violating the constraint)
- ▶ **Iterative method** (subgradient algorithm) to find optimal penalties λ^* ($\Rightarrow$ underestimate of $q^* \leq z^*$)

$$\lambda_{ij}^{t+1} = \lambda_{ij}^t + \rho_t [w_{ij}(\lambda^t) - x_{ij}(\lambda^t)], \quad t = 0, 1, \dots$$

where $\rho_t > 0$ is a step length, decreasing with t

- ▶ Use **feasibility heuristic** from every $[\mathbf{x}(\lambda^t), \mathbf{w}(\lambda^t), \mathbf{y}(\lambda^t)]$ to yield a **feasible solution** to CFL ($\Rightarrow$ overestimate of z^*)
- ▶ E.g., open more depots, send only from open depots,
 $\mathbf{x} := \mathbf{w}, \dots$

Example: $|\mathcal{I}| = 4$, $|\mathcal{J}| = 3$, $\alpha = \frac{1}{2}$

$$(c_{ij}) = \begin{pmatrix} 6 & 2 & 4 \\ 2 & 8 & 4 \\ 16 & 2 & 6 \\ 10 & 12 & 4 \end{pmatrix}, (f_j) = \begin{pmatrix} 11 \\ 16 \\ 21 \end{pmatrix}, (d_i) = \begin{pmatrix} 6 \\ 4 \\ 8 \\ 5 \end{pmatrix}, (b_j) = \begin{pmatrix} 12 \\ 10 \\ 13 \end{pmatrix}$$

The 0/1-knapsack problem

$$q_{xy}(\lambda) = \min \begin{array}{l} \sum_{j=1}^3 v_j(\lambda) \cdot y_j \\ \text{s.t. } 12y_1 + 10y_2 + 13y_3 \geq 23 \\ \mathbf{y} \in \{0, 1\}^3 \end{array} \quad \left| \quad \text{Let } (\lambda_{ij}^t) = \begin{pmatrix} 7 & 0 & 0 \\ 3 & 10 & 2 \\ 5 & 2 & 0 \\ 0 & 7 & 5 \end{pmatrix} \right.$$

Observe: $y_3 = 1$ must hold (why?)

$$\underbrace{\dots \Rightarrow \dots}_{\text{next page}} \quad q_{xy}(\lambda^t) = \min \begin{array}{l} 5y_1 + 8.875y_2 + 18y_3 \\ \text{s.t. } 12y_1 + 10y_2 + 13y_3 \geq 23, \quad \mathbf{y} \in \{0, 1\}^3 \end{array}$$

The value of opening a depot

$$\begin{aligned} v_1(\lambda^t) = 11 + \min \quad & -4x_{11} - 2x_{21} + 3x_{31} + 5x_{41} \\ \text{s.t.} \quad & 6x_{11} + 4x_{21} + 8x_{31} + 5x_{41} \leq 12, \quad \mathbf{x}_1 \in [0, 1]^4 \end{aligned}$$

$$\Rightarrow \boxed{\text{solution } \bar{x}_{11} = \bar{x}_{21} = 1, \bar{x}_{31} = \bar{x}_{41} = 0, \quad v_1(\lambda^t) = 5}$$

$$\begin{aligned} v_2(\lambda^t) = 16 + \min \quad & x_{12} - 6x_{22} - x_{32} - x_{42} \\ \text{s.t.} \quad & 6x_{12} + 4x_{22} + 8x_{32} + 5x_{42} \leq 10, \quad \mathbf{x}_2 \in [0, 1]^4 \end{aligned}$$

$$\Rightarrow \boxed{\text{solution } \bar{x}_{22} = \bar{x}_{42} = 1, \bar{x}_{32} = \frac{1}{8}, \bar{x}_{12} = 0, \quad v_2(\lambda^t) = 8.875}$$

$$\begin{aligned} v_3(\lambda^t) = 21 + \min \quad & 2x_{13} + 0x_{23} + 3x_{33} - 3x_{43} \\ \text{s.t.} \quad & 6x_{13} + 4x_{23} + 8x_{33} + 5x_{43} \leq 13, \quad \mathbf{x}_3 \in [0, 1]^4 \end{aligned}$$

$$\Rightarrow \boxed{\text{solution } \bar{x}_{23} = \bar{x}_{43} = 1, \bar{x}_{13} = \bar{x}_{33} = 0, \quad v_3(\lambda^t) = 18}$$

The solution to the (x, y) -problem for $\lambda = \lambda^t$

- ▶ Open depots: $\mathbf{y}(\lambda^t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
- ▶ Transport goods from open depots to customers (this solution does *not* fulfil the demand constraints (1) for each customer):

$$\mathbf{x}(\lambda^t) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- ▶ Objective value: $q_{\mathbf{xy}}(\lambda^t) = 5 + 0 + 18 = 23$

The w-problem separates into one problem for each customer i

$$q_w(\lambda^t) = \sum_{i=1}^4 q_w^i(\lambda^t)$$

where

$$\begin{aligned} q_w^i(\lambda^t) = \min_w \quad & \sum_{j=1}^3 [(1 - \alpha)c_{ij} + \lambda_{ij}^t] w_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^3 w_{ij} = 1, \quad w_{ij} \geq 0, \quad j = 1, 2, 3 \end{aligned}$$

and $1 - \alpha = \frac{1}{2}$

The solution to the w-problem

$$\begin{aligned} q_w^1(\lambda^t) = & \min 10w_{11} + w_{12} + 2w_{13} \\ \text{s.t. } & w_{11} + w_{12} + w_{13} = 1, \quad w_{1j} \geq 0, \quad j = 1, 2, 3 \end{aligned}$$

$$\Rightarrow \text{solution } w_{12}(\lambda^t) = 1, \quad w_{11}(\lambda^t) = w_{13}(\lambda^t) = 0, \quad q_w^1(\lambda^t) = 1$$

$$\begin{aligned} q_w^2(\lambda^t) = & \min 4w_{21} + 14w_{22} + 4w_{23} \\ \text{s.t. } & w_{21} + w_{22} + w_{23} = 1, \quad w_{2j} \geq 0, \quad j = 1, 2, 3 \end{aligned}$$

$$\Rightarrow \text{solution } w_{21}(\lambda^t) = 1, \quad w_{22}(\lambda^t) = w_{23}(\lambda^t) = 0, \quad q_w^2(\lambda^t) = 4$$

$$\begin{aligned} q_w^3(\lambda^t) = & \min 13w_{31} + 3w_{32} + 3w_{33} \\ \text{s.t. } & w_{31} + w_{32} + w_{33} = 1, \quad w_{3j} \geq 0, \quad j = 1, 2, 3 \end{aligned}$$

$$\Rightarrow \text{solution } w_{32}(\lambda^t) = w_{33}(\lambda^t) = \frac{1}{2}, \quad w_{31}(\lambda^t) = 0, \quad q_w^3(\lambda^t) = 3$$

$$\begin{aligned} q_w^4(\lambda^t) = & \min 5w_{41} + 13w_{42} + 7w_{43} \\ \text{s.t. } & w_{41} + w_{42} + w_{43} = 1, \quad w_{4j} \geq 0, \quad j = 1, 2, 3 \end{aligned}$$

$$\Rightarrow \text{solution } w_{41}(\lambda^t) = 1, \quad w_{42}(\lambda^t) = w_{43}(\lambda^t) = 0, \quad q_w^4(\lambda^t) = 5$$

The solution to the (x, y)- and w-problems

- Send the right amount of goods to each customer (this solution presumes that *all* depots are opened):

$$\mathbf{w}(\lambda^t) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

- Objective value: $q_{\mathbf{w}}(\lambda^t) = 13$
- Total objective value $q(\lambda^t) = q_{\mathbf{xy}}(\lambda^t) + q_{\mathbf{w}}(\lambda^t) = 35$
- Lower bound on the optimal objective value: $z^* \geq 35$

Compute a new λ -vector (here, the steplength $\rho_t = 8$)

$$\begin{aligned} \lambda^{t+1} &= \lambda^t + \rho_t [\mathbf{w}(\lambda^t) - \mathbf{x}(\lambda^t)] \\ &= \begin{pmatrix} 7 - \rho_t & \rho_t & 0 \\ 3 & 10 & 2 - \rho_t \\ 5 & 2 + \frac{\rho_t}{2} & \frac{\rho_t}{2} \\ \rho_t & 7 & 5 - \rho_t \end{pmatrix} = \begin{pmatrix} -1 & 8 & 0 \\ 3 & 10 & -6 \\ 5 & 6 & 4 \\ 8 & 7 & -3 \end{pmatrix} \end{aligned}$$

Feasible solution $\Leftrightarrow \mathbf{x}(\lambda^t) = \mathbf{w}(\lambda^t)$?

If not $\Rightarrow$ Feasibility heuristic

- ▶ Open the depots given by $\mathbf{y}(\lambda^t) \Rightarrow \mathbf{y}^H = \mathbf{y}(\lambda^t) = (1, 0, 1)^T$
- ▶ Transport goods only from opened depots:
 $y_j^H = 0 \Rightarrow x_{ij}^H = 0, i \in \mathcal{I}$
- ▶ Fulfill the demand but do not violate the capacity restrictions

$$\text{Let } \mathbf{x}^H = \begin{pmatrix} \frac{1}{6} & 0 & \frac{5}{6} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow z^H = 6 \cdot \frac{1}{6} + 4 \cdot \frac{5}{6} + 2 + 6 + 10 + 11 + 21 = 52 + \frac{1}{3}$$

$$\Rightarrow z^* \in [35, 52 + \frac{1}{3}] = [q(\lambda^t), z^H]$$

More about the solution method

- ▶ **Choice of step lengths** (ρ_t): Lecture 4 (subgradient optimization, convergence to an optimal value of λ)
- ▶ **Feasibility heuristics** can be made more or less sophisticated
- ▶ There are more ways to **Lagrangian relax continuous constraints** in an optimization problem
- ▶ E.g.: Lagrangian relax (1) or (5) (with multipliers $\mu_i \in \mathbb{R}$ and $\nu_j \in \mathbb{R}_+$, respectively) in the original formulation (CFL)

More solution methods for the CFL

- ▶ There are also other methods for solving CFL
- ▶ E.g., for a fixed value of $\mathbf{y}$, the remaining problem over $\mathbf{x}$ is simple (a transportation problem, network flow)
- ▶ An algorithm can be based on only adjusting $\mathbf{y}$, optimizing over $\mathbf{x}$ for each value of $\mathbf{y}$
- ▶ The problem is then **projected** onto the $\mathbf{y}$ variable space
- ▶ This is the **Benders' decomposition** (later in the course)