Differentiation and Integration of Fourier Series

Some operations that hold for finite sums, may not hold for infinite sums.

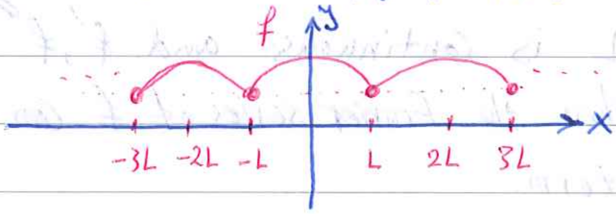
For example

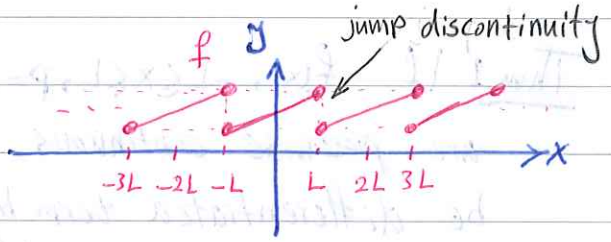
$$\left(\sum_{n=1}^{N} f_n(x)\right)' = \sum_{n=1}^{N} f_n'(x)$$

but

$$\left(\sum_{n=1}^{\infty} f_n(x)\right)' \stackrel{?}{=} \sum_{n=1}^{\infty} f_n'(x).$$

Note: When a periodic function $y=f(x)$, $-L \leq x \leq L$, $p=2L$ is continuous, we have $f(-L)=f(L)$.





Differentiation of Fourier Series

First, by an example, we see that differentiating the Fourier series of $f(x)$, term by term, does not necessarily give the Fourier series of $f'(x)$.

Ex: The Fourier series for $f(x)=x$, $-L \leq x \leq L$, $p=2L$ is

$$x = \frac{2L}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi}{L}x \qquad \left(\begin{array}{l} b_n = \frac{2}{L}\int_0^L x \sin\frac{n\pi}{L}x\,dx = \frac{2L}{n\pi}(-1)^{n+1} \\ a_n = 0,\ n=0,1,\dots \end{array} \right)$$

and $f'(x) = (x)' = 1$ for which the Fourier series is $1=1$.

But differentiating the right side, term by term, gives the series

$$\left(\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{L}x\right)' = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{n\pi}{L}\cos\frac{n\pi}{L}x$$

$$= 2\sum_{n=1}^{\infty}(-1)^{n+1}\cos\frac{n\pi}{L}x \quad (*)$$

That is not the Fourier Series for $f'(x)=1$.

In fact the series $(*)$ does not even converge. For example

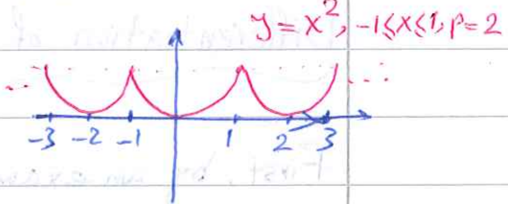
$x=0 \Rightarrow 2\sum_{n=1}^{\infty}(-1)^{n+1}\cos 0 = 2\sum_{n=1}^{\infty}(-1)^{n+1}$ which is not convergent.

Thm 1 If $f(x)$, $-L\leq x\leq L$, $p=2L$ is continuous and f', f'' are piecewise continuous, then the Fourier series of f can be differentiated term by term.

Note: In Ex 1, $f(-L)\neq f(L)$, Since $f(-L)=-L$, $f(L)=L$.

Ex 2 Find the Fourier series of $f(x)=-2x$, $-1\leq x\leq 1$, $p=2$, using the Fourier series of $F(x)=1-x^2$, $-1\leq x\leq 1$, $p=2$.

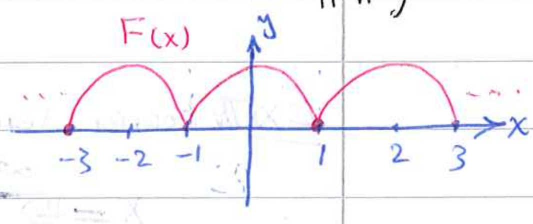
We have the Fourier series of x^2 as



$$x^2 = \frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$$

$$\left(a_0=\frac{1}{L}\int_{-L}^{L}f(x)\,dx = 2\int_0^1 x^2\,dx=\frac{2}{3},\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}x\,dx = 2\int_0^1 x^2\cos n\pi x\,dx=\frac{4(-1)^n}{\pi^2 n^2}\right)$$

Hence



$$F(x)=1-x^2=\frac{2}{3}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$$

Then

$$\underbrace{f(x)}_{=-2x}=F'(x)=\frac{-4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-n\pi)\sin n\pi x$$

$$=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x$$

Integration of Fourier Series

For integration of Fourier series the situation is much better.

Thm 2 If $f(x)$, $-L \leq x \leq L$, $p = 2L$ and f' are p.w. continuous, then the Fourier series of f can be integrated term by term. Moreover the result is a series that converges to the integral of f for $-L \leq x \leq L$.

Ex 3 The Fourier series for $f(x) = 1 - x^2$, $-1 \leq x \leq 1$, $p = 2$ is

$$1 - x^2 = \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x$$

So integrating $\int_{-1}^{x}$ we have

$$\int_{-1}^{x} (1 - y^2)\, dy = \int_{-1}^{x} \left\{ \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi y \right\} dy$$

$$= \left(\frac{2}{3} y \right]_{y=-1}^{y=x} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \int_{-1}^{x} \cos n\pi y \, dy$$

$$= \frac{2}{3}(x+1) - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left(\frac{1}{n\pi} \sin n\pi y \right]_{y=-1}^{y=x}$$

$$= \frac{2}{3}(x+1) - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 \pi} \left(\sin n\pi x - \underbrace{\sin(-n\pi)}_{=0} \right)$$

$$= \frac{2}{3}(x+1) - \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin n\pi x .$$

Note that the right side is not a Fourier series, because of $\frac{2}{3}x$.

Note: Since $\int_{-1}^{x} (1 - y^2)\, dy = x - \frac{x^3}{3} + \frac{2}{3}$, so we can write

$$x - x^3 = \frac{-12}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin n\pi x$$

Convolution and its Laplace transform

Def The convolution of $f, g: \mathbb{R} \to \mathbb{R}$ is (if the integral exists):

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-\tau) g(\tau)\, d\tau$$

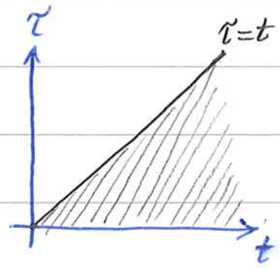
When $f(t) = g(t) = 0$, for $t < 0$, we have

$$(f*g)(t) = \int_{0}^{\infty} f(t-\tau) g(\tau)\, d\tau$$

Thm 1.2 Assume f and g are causal (one-sided) function with $\mathcal{L}\{f\} = F$, $\mathcal{L}\{g\} = G$. Then

$$\mathcal{L}\{(f*g)(t)\} = F(s)\, G(s)$$

Proof



$$\mathcal{L}\{(f*g)(t)\} = \int_0^\infty e^{-st} \left(\int_0^t f(t-\tau) g(\tau)\, d\tau \right) dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(t-\tau) g(\tau)\, d\tau\, dt$$

change order of integrals $$= \int_0^\infty \int_\tau^\infty e^{-st} f(t-\tau) g(\tau)\, dt\, d\tau$$

$$= \int_0^\infty e^{-s\tau} g(\tau) \int_\tau^\infty e^{s\tau} e^{-st} f(t-\tau)\, dt\, d\tau$$

$$= \int_0^\infty e^{-s\tau} g(\tau) \int_\tau^\infty e^{-s(t-\tau)} f(\underbrace{t-\tau}_{=r})\, dt\, d\tau$$

$$= \{ r = t - \tau \to dr = dt \}$$

$$= \int_0^\infty e^{-s\tau} g(\tau) \int_0^\infty e^{-sr} f(r)\, dr\, d\tau = \mathcal{L}\{g(t)\} \mathcal{L}\{f(t)\}$$

$$= F(s)\, G(s).$$

$\square$