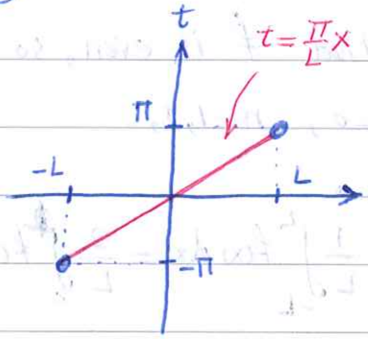
2.2.7 Functions of Arbitrary Period

Consider $y = f(x)$, $-L \leq x \leq L$, $p = 2L$

We change the variable such that

$$[-L, L] \xrightarrow[\text{change of variable}]{} [-\pi, \pi]$$



So by $t = \frac{\pi}{L}x$ $\left(x = \frac{L}{\pi}t\right)$ we define function $g(t)$ as

$$g(t) = f\left(\frac{L}{\pi}t\right) = f(x)$$

Then

$$g(t+2\pi) = f\left(\frac{L}{\pi}(t+2\pi)\right) = f\left(\frac{L}{\pi}t + 2L\right) = f\left(\frac{L}{\pi}t\right) = g(t)$$

That is by the change of variable $t = \frac{\pi}{L}x$,

$$f(x),\ -L \leq x \leq L,\ p = 2L \xrightarrow{\text{transformed to}} g(t),\ -\pi \leq t \leq \pi,\ p = 2\pi$$

Now, based on the Fourier series for $g(t)$,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nt + b_n \sin nt\right)$$

We have the Fourier Series of $f(x)$,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x\right)$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos nt\, dt = \frac{1}{\pi}\int_{-L}^{L} f(x)\cos \frac{n\pi}{L}x \left(\frac{\pi}{L}dx\right)$$

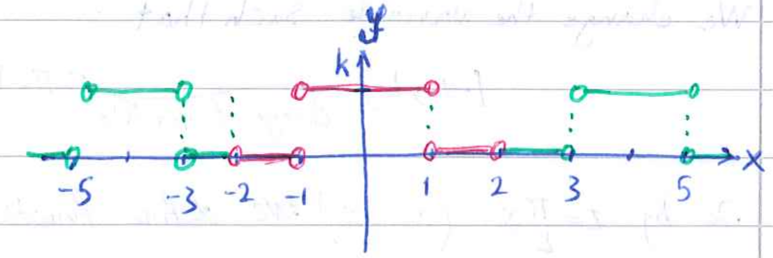
$$\Rightarrow \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos \frac{n\pi}{L}x\, dx, \quad n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin \frac{n\pi}{L}x\, dx, \quad n = 1, 2, \ldots$$

Ex Find the Fourier series of (periodic square wave): (k is a constant)

$$f(x)=\begin{cases} 0 & -2<x<-1 \\ k & -1<x<1 \\ 0 & 1<x<2 \end{cases}, \quad p=4 \quad (p=2L \to L=2)$$



Note that f is even, so

$b_n=0, \quad n=1,2,\dots$

$$a_0=\frac{1}{L}\int_{-L}^{L} f(x)\,dx=\frac{2}{L}\int_0^L f(x)\,dx=\frac{2}{2}\int_0^2 f(x)\,dx=\int_0^1 \underbrace{f(x)}_{=k}dx+\int_1^2 \underbrace{f(x)}_{=0}dx=k$$

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx=\int_0^2 f(x)\cos\frac{n\pi}{2}x\,dx$$

$$=\int_0^1 k\cos\frac{n\pi}{2}x\,dx=k\frac{2}{n\pi}\sin\frac{n\pi}{2}x\Big]_0^1=\frac{2k}{n\pi}\left(\sin\frac{n\pi}{2}-\sin 0\right)$$

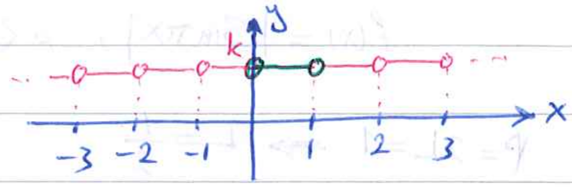
$$=\begin{cases} \frac{2k}{n\pi} & n=1,5,9,\dots \\ -\frac{2k}{n\pi} & n=3,7,11,\dots \\ 0 & n=2,4,6,\dots \end{cases}$$

Hence

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{2}x=\frac{k}{2}+\sum_{n=1}^{\infty}\left(\frac{2k}{n\pi}\sin\frac{n\pi}{2}\right)\cos\frac{n\pi}{2}x$$

$$=\frac{k}{2}+\frac{2k}{\pi}\left(\cos\frac{\pi}{2}x-\frac{1}{3}\cos\frac{3\pi}{2}x+\frac{1}{5}\cos\frac{5\pi}{2}x-\dots\right)$$

Ex Find the Fourier series for $f(x)=k,\ 0<x<1,\ p=1$ (k is constant)



$f(x)$ is even, so $b_n=0,\ n=1,2,\dots$

$p=2L=1 \implies L=\frac{1}{2}$

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)\,dx=\frac{1}{L}\int_0^{2L}f(x)\,dx=\frac{1}{\frac{1}{2}}\int_0^1 \underbrace{f(x)}_{=k}\,dx=2k$$

For $n=1,2,\dots$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}x\,dx=\frac{1}{L}\int_0^{2L}f(x)\cos\frac{n\pi}{L}x\,dx$$

$$=\frac{1}{\frac{1}{2}}\int_0^1 k\cos\frac{n\pi}{\frac{1}{2}}x\,dx=2k\int_0^1\cos 2n\pi x\,dx=2k\,\frac{1}{2n\pi}\sin 2n\pi x\Big|_0^1=0$$

Hence $f(x)=\frac{a_0}{2}=k$ (as we expected)

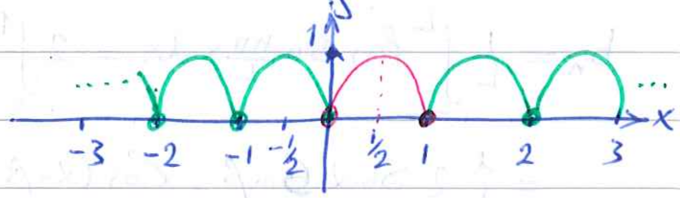
Ex Sketch the graph of these functions. Are these functions equal?

$f(x)=|\sin\pi x|,\ 0<x\leq 1,\ p=1$

$g(x)=|\sin\pi x|,\ -\frac{1}{2}<x\leq\frac{1}{2},\ p=1$

$0<x\leq 1 \implies 0<\pi x\leq\pi \implies \sin\pi x\geq 0$

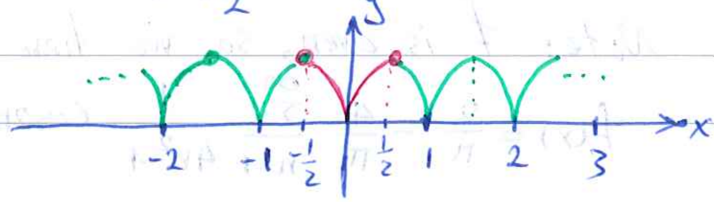
$\implies f(x)=|\sin\pi x| \overset{0<x\leq 1}{=} \sin\pi x,\ p=1$



$-\frac{1}{2}<x\leq\frac{1}{2} \implies -\frac{\pi}{2}<\pi x\leq\frac{\pi}{2}$

$$\implies g(x)=|\sin\pi x|=\begin{cases}-\sin\pi x & -\frac{1}{2}<x\leq 0\\ \sin\pi x & 0<x\leq\frac{1}{2}\end{cases},\quad p=1$$

They are equal ($f=g$).



Ex 14 Find the Fourier series of (the rectified sine wave):

$$f(x) = |\sin \pi x|, \quad 0 < x < 1, \quad p = 1$$

$$p = 2L = 1 \implies L = \frac{1}{2}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^{2L} f(x)\,dx = \frac{1}{\frac{1}{2}}\int_0^1 |\sin \pi x|\,dx = 2\int_0^1 \sin \pi x\,dx$$

$$= -\frac{2}{\pi}\cos \pi x \Big]_0^1 = -\frac{2}{\pi}(\cos \pi - \cos 0) = \frac{4}{\pi}$$

For $n = 1, 2, \dots$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi}{L}x\,dx = 2\int_0^1 \sin \pi x \cos 2n\pi x\,dx$$

$$= \left\{2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)\right\}$$

$$= \int_0^1 \sin(1+2n)\pi x\,dx + \int_0^1 \sin(1-2n)\pi x\,dx$$

$$= \frac{-1}{(1+2n)\pi}\left(\cos(1+2n)\pi x\right]_0^1 - \frac{1}{(1-2n)\pi}\left(\cos(1-2n)\pi x\right]_0^1$$

$$= \frac{-1}{(1+2n)\pi}\Big(\underbrace{\cos(2n+1)\pi}_{=-1} - \cos 0\Big) - \frac{1}{(1-2n)\pi}\Big(\underbrace{\cos(2n-1)\pi}_{=-1} - \cos 0\Big)$$

$$= \frac{2}{(1+2n)\pi} + \frac{2}{(1-2n)\pi} = \frac{2(1-2n) + 2(1+2n)}{(1+2n)(1-2n)\pi} = \frac{4}{(1-4n^2)\pi}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx = 2\int_0^1 \sin \pi x \sin 2n\pi x\,dx$$

$$= \left\{2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)\right\}$$

$$= \int_0^1 \cos(1-2n)\pi x\,dx - \int_0^1 \cos(1+2n)\pi x\,dx = \dots = 0$$

Note: f is even, so we have $b_n = 0$.

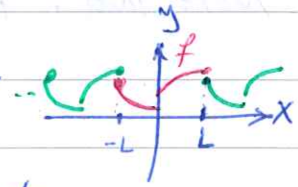
$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos 2n\pi x$$

2.2.8 Sine and Cosine Series

Can we find Fourier series for a non-periodic function $y=f(x),\ 0\leq x\leq L$?

Yes, first we extend it to a periodic function as:

$$F(x)=\begin{cases} f(x) & 0\leq x\leq L \\ g(x) & -L\leq x<0 \end{cases} \qquad p=2L$$



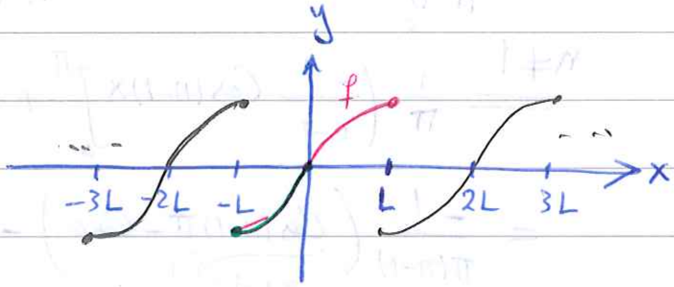
Then $F(x),\ \forall x$, has the Fourier series. Note that

$f(x)=F(x)$, for $0\leq x\leq L$

There are many extensions, among which even or odd extension are more appropriate.

Fourier Sine series (odd extension):

$$F(x)=\begin{cases} f(x) & 0\leq x\leq L \\ -f(-x) & -L\leq x\leq 0 \end{cases}, \quad p=2L$$



Then

$$F(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \quad ,\forall x$$

where

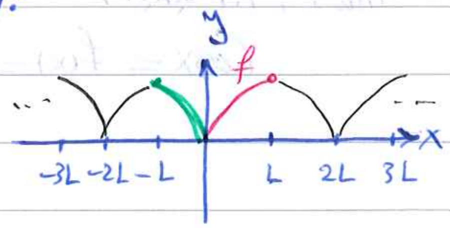
$$b_n=\frac{1}{L}\int_{-L}^{L}F(x)\sin\frac{n\pi}{L}x\,dx=\frac{2}{L}\int_0^L \underbrace{F(x)}_{=f(x)}\sin\frac{n\pi}{L}x\,dx$$

$$\Rightarrow\ b_n=\frac{2}{L}\int_0^1 f(x)\sin\frac{n\pi}{L}x\,dx, \qquad n=1,2,\ldots$$

Note: for $0\leq x\leq L$, $f(x)=F(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$

Fourier Cosine Series (even extension):

$$F(x)=\begin{cases} f(x) & 0\leq x\leq L \\ f(-x) & -L\leq x\leq 0 \end{cases}, \quad p=2L$$



Then

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \quad , \qquad \forall x$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx \; ; \qquad n = 0, 1, 2, \ldots$$

Note that, for $0 \le x \le L$

$$f(x) = F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

Ex 15 Write the Fourier Sine series and Cosine series for $f(x) = \cos x$, $0 \le x \le \pi$.

Sine series: $(a_n = 0, \quad n = 0, 1, \ldots)$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx\,dx = \left\{ 2\sin\alpha\cos\beta = \sin(\alpha-\beta) + \sin(\alpha+\beta) \right\}$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin(n-1)x\,dx + \frac{1}{\pi}\int_0^{\pi} \sin(n+1)x\,dx$$

$$\stackrel{n \neq 1}{=} \frac{1}{\pi}\left(\frac{-1}{n-1}\cos(n-1)x\right]_0^{\pi} + \frac{1}{\pi}\left(\frac{-1}{n+1}\cos(n+1)x\right]_0^{\pi}$$

$$= \frac{-1}{\pi(n-1)}\left(\underbrace{\cos(n-1)\pi}_{=(-1)^{n-1}} - \cos 0\right) - \frac{1}{\pi(n+1)}\left(\underbrace{\cos(n+1)\pi}_{=(-1)^{n-1}} - \cos 0\right)$$

$$= \frac{(-1)^n + 1}{\pi(n-1)} + \frac{(-1)^n + 1}{\pi(n+1)} = \frac{\left((-1)^n+1\right)(n+1) + \left((-1)^n+1\right)(n-1)}{\pi(n-1)(n+1)}$$

$$= \frac{2n\left((-1)^n + 1\right)}{\pi(n^2-1)}$$

$n=1 \Rightarrow b_1 = 0.$

Thus, for $0 < x < \pi$,

$$\cos x = f(x) = \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{n\left((-1)^n+1\right)}{n^2-1}\sin nx.$$

Cosine Series ($b_n = 0$)

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \cos x \, dx = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos x \cos nx \, dx = \left\{ 2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta) \right\}$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(1-n)x \, dx + \frac{1}{\pi}\int_0^{1} \cos(1+n)x \, dx$$

$$\underline{\underline{n \neq 1}} \quad \frac{1}{\pi}\frac{1}{1-n}\sin(1-n)x \Big]_0^{\pi} + \frac{1}{\pi}\frac{1}{1+n}\sin(1+n)x\Big]_0^{\pi}$$

$$= \frac{1}{\pi(1-n)}\Big(\underbrace{\sin(1-n)\pi}_{=0} - \sin 0\Big) + \frac{1}{\pi(1+n)}\Big(\underbrace{\sin(1+n)\pi}_{=0} - \sin 0\Big) = 0$$

$$n = 1 \Rightarrow a_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \cos x \, dx = \frac{2}{\pi}\int_0^{\pi} \cos^2 x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1+\cos 2x}{2} \, dx = \ldots = 1$$

Hence for $0 \leq x \leq \pi$

$$\cos x = f(x) = a_1 \cos x = \cos x.$$

Ex. Fourier Sine series for $f(x) = \sin 10x$, $0 \leq x \leq \pi$?

$a_{10} = 1$
$\Rightarrow f(x) = \sin 10x$
$a_n = 0, n \neq 10$