

## Differentiation and Integration of Fourier Series

Some operations that hold for finite sums, may not hold for infinite sums.

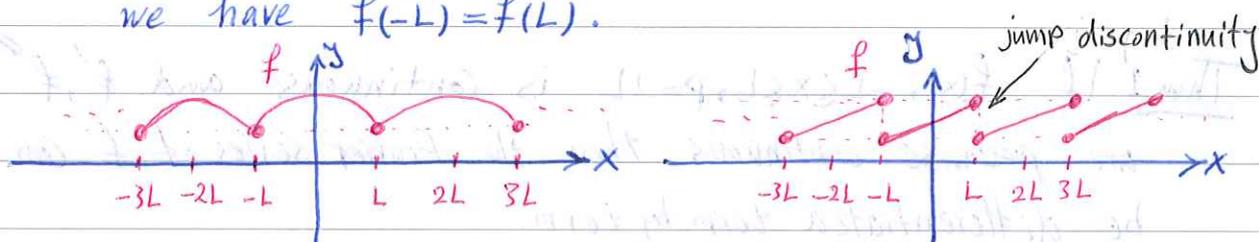
For example

$$\left(\sum_{n=1}^N f_n(x)\right)' = \sum_{n=1}^N f_n'(x)$$

but

$$\left(\sum_{n=1}^{\infty} f_n(x)\right)' \stackrel{?}{=} \sum_{n=1}^{\infty} f_n'(x)$$

Note: When a periodic function  $y=f(x)$ ,  $-L \leq x < L$ ,  $P=2L$  is continuous, we have  $f(-L) = f(L)$ .



## Differentiation of Fourier Series

First, by an example, we see that differentiating the Fourier series of  $f(x)$ , term by term, does not necessarily give the Fourier series of  $f'(x)$ .

Ex: The Fourier series for  $f(x)=x$ ,  $-L \leq x < L$ ,  $P=2L$  is

$$x = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L}$$

$$\left( \begin{array}{l} b_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L} dx = \frac{2L(-1)^{n+1}}{n\pi} \\ a_n = 0, n=0,1, \dots \end{array} \right)$$

and  $f'(x) = (x)' = 1$  for which the Fourier series is  $1=1$ .

But differentiating the right side, term by term, gives the series

$$\left( \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L} \right)' = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{n\pi}{L} \cos \frac{n\pi x}{L}$$

$$= 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos \frac{n\pi x}{L} \quad (*)$$

That is not the Fourier series for  $f'(x)=1$ .

In fact the series  $(*)$  does not even converge. For example

$$x=0 \Rightarrow 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos 0 = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \text{ which is not convergent.}$$

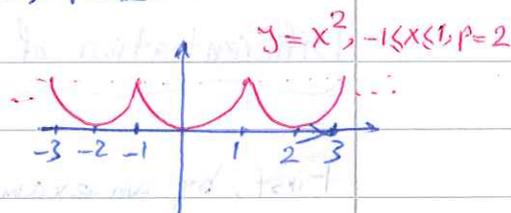
Thm 1 If  $f(x)$ ,  $-L \leq x \leq L$ ,  $p=2L$  is continuous and  $f', f''$  are piecewise continuous, then the Fourier series of  $f$  can be differentiated term by term.

Note: In Ex 1,  $f(-L) \neq f(L)$ , since  $f(-L)=-L$ ,  $f(L)=L$ .

Ex 2 Find the Fourier series of  $f(x)=-2x$ ,  $-1 \leq x \leq 1$ ,  $p=2$ , using the Fourier series of  $F(x)=1-x^2$ ,  $-1 \leq x \leq 1$ ,  $p=2$ .

We have the Fourier series of  $x^2$  as

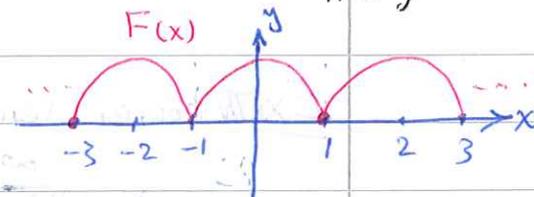
$$x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x$$



$$\left( a_0 = \frac{1}{L} \int_{-L}^L f(x) dx = 2 \int_0^1 x^2 dx = \frac{2}{3}, \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx = 2 \int_0^1 x^2 \cos n\pi x dx = \frac{4(-1)^n}{\pi^2 n^2} \right)$$

Hence

$$F(x) = 1 - x^2 = \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x$$



Then

$$\underbrace{f(x)}_{=-2x} = F'(x) = \frac{-4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-n\pi) \sin n\pi x$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin n\pi x$$

## Integration of Fourier Series

For integration of Fourier series the situation is much better.

Thm 2 If  $f(x)$ ,  $-L \leq x \leq L$ ,  $p=2L$  and  $f'$  are p.w. continuous, then the Fourier series of  $f$  can be integrated term by term. Moreover the result is a series that converges to the integral of  $f$  for  $-L \leq x \leq L$ .

Ex 3 The Fourier series for  $f(x) = 1-x^2$ ,  $-1 \leq x \leq 1$ ,  $p=2$  is

$$1-x^2 = \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x$$

So integrating  $\int_{-1}^x$  we have

$$\begin{aligned} \int_{-1}^x (1-y^2) dy &= \int_{-1}^x \left\{ \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi y \right\} dy \\ &= \left( \frac{2}{3} y \right) \Big|_{y=-1}^{y=x} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \int_{-1}^x \cos n\pi y dy \\ &= \frac{2}{3} (x+1) - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left( \frac{1}{n\pi} \sin n\pi y \right) \Big|_{y=-1}^{y=x} \\ &= \frac{2}{3} (x+1) - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 \pi} \left( \sin n\pi x - \underbrace{\sin(-n\pi)}_{=0} \right) \\ &= \frac{2}{3} (x+1) - \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin n\pi x. \end{aligned}$$

~~Note~~

Note that the right side is not a Fourier series, because of  $\frac{2}{3}x$ .

Note: Since  $\int_{-1}^x (1-y^2) dy = x - \frac{x^3}{3} + \frac{2}{3}$ , so we can write

$$x - x^3 = \frac{-12}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin n\pi x$$

Integration of  $\frac{1}{x^2}$

Let  $u = x^{-1}$  then  $du = -x^{-2} dx = -\frac{1}{x^2} dx$

Thus  $\int \frac{1}{x^2} dx = \int -du = -u + C = -\frac{1}{x} + C$

Ex: The power series for  $\ln(1+x)$  is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

is integrated  $\int \ln(1+x) dx$

$$\int \ln(1+x) dx = \int \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \right) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{(n+1)n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{(n+1)n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{(n+1)n}$$

Integration of  $\frac{1}{1-x}$

Let  $u = 1-x$  then  $du = -dx$

$$\int \frac{1}{1-x} dx = \int -\frac{1}{u} du = -\ln|u| + C = -\ln|1-x| + C$$

## Convolution and its Laplace transform

Def The convolution of  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  is (if the integral exists):

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau) d\tau$$

When  $f(t) = g(t) = 0$ , for  $t < 0$ , we have

$$(f * g)(t) = \int_0^{\infty} f(t-\tau)g(\tau) d\tau$$

Thm 1.2 Assume  $f$  and  $g$  are causal (one-sided) function with  
 $\mathcal{L}\{f\} = F$ ,  $\mathcal{L}\{g\} = G$ . Then

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$$

Proof

$$\mathcal{L}\{(f * g)(t)\} = \int_0^{\infty} e^{-st} \left( \int_0^t f(t-\tau)g(\tau) d\tau \right) dt$$

change order of integrals  $= \int_0^{\infty} \int_0^t e^{-st} f(t-\tau)g(\tau) d\tau dt$

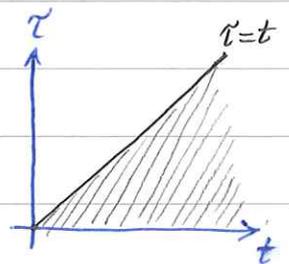
change order of integrals  $= \int_0^{\infty} \int_{\tau}^{\infty} e^{-st} f(t-\tau)g(\tau) dt d\tau$

$$= \int_0^{\infty} e^{-s\tau} g(\tau) \int_{\tau}^{\infty} e^{-st} e^{-st} f(t-\tau) dt d\tau$$

$$= \int_0^{\infty} e^{-s\tau} g(\tau) \int_{\tau}^{\infty} e^{-s(t-\tau)} \underbrace{f(t-\tau)}_{=r} dt d\tau$$

$$= \{r = t - \tau \rightarrow dr = dt\}$$

$$= \int_0^{\infty} e^{-s\tau} g(\tau) \underbrace{\int_0^{\infty} e^{-sr} f(r) dr}_{\mathcal{L}\{f(t)\}} d\tau = \mathcal{L}\{g(t)\} \mathcal{L}\{f(t)\} = F(s)G(s).$$



□

substitution and the inverse transformation

Let the substitution be  $u = g(x)$  and  $du = g'(x) dx$

$$\int f(g(x)) g'(x) dx = \int f(u) du$$

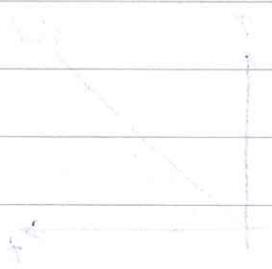
When this substitution is used, we have

$$\int f(g(x)) g'(x) dx = \int f(u) du$$

After the integral has been evaluated, we must substitute back in terms of  $x$

$$\int f(g(x)) g'(x) dx = F(g(x)) + C$$

$$= F(u) + C$$



$$\int f(g(x)) g'(x) dx = \int f(u) du$$

$$= \int f(u) du$$

$$= F(u) + C$$

$$= F(g(x)) + C$$

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