9.2 The wave equation in 1D

Consider $(u = u(x,t))$,

$$\begin{cases} \text{DE} & \ddot{u} - u'' = 0, & 0<x<1, \ t>0 \\ \text{BC} & u(0,t) = u(1,t) = 0 & t>0 \\ \text{IC} & u(x,0) = u_0(x), \ \dot{u}(x,0) = v_0(x) & 0<x<1 \end{cases} \qquad (1)$$

Thm 9.5 We have (the conservation of energy):

$$\frac{1}{2}\|\dot{u}(\cdot,t)\|^2 + \frac{1}{2}\|u'(\cdot,t)\|^2 = \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\|u_0'\|^2 = \text{Constant}$$

Proof Multiply DE by $\dot{u}$ and integrate over $(0,1)$:

$$\int_0^1 \underbrace{\ddot{u}\,\dot{u}}_{=\frac{1}{2}\frac{d}{dt}(\dot{u})^2} dx - \int_0^1 u''\,\dot{u}\,dx = 0$$

$$\xRightarrow{\text{I.P.}} \frac{1}{2}\int_0^1 \frac{d}{dt}(\dot{u})^2 dx - \Big[u'(x,t)\,\dot{u}(x,t)\Big]_{x=0}^{x=1} + \int_0^1 \overbrace{u'\,\dot{u}'}^{\frac{1}{2}\frac{d}{dt}(u')^2} dx = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|\dot{u}\|^2 - \underset{?}{u'(1,t)}\underbrace{\dot{u}(1,t)}_{=0} + \underset{?}{u'(0,t)}\underbrace{\dot{u}(0,t)}_{=0} + \frac{1}{2}\frac{d}{dt}\|u'\|^2 = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|\dot{u}\|^2 + \frac{1}{2}\frac{d}{dt}\|u'\|^2 = 0$$

$$\xRightarrow{\int_0^t} \frac{1}{2}\|\dot{u}(t)\|^2 - \frac{1}{2}\|\underbrace{\dot{u}(0)}_{=v_0}\| + \frac{1}{2}\|u'(t)\|^2 - \frac{1}{2}\|u'(0)\|^2$$

$\int \frac{d}{dt}\square\, dt = \square$

$$\Rightarrow \underbrace{\frac{1}{2}\|\dot{u}(t)\|^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}\|u'(t)\|^2}_{\text{Potential energy}} = \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\|u_0'\|^2 = \text{Constant}$$

9.2.1 Wave equation as a system of First order eq.

Consider the (homogeneous) wave eq.:

$$\begin{cases} \ddot{u} - u'' = 0 & 0<x<1,\ t>0 \\ u(0,t)=0,\ u'(1,t)=g(t) & t>0 \\ u(x,0)=u_0(x),\ \dot{u}(x,0)=v_0(x), & 0<x<1 \end{cases}$$

Change of variables $\dot{u} = v$, (Note that $\ddot{u} = \dot{v}$),

$$\begin{cases} \dot{u} - v = 0 \\ \dot{v} - u'' = 0 \end{cases} \implies \begin{cases} \dot{u} + (0)u + (-1)v = 0 \\ \dot{v} + \left(-\frac{\partial^2}{\partial x^2}\right)u + (0)v = 0 \end{cases}$$

$$\implies \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -1 \\ -\frac{\partial^2}{\partial x^2} & 0 \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{=w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \dot{w} + Aw = 0$$

9.2.2 The FEM for the wave eq.

Solve the following wave equation by cG(1)-cG(1) method.

$$
\begin{cases}
\ddot{u} - u'' = 0, & 0<x<1,\ t>0 & \text{DE} \\
u(0,t)=0,\ u'(1,t)=g(t) & t>0 & \text{B.C.} \\
u(x,0)=u_0(x),\ \dot{u}(x,0)=v_0(x), & 0<x<1 & \text{I.C.}
\end{cases}
$$

(u_0 is initial displacement
v_0 〃 〃 velocity)

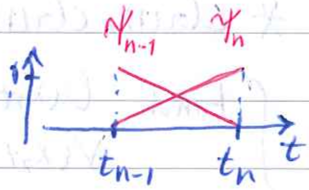
* Change of variable $\dot{u} = v$, ← velocity

$$
\begin{cases}
\dot{u} - v = 0 & 0<x<1,\ t>0 & \text{DE} \\
\dot{v} - u'' = 0 & 0<x<1,\ t>0 & \text{DE} \\
u(0,t)=0,\ u'(1,t)=g(t) & t>0 & \text{B.C.} \\
u(x,0)=u_0(x),\ v(x,0)=v_0(x) & 0<x<1 & \text{I.C.}
\end{cases}
$$

* Consider a partition for the time interval $[0,\infty)$,

$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots$

with $I_n = (t_{n-1}, t_n]$, $k_n = t_n - t_{n-1}$, $n = 1, 2, \ldots$



* VF: Multiply the first DE by a test function $\varphi = \varphi(x,t)$,
〃 〃 second 〃 〃 〃 〃 $\tilde{\varphi} = \tilde{\varphi}(x,t)$ with $\tilde{\varphi}(0,t) = 0$ (Dirichlet B.C.)

Then integrate over $(0,1)$ and I_n,

$$
\begin{cases}
\int_{I_n}\int_0^1 \dot{u}\varphi\,dx\,dt - \int_{I_n}\int_0^1 v\varphi\,dx\,dt = 0, & \forall\varphi \\
\int_{I_n}\int_0^1 \dot{v}\tilde{\varphi}\,dx\,dt - \int_{I_n}\int_0^1 u''\tilde{\varphi}\,dx\,dt = 0, & \forall\tilde{\varphi},\ \tilde{\varphi}(0,t)=0
\end{cases}
$$

Now, integrate by parts in the second eq.

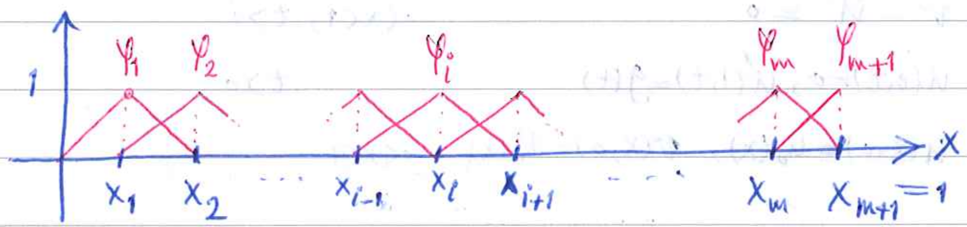
$$
\begin{cases}
\int_{I_n}\int_0^1 \dot{u}\varphi\,dx\,dt - \int_{I_n}\int_0^1 v\varphi\,dx\,dt = 0, & \forall\varphi \\
\int_{I_n}\int_0^1 \dot{v}\tilde{\varphi}\,dx\,dt - \int_{I_n}\underbrace{\left[u'\tilde{\varphi}\right]_{x=0}^{x=1}}_{=\underbrace{u'(1,t)}_{=g(t)}\tilde{\varphi}(1,t) - u'(0,t)\underbrace{\tilde{\varphi}(0,t)}_{=0}} dt + \int_{I_n}\int_0^1 u'\tilde{\varphi}'\,dx\,dt = 0, & \forall\tilde{\varphi},\ \tilde{\varphi}(0,t)=0
\end{cases}
$$

So VF is:

$$\begin{cases} \text{Find } u=u(x,t) \text{ and } v=v(x,t), \text{ such that } u(0,t)=0, \text{ and} \\ \int_{I_n}\int_0^1 \dot{u}\varphi\,dx\,dt - \int_{I_n}\int_0^1 v\varphi\,dx\,dt = 0, \quad \forall\varphi \\ \int_{I_n}\int_0^1 \dot{v}\tilde{\varphi}\,dx\,dt + \int_{I_n}\int_0^1 u'\tilde{\varphi}'\,dx\,dt = \int_{I_n} g(t)\tilde{\varphi}(1,t)\,dt; \quad \forall\tilde{\varphi},\ \tilde{\varphi}(0,t)=0 \end{cases}$$

* Consider a partition for the space domain $(0,1)=\Omega$,

$\mathcal{T}_h: \ 0=x_0<x_1< \cdots <x_m<x_{m+1}=1$



* cG(1)-cG(1) is:

$$\begin{cases} \text{Find } U(x,t)=U_{n-1}(x)\psi_{n-1}(t)+U_n(x)\psi_n(t) \\ \qquad V(x,t)=V_{n-1}(x)\psi_{n-1}(t)+V_n(x)\psi_n(t), \text{ such that for } n=1,2,\dots \\ \int_{I_n}\int_0^1 \dot{U}\varphi_i\,dx\,dt - \int_{I_n}\int_0^1 V\varphi_i\,dx\,dt = 0, \quad i=1,2,\dots,m+1 \\ \int_{I_n}\int_0^1 \dot{V}\varphi_i\,dx\,dt + \int_{I_n}\int_0^1 U'\varphi_i'\,dx\,dt = \int_{I_n} g(t)\,\varphi_i(\underset{\overset{\|}{x_{m+1}}}{1})\,dt, \quad i=1,2,\dots,m+1 \end{cases}$$

Where $\psi_{n-1}(t)=\dfrac{t-t_n}{-k_n}$, $\quad \psi_n(t)=\dfrac{t-t_{n-1}}{k_n}$

and

$$U_{\tilde{n}}(x)=\sum_{j=1}^{m+1} U_{\tilde{n},j}\,\varphi_j(x), \quad \tilde{n}=n-1 \text{ or } \tilde{n}=n$$

$$V_{\tilde{n}}(x)=\sum_{j=1}^{m+1} V_{\tilde{n},j}\,\varphi_j(x), \quad \tilde{n}=n-1 \text{ or } \tilde{n}=n$$

Hence

$$\left\{\begin{aligned} &\int_{I_n}\overbrace{\int_0^1 \frac{U_n(x)-U_{n-1}(x)}{k_n}\varphi_i(x)\,dx}^{\text{independent of } t}\,dt - \int_{I_n}\int_0^1 V_{n-1}(x)\psi_{n-1}(t)\varphi_i(x)\,dx\,dt \\ &\qquad - \int_{I_n}\int_0^1 V_n(x)\psi_n(t)\varphi_i(x)\,dx\,dt = 0, \quad i=1,\dots,m+1 \\ &\int_{I_n}\overbrace{\int_0^1 \frac{V_n(x)-V_{n-1}(x)}{k_n}\varphi_i(x)\,dx}^{\text{independent of } t}\,dt + \int_{I_n}\int_0^1 U_{n-1}'(x)\psi_{n-1}(t)\varphi_i'(x)\,dx\,dt \\ &\qquad + \int_{I_n}\int_0^1 U_n'(x)\psi_n(t)\varphi_i'(x)\,dx\,dt = \int_{I_n} g(t)\,\varphi_i(1)\,dt, \quad i=1,\dots,m+1 \end{aligned}\right.$$

$$\left\{\begin{aligned} &\underbrace{\int_0^1 U_n(x)\varphi_i(x)\,dx}_{\tilde{M}U_n} - \underbrace{\int_0^1 U_{n-1}(x)\varphi_i(x)\,dx}_{\tilde{M}U_{n-1}} - \frac{k_n}{2}\overbrace{\int_0^1 V_{n-1}(x)\varphi_i(x)\,dx}^{\tilde{M}V_{n-1}} \\ &\qquad - \frac{k_n}{2}\underbrace{\int_0^1 V_n(x)\varphi_i(x)\,dx}_{\tilde{M}V_n} = 0, \quad i=1,\dots,m+1 \\ &\underbrace{\int_0^1 V_n(x)\varphi_i(x)\,dx}_{\tilde{M}V_n} - \underbrace{\int_0^1 V_{n-1}(x)\varphi_i(x)\,dx}_{\tilde{M}V_{n-1}} + \frac{k_n}{2}\underbrace{\int_0^1 U_{n-1}'(x)\varphi_i'(x)\,dx}_{\tilde{S}U_{n-1}} \\ &\qquad + \frac{k_n}{2}\underbrace{\int_0^1 U_n'(x)\varphi_i'(x)\,dx}_{\tilde{S}U_n} = \underbrace{\int_{I_n} g(t)\,dt\,\varphi_i(\overset{x_{m+1}}{\overset{\parallel}{1}})}_{g_n}, \quad i=1,\dots,m+1 \end{aligned}\right.$$

Hence

$$\left\{\begin{aligned} &\tilde{M}U_n - \frac{k_n}{2}\tilde{M}V_n = \tilde{M}U_{n-1} + \frac{k_n}{2}\tilde{M}V_{n-1} \\ &\frac{k_n}{2}\tilde{S}U_n + \tilde{M}V_n = -\frac{k_n}{2}\tilde{S}U_{n-1} + \tilde{M}V_{n-1} + g_n \end{aligned}\right.$$

This is a system of $2(m+1)$ equations with $2(m+1)$ unknowns $U_{n,1},\dots,U_{n,m+1}$ and $V_{n,1},\dots,V_{n,m+1}$.

$$\begin{bmatrix} \tilde{M} & -\frac{k_n}{2}\tilde{M} \\ \frac{k_n}{2}\tilde{S} & \tilde{M} \end{bmatrix}\begin{bmatrix} U_n \\ V_n \end{bmatrix} = \begin{bmatrix} \tilde{M} & \frac{k_n}{2}\tilde{M} \\ -\frac{k_n}{2}\tilde{S} & \tilde{M} \end{bmatrix}\begin{bmatrix} U_{n-1} \\ V_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ g_n \end{bmatrix}, \quad n=1,2,\dots$$

with initial value:

$$U_0 = \begin{bmatrix} U_{0,1} \\ \vdots \\ U_{0,m+1} \end{bmatrix} = \begin{bmatrix} u_0(x_1) \\ \vdots \\ u_0(x_{m+1}) \end{bmatrix}_{(m+1)\times 1}, \quad V_0 = \begin{bmatrix} V_{0,1} \\ \vdots \\ V_{0,m+1} \end{bmatrix} = \begin{bmatrix} v_0(x_1) \\ \vdots \\ v_0(x_{m+1}) \end{bmatrix}_{(m+1)\times 1}$$

and

$$g_n = \begin{bmatrix} g_{n,1} \\ \vdots \\ g_{n,m} \\ g_{n,m+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \int_{I_n} g(t)\,dt \end{bmatrix}$$

Since for $i = 1, \dots, m$

$$g_{n,i} = \int_{I_n} g(t)\,dt \, \underbrace{\varphi_i(x_{m+1})}_{=0}$$

Note: with uniform (space) mesh

$$\tilde{S} = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, \quad \tilde{M} = \frac{h}{6} \begin{bmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & & 4 & 1 \\ 0 & \cdots & 0 & 1 & 2 \end{bmatrix}$$