9.1.2 FEM for the heat equation

Consider 1D heat equation:

$$\begin{cases} \dot{u} - u'' = f, & 0<x<1,\ t>0 \\ u(0,t) = u(1,t) = 0 & t>0 \\ u(x,0) = u_0(x) & 0<x<1 \end{cases} \qquad (1)$$

We solve by cG(1)-cG(1), that is a FEM using continuous piecewise linear ~~...~~ polynomials, both in time and space variables.

* Consider a partition for time ~~variable~~ interval $[0,\infty)$:

$0 = t_0 < t_1 < \dots < t_{n-1} < t_n < \dots$

with subintervals $I_n = (t_{n-1}, t_n]$, and time mesh length $k_n = t_n - t_{n-1}$.

To write the VF for (1), multiply DE by a test function $v = v(x,t)$, with $v(0,t) = v(1,t) = 0$ (because of the Dirichlet B.C.) Then integrate over $(0,1)$ and I_n, and integrate by parts w.r.t the space variable x:

$$\int_{I_n}\int_0^1 \dot{u} v \, dx\, dt - \int_{I_n}\int_0^1 u'' v \, dx\, dt = \int_{I_n}\int_0^1 f v \, dx\, dt$$

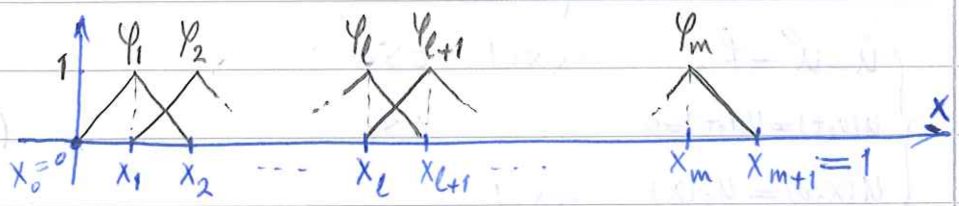
$$\xRightarrow[I.P]{} \int_{I_n}\int_0^1 \dot{u} v \, dx\, dt - \int_{I_n} \underbrace{[u'v]_{x=0}^{x=1}}_{= u'(1,t)\underbrace{v(1,t)}_{=0} - u'(0,t)\underbrace{v(0,t)}_{=0} = 0} dt + \int_{I_n}\int_0^1 u'v' \, dx\, dt = \int_{I_n}\int_0^1 f v \, dx\, dt$$

* So VF is:

$$\begin{cases} \text{Find } u = u(x,t) \text{ such that } u(0,t) = u(1,t) = 0 \text{ and} \\ \int_{I_n}\int_0^1 \dot{u} v \, dx\, dt + \int_{I_n}\int_0^1 u'v' \, dx\, dt = \int_{I_n}\int_0^1 f v \, dx\, dt, \quad \forall v,\ v(0,t) = v(1,t) = 0 \end{cases}$$

* Consider a partition for the space interval $[0,1] = \Omega$,

$$\mathcal{T}_h : 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$$

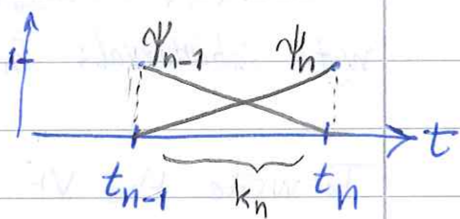


* cG(1)-cG(1) is:

Find $U(x,t) = U_{n-1}(x)\psi_{n-1}(t) + U_n(x)\psi_n(t)$, such that for $n = 1, 2, \ldots$

$$\int_{I_n}\int_0^1 \dot{U}\varphi_i \, dx\,dt + \int_{I_n}\int_0^1 U'\varphi_i' \, dx\,dt = \int_{I_n}\int_0^1 f\varphi_i \, dx\,dt, \quad i = 1, 2, \ldots, m$$

Where $\psi_{n-1}(t) = \dfrac{t - t_n}{-k_n}$, $\psi_n(t) = \dfrac{t - t_{n-1}}{k_n}$



and

$$U_{n-1}(x) = U_{n-1,1}\varphi_1(x) + U_{n-1,2}\varphi_2(x) + \cdots + U_{n-1,m}\varphi_m(x) = \sum_{j=1}^{m} U_{n-1,j}\varphi_j(x)$$

$$U_n(x) = U_{n,1}\varphi_1(x) + U_{n,2}\varphi_2(x) + \cdots + U_{n,m}\varphi_m(x) = \sum_{j=1}^{m} U_{n,j}\varphi_j(x)$$

Note: 1. $\varphi_i = \varphi_i(x)$

2. $\dot{U}(x,t) = U_{n-1}(x)\underbrace{\dot{\psi}_{n-1}(t)}_{= -1/k_n} + U_n(x)\underbrace{\dot{\psi}_n(t)}_{= 1/k_n} = \dfrac{U_n(x) - U_{n-1}(x)}{k_n}$

3. $U'(x,t) = U_{n-1}'(x)\psi_{n-1}(t) + U_n'(x)\psi_n(t)$

$U_n'(x) = \sum_{j=1}^{m} U_{n,j}\varphi_j'(x)$

4. We need to find m unknows $U_{n,1}, \ldots, U_{n,m}$.

5. $\int_{I_n} dt = k_n = |I_n|$

6. $\int_{I_n}\psi_{n-1}(t)\,dt = \int_{I_n}\psi_n(t)\,dt = \dfrac{k_n}{2} = \text{Area(triangle)}$

Hence

$$\int_{I_n}\int_0^1 \overbrace{\frac{U_n(x)-U_{n-1}(x)}{k_n}}^{=\text{not depend on } t}\varphi_i(x)\,dx\,dt + \int_{I_n}\int_0^1 U_{n-1}'(x)\,\psi_{n-1}(t)\,\varphi_i'(x)\,dx\,dt$$

$$+\int_{I_n}\int_0^1 U_n'(x)\,\psi_n(t)\,\varphi_i'(x)\,dx\,dt = \int_{I_n}\int_0^1 f\varphi_i\,dx\,dt \qquad i=1,\dots,m$$

Then

$$\overbrace{\int_0^1 U_n\varphi_i\,dx}^{=MU_n} - \overbrace{\int_0^1 U_{n-1}\varphi_i\,dx}^{=MU_{n-1}} + \overbrace{\int_{I_n}\psi_{n-1}\,dt}^{=\frac{k_n}{2}}\overbrace{\int_0^1 U_{n-1}'\varphi_i'\,dx}^{=SU_{n-1}}$$

$$+\underbrace{\int_{I_n}\psi_n\,dt}_{=\frac{k_n}{2}}\underbrace{\int_0^1 U_n'\varphi_i'\,dx}_{=SU_n} = \overbrace{\int_{I_n}\int_0^1 f\varphi_i\,dx\,dt}^{=F_n} \qquad i=1,\dots,m$$

$$\int_0^1\left(\sum_{j=1}^m U_{n,j}\varphi_j\right)\varphi_i\,dx = \sum_{j=1}^m\left(\int_0^1\varphi_j\varphi_i\,dx\right)U_{n,j}$$

Implies the recursive system of equation (in matrix form):

mass matrix   stiffness matrix

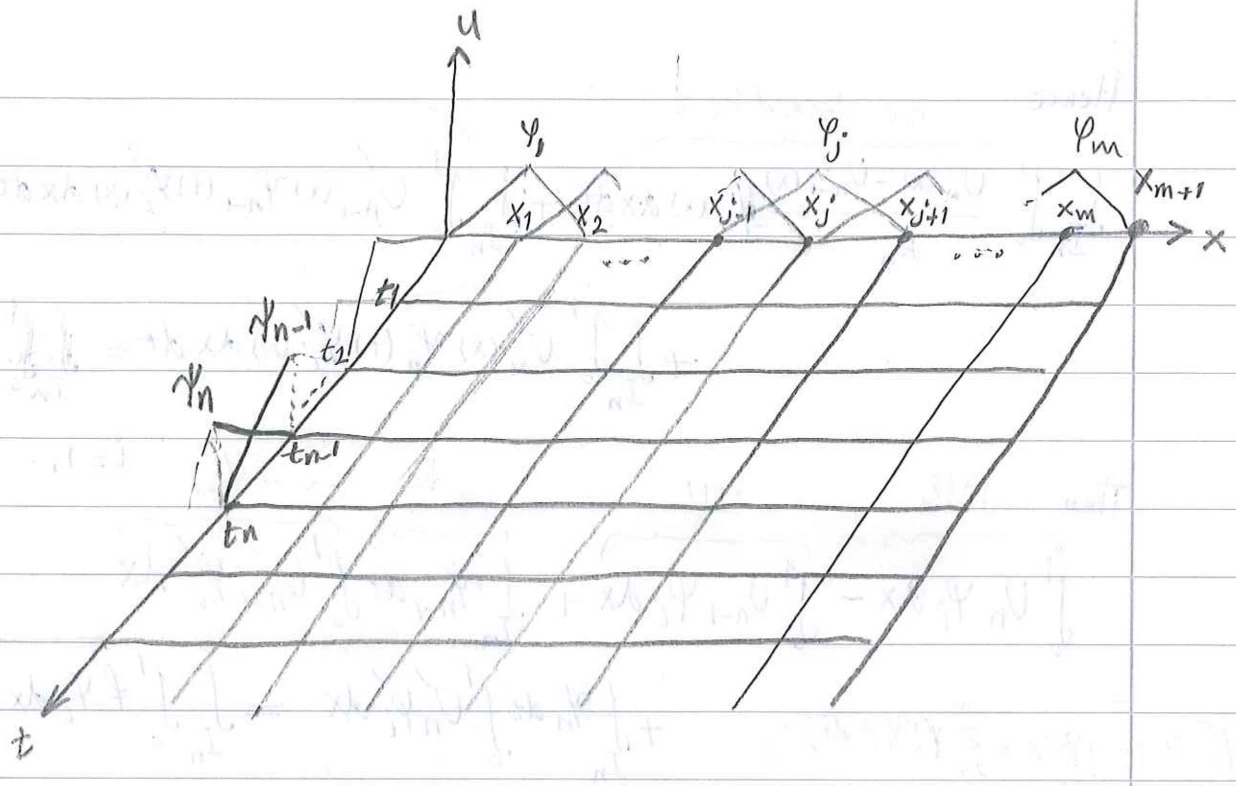
$$\left(M+\frac{k_n}{2}S\right)U_n = \left(M-\frac{k_n}{2}S\right)U_{n-1} + F_n \qquad (2)$$

Where
$$U_n = \begin{bmatrix} U_{n,1}\\ \vdots \\ \vdots \\ U_{n,m}\end{bmatrix}_{m\times 1}, \quad F_n = \begin{bmatrix} F_{n,1}\\ F_{n,2}\\ \vdots \\ F_{n,m}\end{bmatrix}_{m\times 1} = \begin{bmatrix} \int_{I_n}\int_0^1 f\varphi_1\,dx\,dt\\ \int_{I_n}\int_0^1 f\varphi_2\,dx\,dt\\ \vdots \\ \int_{I_n}\int_0^1 f\varphi_m\,dx\,dt\end{bmatrix}_{m\times 1}$$

Note: (2) is in fact the Crank-Nicolson method. So with initial data

$$U_0 = \begin{bmatrix} U_{0,1}\\ U_{0,2}\\ \vdots \\ U_{0,m}\end{bmatrix} = \begin{bmatrix} u_0(x_1)\\ u_0(x_2)\\ \vdots \\ u_0(x_m)\end{bmatrix}$$

$$U_n = \left(M+\frac{k_n}{2}S\right)^{-1}\left(M-\frac{k_n}{2}S\right)U_{n-1} + \left(M+\frac{k_n}{2}S\right)^{-1}F_n, \quad n=1,2,\dots$$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9.2 The wave equation in 1D

Consider $(u = u(x,t))$,

$$\begin{array}{ll} \text{DE} & \\ \text{BC} & \\ \text{IC} & \end{array}\begin{cases} \ddot{u} - u'' = 0, & 0<x<1,\ t>0 \\ u(0,t) = u(1,t) = 0 & t>0 \\ u(x,0) = u_0(x),\ \dot{u}(x,0) = v_0(x) & 0<x<1 \end{cases} \qquad (1)$$

Thm 9.5 We have (the conservation of energy):

$$\frac{1}{2}\|\dot{u}(\cdot,t)\|^2 + \frac{1}{2}\|u'(\cdot,t)\|^2 = \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\|u_0'\|^2 = \text{Constant}$$

Proof Multiply DE by $\dot{u}$ and integrate over $(0,1)$:

$$\int_0^1 \underbrace{\ddot{u}\,\dot{u}}_{=\frac{1}{2}\frac{d}{dt}(\dot{u})^2} dx - \int_0^1 u''\dot{u}\,dx = 0$$

$$\overset{\text{I.P.}}{\Longrightarrow} \frac{1}{2}\int_0^1 \frac{d}{dt}(\dot{u})^2 dx - \Big[u'(x,t)\dot{u}(x,t)\Big]_{x=0}^{x=1} + \int_0^1 \overbrace{u'\dot{u}'}^{\frac{1}{2}\frac{d}{dt}(u')^2} dx = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|\dot{u}\|^2 - \underset{?}{u'(1,t)}\underbrace{\dot{u}(1,t)}_{=0} + \underset{?}{u'(0,t)}\underbrace{\dot{u}(0,t)}_{=0} + \frac{1}{2}\frac{d}{dt}\|u'\|^2 = 0$$

$$\Longrightarrow \frac{1}{2}\frac{d}{dt}\|\dot{u}\|^2 + \frac{1}{2}\frac{d}{dt}\|u'\|^2 = 0$$

$$\overset{\int_0^t}{\Longrightarrow} \frac{1}{2}\|\dot{u}(t)\|^2 - \frac{1}{2}\|\underbrace{\dot{u}(0)}_{=v_0}\| + \frac{1}{2}\|u'(t)\|^2 - \frac{1}{2}\|u'(0)\|^2$$

$\int \frac{d}{dt}\square\, dt = \square$

$$\Longrightarrow \underbrace{\frac{1}{2}\|\dot{u}(t)\|^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}\|u'(t)\|^2}_{\text{potential energy}} = \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\|u_0'\|^2 = \text{Constant}$$

9.2.1 Wave equation as a system of first order eq.

Consider the (homogeneous) wave eq.:

$$\begin{cases} \ddot{u} - u'' = 0 & 0<x<1, \quad t>0 \\ u(0,t)=0, \; u'(1,t)=g(t) & t>0 \\ u(x,0)=u_0(x), \; \dot{u}(x,0)=v_0(x), & 0<x<1 \end{cases}$$

Change of variables $\dot{u} = v$, (Note that $\ddot{u} = \dot{v}$),

$$\begin{cases} \dot{u} - v = 0 \\ \dot{v} - u'' = 0 \end{cases} \Longrightarrow \begin{cases} \dot{u} + (0)u + (-1)v = 0 \\ \dot{v} + \left(-\frac{\partial^2}{\partial x^2}\right)u + (0)v = 0 \end{cases}$$

$$\Longrightarrow \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -1 \\ -\frac{\partial^2}{\partial x^2} & 0 \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{=w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \dot{w} + Aw = 0$$