Review : FEM

Consider the BVP

$$\begin{cases} -u'' = f, & 0<x<1 \\ u(0)=0,\ u(1)=0 \end{cases}$$

$\xleftarrow[a=1]{} \quad -(au')' = f$

VF: $V^0 = \{v \mid v, v' \in L_2(0,1),\ v(0)=v(1)=0\} = H_0^1$

$$\begin{cases} \text{Find } u \in V^0 \text{ s.t.} \\ \int_0^1 u'v'\,dx = \int_0^1 f v\,dx, \quad \forall v \in V^0 \end{cases}$$

FEM: $\mathcal{T}_h:\ 0 = x_0 < x_1 < \dots < x_m < x_{m+1} = 1$

$$V_h^0 = \{v \mid v \text{ is cont. p.w. linear on } \mathcal{T}_h,\ v(0)=v(1)=0\} \underset{= \operatorname{Span}\{\varphi_1,\dots,\varphi_m\}}{\overset{\text{subspace}}{\subset}} V^0$$

$$\begin{cases} \text{Find } U \in V_h^0 \text{ s.t.} \\ \int_0^1 U'\chi'\,dx = \int_0^1 f\chi\,dx, \quad \forall \chi \in V_h^0 \end{cases} \xrightarrow{\text{linear system}} S\xi = b$$

$$U(x) = \sum_{j=1}^{m} \xi_j \varphi_j(x)$$

Galerkin orthogonality: $\int_0^1 (u' - U')\chi'\,dx = 0, \ \forall \chi \in V_h^0$

Error estimates

L_2- inner product : $(u,v) = \int_0^1 uv\,dx$

L_2- norm : $\|u\| = \sqrt{(u,u)} = \left(\int_0^1 (u(x))^2\,dx\right)^{1/2}$

Energy-inner product: $(u,v)_E = \int_0^1 u'v'\,dx$

Energy-norm : $\|u\|_E = \sqrt{(u,u)_E} = \left(\int_0^1 (u')^2\,dx\right)^{1/2}$

Note: $\|u\|_E = \|u'\|$

Thm (Thm 7.1 & 7.2)

(a) $\|u-U\|_E \leq \|u-\chi\|_E, \quad \forall \chi \in V_h^0$

(b) $\|u-U\|_E \leq c_i \|hu''\|$

Proof For any $\chi \in V_h^0$,

(a) $$\|u-U\|_E^2 = \int_0^1 (u'-U')^2 dx = \int_0^1 (u'-U')(u'-U')dx$$

$$= \int_0^1 (u'-U')(\underbrace{u'-\chi'}+\underbrace{\chi'-U'})dx$$

$$= \int_0^1 (u'-U')(u'-\chi')dx + \underbrace{\int_0^1 (u'-U')(\underbrace{\chi-U}_{\in V_h^0})' dx}_{=0}$$

$$\leq \sqrt{\int_0^1 (u'-U')^2 dx} \sqrt{\int_0^1 (u'-\chi')^2 dx}$$

$$= \|u-U\|_E \|u-\chi\|_E$$

$$\Rightarrow \|u-U\|_E^{\cancel{2}} \leq \cancel{\|u-U\|_E} \|u-\chi\|_E \Rightarrow \|u-U\|_E \leq \|u-\chi\|_E.$$

(b) Set $\chi = \pi_h u$

$$\|u-U\|_E \leq \|u-\pi_h u\|_E = \|(u-\pi_h u)'\| \overset{\text{Thm 5.3}}{\leq} c_i \|hu''\|. \quad \square$$

Other boundary conditions

Consider the BVP

$$\begin{cases} -u'' + u = f, & 0<x<1 \\ u'(0)=0, \ u(1)=3 \end{cases}$$

VF: $V = \{v \mid v, v' \in L_2, \ v(1)=3\}$

$\tilde{V} = \{v \mid v, v' \in L_2, \ v(1)=0\}$

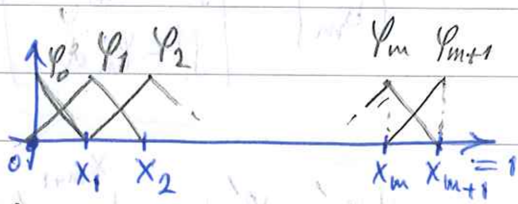
$$-\int_0^1 u''v\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx$$

$$\xRightarrow[\text{I.P.}]{v(1)=0} -\underbrace{u'(1)v(1)}_{=0} + \underbrace{u'(0)}_{=0}v(0) + \int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx$$

VF is:

$$\begin{cases} \text{Find } u \in V \text{ s.t.} \\ \int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx, \quad \forall v \in \tilde{V} \end{cases}$$

FEM: $\mathcal{T}_h : 0 = x_0 < x_1 < \dots < x_m < x_{m+1} = 1$



$V_h = \{v \mid v \text{ is cont. p.w. linear on } \mathcal{T}_h, \ v(1)=3\}$

$= \text{Span}\{\varphi_0, \varphi_1, \dots, \varphi_m, \varphi_{m+1}\} \Rightarrow \dim(V_h) = m+2$

$\tilde{V}_h = \{v \mid v \text{ is cont. p.w. linear on } \mathcal{T}_h, \ v(1)=0\} = \text{Span}\{\varphi_0, \varphi_1, \dots, \varphi_m\}$

$\Rightarrow \dim(\tilde{V}_h) = m+1$

$$\begin{cases} \text{Find } U \in V_h \text{ s.t.} \\ \int_0^1 U'\chi'\,dx + \int_0^1 U\chi\,dx = \int_0^1 f\chi\,dx, \quad \forall \chi \in \tilde{V}_h \end{cases}$$

$$U \in V_h \Rightarrow U(x) = \xi_0\varphi_0(x) + \xi_1\varphi_1(x) + \dots + \xi_m\varphi_m(x) + \xi_{m+1}\varphi_{m+1}(x)$$

$$\xRightarrow[U(1)=U(x_{m+1})=3]{} 3 = U(x_{m+1}) = 0 + \dots + 0 + \xi_{m+1}$$

$$\Rightarrow U(x) = \sum_{j=0}^{m} \xi_j\varphi_j(x) + 3\varphi_{m+1}(x)$$

Then take $\chi=\varphi_i, i=0,1,\dots,m$ we have

$$\int_0^1\left(\sum_{j=0}^m \xi_j\varphi_j'(x)+3\varphi_{m+1}'(x)\right)\varphi_i'(x)\,dx+\int_0^1\left(\sum_{j=0}^m \xi_j\varphi_j(x)+3\varphi_{m+1}(x)\right)\varphi_i(x)\,dx$$
$$=\int_0^1 f\varphi_i\,dx,\quad i=0,1,\dots,m$$

$$\sum_{j=0}^m\left(\int_0^1\varphi_j'\varphi_i'\,dx\right)\xi_j+\sum_{j=0}^m\left(\int_0^1\varphi_j\varphi_i\,dx\right)\xi_j=\int_0^1 f\varphi_i\,dx-3\int_0^1\varphi_{m+1}'\varphi_i'\,dx-3\int_0^1\varphi_{m+1}\varphi_i\,dx,$$
$$i=0,1,\dots,m$$

$$\Rightarrow\quad S\xi+M\xi=b$$

Where, with uniform mesh,

$$S=\frac{1}{h}\begin{bmatrix}1&-1&&&\\-1&2&-1&&\\&\ddots&\ddots&\ddots&\\&&&&-1\\&&&-1&2\end{bmatrix}_{(m+1)\times(m+1)},\qquad M=\frac{h}{6}\begin{bmatrix}2&1&&&\\1&4&1&&\\&\ddots&\ddots&\ddots&\\&&&&1\\&&&1&4\end{bmatrix}_{(m+1)\times(m+1)}$$

$$b=\begin{bmatrix}b_0\\b_1\\\vdots\\b_{m-1}\\b_m\end{bmatrix}=\begin{bmatrix}\int_0^1 f\varphi_0\,dx\\\int_0^1 f\varphi_1\,dx\\\vdots\\\int_0^1 f\varphi_{m-1}\,dx\\\int_0^1 f\varphi_m\,dx\end{bmatrix}-3\begin{bmatrix}0\\0\\\vdots\\0\\-\frac{1}{h}\end{bmatrix}-3\begin{bmatrix}0\\0\\\vdots\\0\\-\frac{h}{6}\end{bmatrix}$$

$$\int_0^1\varphi_{m+1}'\varphi_m'\,dx=\int_{x_m}^{x_{m+1}}\left(\frac{1}{h}\right)\left(-\frac{1}{h}\right)dx=-\frac{1}{h}$$

$$\int_0^1\varphi_{m+1}\varphi_m\,dx=\int_{x_m}^{x_{m+1}}\left(\frac{x-x_m}{h}\right)\left(\frac{x-x_{m+1}}{-h}\right)dx=\frac{-1}{h^2}\int_0^h t(t-h)\,dt=-\frac{h}{6}$$