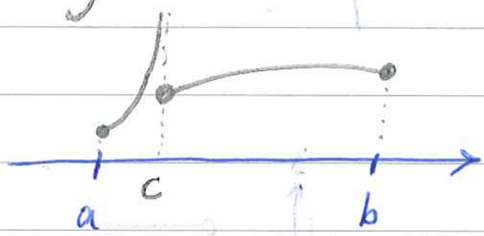
Chapter 1 : Laplace Transform

1.1 The Laplace Transform

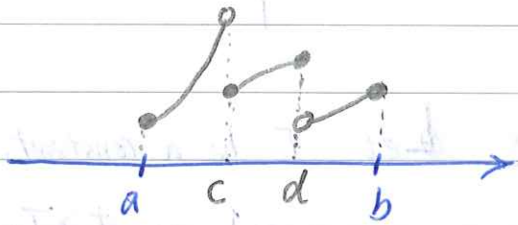
Preliminaries:

Def: A function $y=f(t)$, $t\in[a,b]$ is piecewise continuous, if the number of discontinuous point are finite, and its left and right limits exist at discontinuity point.

Note: Any continuous function is p.w. continuous



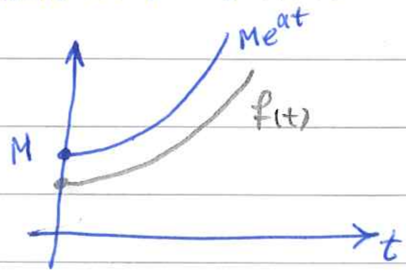
Not p.w. Cont. $\left(\lim\limits_{t\to c^-} f(t)=\infty\right)$



P.W. Cont.

Def A function $y=f(t)$, $t\geq 0$ is of exponential order if, there exist constants M, a such that:

$$|f(t)|\leq Me^{at}, \quad t\geq 0$$



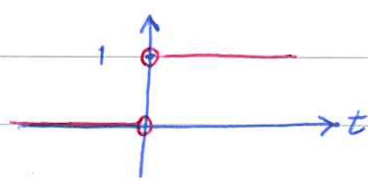
Ex. $f(t)=e^{ct}$, $t\geq 0$ (c is constant)

$f(t)=\sin x$, $t\geq 0$, $f(t)=t^3$, $t\geq 0$, $f(t)=t^{1000}$, $t\geq 0$

Def A function $y=f(t)$ is causal (one-sided) if $f(t)=0$ for $t<0$.

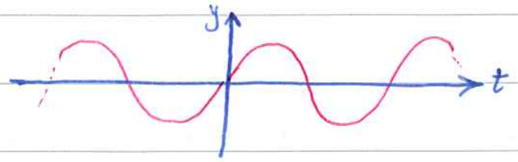
Ex The Heaviside function

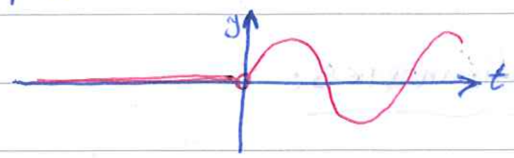
$$\theta(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$$



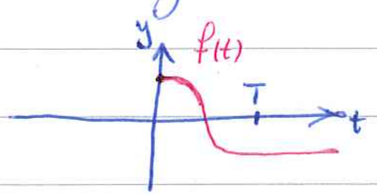
Ex For $f(t) = \sin t$, what is $\theta(t) f(t)$?

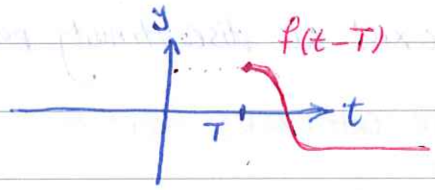
$$\theta(t) f(t) = \begin{cases} f(t) & t>0 \\ 0 & t<0 \end{cases} = \begin{cases} \sin t & t>0 \\ 0 & t<0 \end{cases}$$





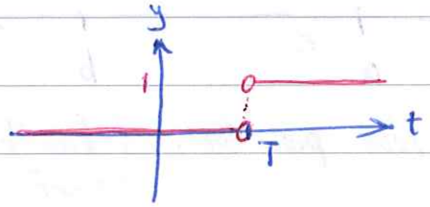
Recall: Shifting a function (independent variable):





Ex Let T be a constant. Then

$$\theta(t-T) = \begin{cases} 1 & t>T \\ 0 & t<T \end{cases}$$



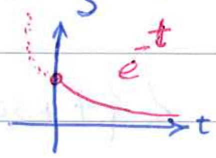
<u>Def</u>: The Laplace transform of $y = f(t)$, $t \geq 0$ is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

if the integral converges

<u>Ex</u> $\mathcal{L}\{\theta(t)\} = \Theta(s) = \int_0^\infty \theta(t) e^{-st}\, dt = \left[-\frac{1}{s} e^{-st}\right]_{t=0}^{\infty} = -\frac{1}{s}\left[e^{-st}\right]_{t=0}^{\infty}$

$$= -\frac{1}{s}\left(\lim_{t\to\infty} e^{-st} - e^{0}\right) \overset{\text{if } s>0}{=\!=\!=} -\frac{1}{s}(0-1) = \frac{1}{s},$$



So $\mathcal{L}\{\theta(t)\} = \frac{1}{s}$, $s > 0$

<u>Ex</u> $\mathcal{L}\{e^{ct}\} = ?$ (c is constant) (by default we assume $t \geq 0$)

$$\mathcal{L}\{e^{ct}\} = \int_0^\infty e^{ct} e^{-st}\, dt = \int_0^\infty e^{(c-s)t}\, dt = \left[\frac{1}{c-s} e^{(c-s)t}\right]_{t=0}^{\infty}$$

$$= \frac{1}{c-s}\left[\lim_{t\to\infty} e^{(c-s)t} - e^{0}\right] \overset{c-s<0}{=\!=\!=} \frac{1}{c-s}[0-1] = \frac{1}{s-c}$$

$\Rightarrow$ $\mathcal{L}\{e^{ct}\} = \frac{1}{s-c}$, $s > c$

<u>Thm</u>: Assume $f(t)$, $t \geq 0$ is p.w. continuous and it is of exponential order. Then its Laplace tranform $\mathcal{L}\{f(t)\}$ exists.

<u>Proof</u>. Let M, a be constants such that

$$|f(t)| \leq M e^{at}, \quad t \geq 0.$$

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt \leq \int_0^\infty |f(t)| e^{-st}\, dt \leq M \int_0^\infty e^{at} e^{-st}\, dt$$

$$= M\left[\frac{1}{a-s} e^{(a-s)t}\right]_{t=0}^{\infty} \overset{a-s<0}{=\!=\!=} \frac{M}{a-s}(0-1) = \frac{M}{s-a} < \infty. \quad \square$$

General Properties of the Laplace Transform

Thm 2 Laplace tranform is a linear operator: (a, b are constant)

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

Proof. $\mathcal{L}\{af(t)+bg(t)\} = \int_0^1 \{af(t)+bg(t)\}e^{-st}\,dt$

$$= a\int_0^1 f(t)e^{-st}\,dt + b\int_0^1 g(t)e^{-st}\,dt = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}. \quad \square$$

Ex $\mathcal{L}\{4+3e^{-2t}\} = 4\mathcal{L}\{1\} + 3\mathcal{L}\{e^{-2t}\} = \frac{4}{s} + 3\frac{1}{s-(-2)} = \frac{4}{s} + \frac{3}{s+2}$

Ex 3 $\mathcal{L}\{\sinh t\} = ?$, $\mathcal{L}\{\sinh at\} = ?$

$$\mathcal{L}\{\sinh t\} = \mathcal{L}\left\{\frac{e^{t}-e^{-t}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{t}\} - \frac{1}{2}\mathcal{L}(e^{-t}) = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$= \frac{1}{s^2-1}$$

$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \cdots = \frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a} = \frac{a}{s^2-a^2}$$

Ex 4 $\mathcal{L}\{\sin \omega t\} = ?$, $\mathcal{L}\{\cos \omega t\} = ?$

(Recall: $e^{i\alpha} = \cos\alpha + i\sin\alpha$)

$$\mathcal{L}\{e^{i\omega t}\} = \frac{1}{s-i\omega} = \frac{1}{s-i\omega}\,\frac{s+i\omega}{s+i\omega} = \frac{s+i\omega}{s^2-(i\omega)^2} \overset{i^2=-1}{=} \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}$$

also

$$\mathcal{L}\{e^{i\omega t}\} = \mathcal{L}\{\cos\omega t + i\sin\omega t\} = \mathcal{L}\{\cos\omega t\} + i\mathcal{L}\{\sin\omega t\}$$

Hence

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}, \quad \mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$$

Thm 3 (Shifting rule) $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$, for $s>c$.

Proof $\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{ct}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(s-c)t}\,dt = F(s-c)$. $\square$

Ex5 $\mathcal{L}\{3e^{-2t}\cos 5t\} = 3\mathcal{L}\{e^{-2t}\cos 5t\}$

Since $\mathcal{L}\{\cos 5t\} = \dfrac{s}{s^2+5^2}$

So
$$\mathcal{L}\{3e^{-2t}\cos 5t\} = 3\,\frac{s-(-2)}{(s-(-2))^2+25} = \frac{3(s+2)}{(s+2)^2+25}$$

Thm 4 (Shifting rule) If $f(t-T)$ is zero for $t\leq T$, then
$$\mathcal{L}\{f(t-T)\} = e^{-Ts}F(s), \quad \text{for } s>0.$$

Proof
$$\mathcal{L}\{f(t-T)\} = \int_0^\infty f(t-T)e^{-st}\,dt = \int_T^\infty f(t-T)e^{-st}\,dt$$
$$= \left\{\begin{array}{l}\tau = t-T\\ d\tau = dt\end{array}\right\} = \int_0^\infty f(\tau)e^{-s(\tau+T)}\,d\tau$$
$$= e^{-sT}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau, \quad \text{for } s>0. \ \square$$

Note: For a given $f(t-T)$: $\mathcal{L}\{\theta(t-T)f(t-T)\} = e^{-Ts}F(s)$.

Ex $f(t) = \sin t\cdot\theta(t)$
$$\mathcal{L}\{f(t-5)\} = \mathcal{L}\{\theta(t-5)\sin(t-5)\} = e^{-5s}\frac{1}{s^2+1}.$$
$\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$

Thm 5 (Derivative of Laplace transform):

$$\mathcal{L}\{t f(t)\} = -F'(s)$$

Proof $F'(s) = \frac{d}{ds}\int_0^\infty f(t) e^{-st} dt \xrightarrow[\text{independent}]{s \text{ and } t \text{ are}} \int_0^\infty f(t) \underbrace{\frac{\partial}{\partial s} e^{-st}}_{= -te^{-st}} dt$

$$= -\int_0^\infty \underbrace{t f(t) e^{-st}} dt = -\mathcal{L}\{t f(t)\}. \square$$

Note: $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$

Ex 6 $\mathcal{L}\{t \sinh t\} = ?$

$$\mathcal{L}\{\sinh t\} = \frac{1}{s^2-1} \Rightarrow \mathcal{L}\{t \sinh t\} = -\left(\frac{1}{s^2-1}\right)' = \frac{2s}{(s^2-1)^2}$$

Ex $\mathcal{L}\{te^{2t}\} = ?$

$$\mathcal{L}\{te^{2t}\} \overset{\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}}{=} \overset{\text{Thm 5}}{=} -\left(\frac{1}{s-2}\right)' = \frac{1}{(s-2)^2}$$

$$\mathcal{L}\{te^{2t}\} \underset{\mathcal{L}\{t\} = \frac{1}{s^2}}{=} \underset{\text{Thm 3}}{=} \frac{1}{(s-2)^2}$$

Thm 6 (Integral of Laplace transform) Let $\lim_{t \to 0} \frac{f(t)}{t} < \infty$. Then

$$\mathcal{L}\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty F(w)\, dw$$

Proof. $\int_s^\infty F(w)\,dw = \int_s^\infty \int_0^\infty f(t) e^{-wt} dt\, dw = \int_0^\infty \int_s^\infty f(t) e^{-wt} dw\, dt$

$$= \int_0^\infty f(t) \left(-\frac{1}{t} e^{-wt}\right]_{w=s}^{\infty} dt = \int_0^\infty f(t) \left(-\frac{1}{t}\right)(0 - e^{-st}) dt$$

$$= \int_0^\infty \frac{1}{t} f(t) e^{-st} dt = \mathcal{L}\left\{\frac{1}{t} f(t)\right\}. \square$$

Ex 7 $\mathcal{L}\{\frac{\sin t}{t}\} = ?$. $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$, $\lim_{t\to 0} \frac{\sin t}{t} = 1$

$$\mathcal{L}\{\tfrac{1}{t}\sin t\} = \int_s^\infty \frac{dw}{w^2+1} = tg^{-1}w\Big]_{w=s}^\infty = \frac{\pi}{2} - tg^{-1}s.$$

Note: $\int_0^\infty \frac{\sin t}{t}\,dt = \int_0^\infty \frac{\sin t}{t} e^{-(0)t}\,dt = \mathcal{L}\{\frac{\sin t}{t}\}\Big|_{s=0}$

$$= \frac{\pi}{2} - tg^{-1}(0) = \frac{\pi}{2}$$

Thm 7 (Laplace transform of derivative) Let f and f' be continuous and of exponential order. Then

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Proof $\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,dt \overset{I.P}{=\!=\!=}$ $\left\{ \begin{array}{ll} u = e^{-st} & \rightarrow du = -se^{-st}dt \\ dv = f'(t)dt & v = f \end{array} \right\}$

$$= f(t)e^{-st}\Big]_{t=0}^\infty - \int_0^\infty f(t)(-s)e^{-st}\,dt$$

$$= (0 - f(0)e^0) + s\int_0^\infty f(t)e^{-st}\,dt = s\mathcal{L}\{f(t)\} - f(0). \quad \square$$

Note: $\mathcal{L}\{f''\} = \mathcal{L}\{(f')'\} = s\mathcal{L}\{f'\} - f'(0) = s(s\mathcal{L}\{f\} - f(0)) - f'(0)$

$$= s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

Thm 8 (Laplace transform of integral) $\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}.$

Proof Set $h(t) = \int_0^t f(\tau)d\tau$. Then $h'(t) = f(t)$ and $h(0) = 0$.

$$\mathcal{L}\{h'\} = \mathcal{L}\{f\} = s\mathcal{L}\{h\} - \underbrace{h(0)}_{=0} = s\mathcal{L}\{\int_0^t f(\tau)d\tau\} \Rightarrow \mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}.$$

$\square$

1.2 The Inverse Laplace Transform

$$\mathcal{L}\{f(t)\} = F(s)$$
$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Note: $\mathcal{L}^{-1}$ is a linear operator. That is, for constants a, b:

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}$$
$$= af(t) + bg(t).$$

Ex $\mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{3}{s^2} + \frac{4s}{s^2+1}\right\} = ?$

$$? = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$
$$= 2\theta(t) - 3t + 4\cos t$$

Ex 8 $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^4}\right\} = ?$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$, $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{3!}t^3 = \frac{1}{6}t^3$

$$\Rightarrow ? = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^4}\right\} = \theta(t-1)(t-1) - \frac{1}{6}\theta(t-2)(t-2)^3.$$

1.2.1 Method of Partial Fractions

Consider $F(s) = \frac{Q(s)}{P(s)}$, degree(Q) < degree(P)

polynomials

I. $P(s)$ is Quadratic with two distinct real roots:

$$F(s) = \frac{2s-8}{s^2-5s+6} = \frac{2s-8}{(s-2)(s-3)} \underset{=}{\text{need}} \frac{A}{s-2} + \frac{B}{s-3}$$

$s=2,3$ are roots

$$\underline{\text{Find } A, B} \quad \frac{A(s-3)+B(s-2)}{(s-2)(s-3)}$$

$$= \frac{(A+B)s+(-3A-2B)}{(s-2)(s-3)}$$

Hence $\begin{cases} A+B=2 \\ -3A-2B=-8 \end{cases} \Longrightarrow \begin{cases} A=4 \\ B=-2 \end{cases}$

$$\Rightarrow F(s) = \frac{4}{s-2} + \frac{-2}{s-3} \Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= 4e^{2t} - 2e^{3t}$$

II. $P(s)$ is Quadratic with a double root:

$$F(s) = \frac{s+1}{(s+2)^2}$$

$$\underset{=}{\text{need}} \quad \frac{A}{s+2} + \frac{B}{(s+2)^2} = \frac{As+(2A+B)}{(s+2)^2}$$

Hence $\begin{cases} A=1 \\ 2A+B=1 \end{cases} \Longrightarrow \begin{cases} A=1 \\ B=-1 \end{cases}$

$$\Rightarrow F(s) = \frac{1}{s+2} - \frac{1}{(s+2)^2} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = e^{-2t} - e^{-2t}t$$

$\mathcal{L}\{t\} = \frac{1}{s^2}$

III $P(s)$ is Quadratic with Complex roots:

$$F(s) = \frac{s+1}{s^2+4s+5} = \frac{s+1}{s^2+4s+4+1} = \frac{s+1}{(s+2)^2+1} = \frac{s+1+1-1}{(s+2)^2+1}$$

$$= \frac{s+2}{(s+2)^2+1} - \frac{1}{(s+2)^2+1} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = e^{-2t}\cos t - e^{-2t}\sin t.$$

$\underline{\text{Ex 9}}$ $\quad \mathcal{L}^{-1}\left\{ \dfrac{s+2}{s^3 - s^2 + s - 1} \right\} = ?$

$$\underbrace{s^3 - s^2} + \underbrace{s - 1} = s^2(s-1) + (s-1) = (s^2+1)(s-1)$$

$$\Rightarrow F(s) = \frac{s+2}{(s-1)(s^2+1)} \xrightarrow{\text{need}} \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$\xrightarrow[A,B,C]{\text{Find}} \frac{(A+B)s^2 + (C-B)s + (A-C)}{(s-1)(s^2+1)}$$

Hence $\begin{cases} A+B=0 \\ C-B=1 \\ A-C=2 \end{cases} \Longrightarrow \begin{cases} A = 3/2 \\ B = -3/2 \\ C = -1/2 \end{cases}$

So

$$\mathcal{L}^{-1}\{F(s)\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{3}{2}e^t - \frac{3}{2}\cos t - \frac{1}{2}\sin t.$$

1.3 Applications of Laplace Transforms

Solve IVP and Integral equations.

1. Take the Laplace transform, to obtain an algebraic eq.
2. Solve the algebraic eq.
3. Take the inverse Laplace transform.

1.3.1 IVP

Ex Solve $\begin{cases} y'(t) + 2y(t) = 12e^{3t} \\ y(0) = 3 \end{cases}$

$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{12e^{3t}\}$

$\Rightarrow s\mathcal{L}\{y\} - y(0) + 2\mathcal{L}\{y\} = 12\frac{1}{s-3}$

$\overset{Y(s)=\mathcal{L}\{y(t)\}}{\Longrightarrow} (s+2)Y(s) = \frac{12}{s-3} + 3 = \frac{3(1+s)}{s-3}$

$\Rightarrow Y(s) = \frac{3(s+1)}{(s+2)(s-3)}$

To take $\mathcal{L}^{-1}$, we need to decompose $Y(s)$ into partial fractions.

$$\frac{3s+3}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3} = \frac{(A+B)s + (-3A+2B)}{(s+2)(s-3)}$$

$\Rightarrow \begin{cases} A+B = 3 \\ -3A+2B = 3 \end{cases} \Rightarrow \begin{cases} A = 3/5 \\ B = 12/5 \end{cases}$

Hence $Y(s) = \frac{3/5}{s+2} + \frac{12/5}{s-3} \Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{3}{5}e^{-2t} + \frac{12}{5}e^{3t}.$

Ex 10 Solve (for $t>0$) $\begin{cases} y''(t) + 4y'(t) + 3y(t) = 0, & t>0 \\ y(0) = 3, \ y'(0) = 1 \end{cases}$

$\overset{\mathcal{L}}{\Rightarrow} \mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{0\}$

$\overset{Y(s)=\mathcal{L}\{y(t)\}}{\Longrightarrow} s^2Y(s) - s\underbrace{y(0)}_{=3} - \underbrace{y'(0)}_{=1} + 4(sY(s) - \underbrace{y(0)}_{=3}) + 3Y(s) = 0$

$\Rightarrow Y(s) = \frac{3s+13}{s^2+4s+3} = \frac{3s+13}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1} = \frac{(A+B)s + A + 3B}{(s+3)(s+1)}$

$$\Rightarrow \begin{cases} A = -2 \\ B = 5 \end{cases}$$

Hence

$$Y(s) = -\frac{2}{s+3} + \frac{5}{s+1} \Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = -2e^{-3t} + 5e^{-t}.$$

1.3.2 Integral Equations

Ex Solve
$$\begin{cases} i(t) + \int_0^t i(\tau)\,d\tau = v(t) \\ i(0) = 0 \end{cases}$$

where $v(t) = \theta(t-1) - \theta(t-2)$. ($i(t)$ is the current in an electric RC-circuit.)

$$\mathcal{L}\{i\} + \mathcal{L}\left\{\int_0^t i(\tau)\,d\tau\right\} = \mathcal{L}\{\theta(t-1) - \theta(t-2)\}$$

$$\xrightarrow[\text{Thm } 8]{I = \mathcal{L}\{i\}} I(s) + \frac{I(s)}{s} = \mathcal{L}\{\theta(t-1)\} - \mathcal{L}\{\theta(t-2)\}$$

$$\xrightarrow[\text{Thm } 4]{\mathcal{L}\{\theta(t)\} = 1/s} \frac{1+s}{s} I(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$\Rightarrow I(s) = \frac{e^{-s}}{s+1} - \frac{e^{-2s}}{s+1} \Rightarrow i(t) = \mathcal{L}^{-1}\{I(s)\} = e^{-(t-1)}\theta(t-1) - e^{-(t-2)}\theta(t-2)$$

Hence

$$i(t) = \begin{cases} 0 & t<1 \\ e^{-(t-1)} & 1<t<2 \\ e^{-(t-1)} - e^{-(t-2)} & 2<t \end{cases} = \begin{cases} 0 & t<1 \\ e e^{-t} & 1<t<2 \\ (e-e^{2})e^{-t} & 2<t \end{cases}$$

Note: To compute $\mathcal{L}\{v(t)\}$, we could also use

$$v(t) = \theta(t-1) - \theta(t-2) = \begin{cases} 0 & t<1 \\ 1 & 1<t<2 \\ 0 & 2<t \end{cases}$$

Then

$$\mathcal{L}\{v(t)\} = V(s) = \int_0^\infty v(t) e^{-st}\, dt = \int_1^2 e^{-st}\, dt = -\frac{e^{-2s}}{s} + \frac{e^{-s}}{s}.$$