TMA 683: Tillämpad matematik 2018/19

This course has two parts:

1. Approximate solution of Differential Equations (DE) $\begin{cases} \text{FEM} \\ \text{FDM} \end{cases}$

2. Fourier methods $\begin{cases} \text{Laplace transform} \\ \text{Fourier series} \\ \text{Separation of variables} \end{cases}$

Differential Equation (DE): is a relation between an unknown function u and its derivative(s)

DE $\begin{cases} \text{Ordinary Differential Equation (ODE): } u=u(x) \text{ is a function of one variable} \\ \text{Partial Differential Equation (PDE): } u=u(x,y) \text{ or } u=u(x,y,z) \text{ or } u=u(x_1,x_2,\dots,x_n) \\ \quad \text{a function of more than one variable} \end{cases}$

<u>Ex.</u>

ODE: $u=u(x) \rightarrow$

1. $-u'(x)=2x, \quad x\in(0,1)$

2. $u(x)+2xu'(x)=-\sin x, \quad x\in(-\pi,\pi)$

3. $\begin{cases} u'(x)-3u(x)=0, & x\in(0,L) \\ u(0)=1 \end{cases}$

4. $\begin{cases} u''(x)-u(x)=x^2-3x, & x\in(0,1) \\ u(0)=0,\ u(1)=0 \end{cases}$

$u=u(t) \rightarrow$

5. $\begin{cases} u''(t)-u(t)=t^2-3t, & t\in(0,T) \\ u(0)=0,\ u'(0)=1 \end{cases}$

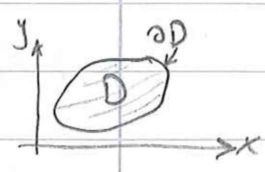
6. $\begin{cases} u''(t)=-2t, & t\in(0,2) \\ u(0)=0,\ u'(0)=1 \end{cases}$

7. $u'(t)-2u(t)u''(t)=3tu(t), \quad t\in(0,T)$

PDE: $u=u(x,y) \rightarrow$ 8. $u(x,y)+u_x(x,y)+2u_y(x,y)=x^2+3y\sin x$, $(x,y)\in D$

9. $-u_{xx}-u_{yy}=x+y$, $(x,y)\in D$

10. $\begin{cases} u^{(x,y)}+u_{xy}^{(x,y)}=f(x,y), & (x,y)\in D \\ u(x,y)=g(x,y), & (x,y)\in \partial D \end{cases}$



↘ boundary of D

$u=u(x,t) \rightarrow$ 11. $u_t-u_{xx}=f(x,y)$, $(x,y)\in D$ (heat eq.)

12. $u_{tt}-u_{xx}=f(x,y)$, $(x,y)\in D$ (wave eq.)

Note: A solution of a differential equation is a function that satisfies the differential eq.

Ex. $u(x)=-x^2$ is a solution of 1. Since $u'(x)=-2x \Rightarrow -u'(x)=2x$

$u(x)=-x^2+3$ 〃 〃 〃 〃 〃 〃 〃 〃 〃

$\underbrace{u(x)=-x^2+C}$ 〃 〃 〃 〃 〃 for any constant C, Since 〃 〃 〃 〃

a general solution (it contains parameter C)

$u(x)=x-x^2$ is a solution of 4.

Since $\begin{cases} u''(x)-u(x)=-2x-(x-x^2)=x^2-3x \\ u(0)=0-(0)^2=0, \quad u(1)=1-(1)^2=0 \end{cases}$

Note: Usually it is not easy or even not possible to find the exact (analytic) solution to a DE. So we need to find an approximate solution by a numerical method.

We use finite element method (FEM) and finite difference method (FDM).

Note: The exact solution of Ex. 4 is $u(x)=x-x^2$, and we can consider x as the space variable. Ex. 4 is an example of a boundary value problem (BVP). The exact solution of Ex. 5 is also

$u(t)=t-t^2$, and we can consider t as the time variable. Ex. 5 is an example of an initial value problem (IVP).

Notation: Laplace operator

(We can write $\Delta = \nabla^2$)

$$\begin{cases} \text{in 2-dim (two variables } x,y): & \Delta u(x,y) = u_{xx}(x,y) + u_{yy}(x,y) \\ \text{in 3-dim (three variables } x,y,z): & \Delta u(x,y,z) = u_{xx} + u_{yy} + u_{zz} \\ \text{in n-dim}: & \Delta u(x_1,\dots,x_n) = u_{x_1x_1} + \cdots + u_{x_nx_n} \end{cases}$$

Ex: $u(x,y) = x^2+y^3+2xy \rightarrow \begin{cases} u_x = 2x+2y, & u_{xx}=2 \\ u_y = 3y^2+2x, & u_{yy}=6y \end{cases}$

$$\Delta u = u_{xx}+u_{yy} = 2+6y$$

Notation: If we denote $D=\frac{d}{dx}$, we can write $u'(x) = \frac{d}{dx}u(x) = Du(x)$

and for example, we can write $u''+2xu'+3u=0$ as

$$D^2u+3xDu+3=0 \quad \text{or} \quad (D^2+3xD+I)u=0$$

the identity operator

$Iu=u$, $I\ast=\ast$

Different types of boundary conditions

We can complement a DE with boundary condition(s). (B.C.).

Ex.
$$\begin{cases} \text{DE} & -u''(x)=f(x), \quad x\in(0,1) \\ \text{B.C.} & u(0)=0, \quad u(1)=5 \end{cases}$$

Dirichlet B.C (Solution is known at the boundary)

$$\begin{cases} \text{DE} & -u''(x)=f(x), \quad x\in(0,1) \\ \text{B.C.} & u'(0)=1, \quad 2u'(1)=3 \end{cases}$$
→ Neumann B.C. (the derivative of the solution in certain direction is known)

$$\begin{cases} -u''(x)=f(x), \quad x\in(0,1) \\ u'(0)+2(u-3)=5, \quad u(1)=4 \end{cases}$$

Robin B.C. Dirichlet B.C.

[illegible] initial value problem (IVP).

Laplacian operator: in 2-dim: $\Delta u(x,y) = u_{xx}(x,y) + u_{yy}(x,y)$

in 3-dim: $\Delta u(x,y,z) = u_{xx} + u_{yy} + u_{zz}$

(we can write $\Delta = \nabla^2$)

in n-dim: $\Delta u(x_1, \dots, x_n) = u_{x_1x_1} + \dots + u_{x_nx_n}$

Ex: $u(x,y) = x^2 + y^3$, $u_x = 2x$, $u_{xx} = 2$, $u_y = 3y^2$, $u_{yy} = 6y$

$\Delta u = u_{xx} + u_{yy} = 2 + 6y$

Notation: If we denote $D = \frac{d}{dx}$, we can write $u'(x) = \frac{d}{dx}u = Du$

For example, we can write $u'' + 3u' + 2u = 0$

$D^2u + 3Du + 2u = 0$ or $(D^2 + 3D + 2)u = 0$

Different types of boundary conditions

We can supplement an ODE with boundary conditions (B.C.)

Ex: (i) $-u''(x) = f(x)$, $x \in (0,1)$

[illegible]

2.2 Some equations of mathematical physics

Let $u=u(x)$ or $u=u(x,y)$ or $u=u(x,y,z)$. The stationary diffusion equation (the stationary heat equation)

$$-u''(x)=f(x).$$

$$-\left(u_{xx}(x,y)+u_{yy}(x,y)\right)=f(x,y) \quad \text{or} \quad -\Delta u=f$$

$$-\left(u_{xx}+u_{yy}+u_{zz}\right)=f \quad \text{or} \quad -\Delta u=f$$

Let $u=u(x,t)$ or $u=u(x,y,t)$. The diffusion equation (the heat eq.)

$$u_t(x,t)-u_{xx}(x,t)=f(x,t)$$

$$u_t(x,y,t)-\left(u_{xx}(x,y,t)+u_{yy}(x,y,t)\right)=f(x,y,t) \quad \text{or} \quad u_t-\Delta u=f$$

and the wave equation is

$$u_{tt}-\Delta u=f$$

To complement the PDE, we consider boundary conditions or boundary + initial conditions.

(a) $\begin{cases} -u''(x)=f(x), & x\in(0,1) \\ u(0)=2,\ u'(1)=3 \end{cases}$

(b) $\begin{cases} -\Delta u=f, & \text{in } \Omega \\ u=0 & \text{on } \underbrace{\partial\Omega}_{\text{boundary of } \Omega} \end{cases}$

(c) $\begin{cases} u_t(x,t)-u_{xx}(x,t)=f(x,t), & x\in(0,L),\ t\in(0,T] \\ u'(0,t)=0,\quad u(1,t)=5t, & t\in(0,T] \\ u(x,0)=3x & x\in(0,L) \end{cases}$

(d) $\begin{cases} u_{tt}(x,t)-u_{xx}(x,t)=f(x,t), & x\in(0,L),\ t\in(0,T] \\ u(0,t)=\sin t,\quad u(1,t)=0 & t\in(0,T] \\ u(x,0)=x & x\in(0,L) \\ u_t(x,0)=0 & x\in(0,L) \end{cases}$

Chapter 3: Mathematical tools

3.1 Vector spaces (linear spaces)

Def A set V of functions or vectors is called a linear space (vector space), if for all $u, v, w \in V$ and all $\alpha, \beta \in \mathbb{R}$, we have

$(V,+)$ is a group
$$\begin{cases} (1)\ u + \alpha v \in V & \text{(linearity)} \\ (2)\ (u+v)+w = u+(v+w) & \text{(associativity)} \\ (3)\ \exists 0 \in V \text{ such that } u+0 = 0+u = u & \text{(Identity)} \\ (4)\ \forall u \in V,\ \exists (-u) \in V, \text{ such that } u+(-u)=0 & \text{(Invertibility)} \end{cases}$$

(5) $u+v = v+u$ (commutativity or abelian)

(6) $(\alpha+\beta)u = \alpha u + \beta u$

(7) $\alpha(u+v) = \alpha u + \alpha v$

(8) $\alpha(\beta u) = (\alpha\beta)u$

(9) $1u = u$

Ex1 $V = \mathbb{R}^2 = \{\langle x, y\rangle \mid x, y \in \mathbb{R}\}$ is a linear space

$V = \mathbb{R}^3$ is also a linear space. In general, $V = \mathbb{R}^n$, $n \in \mathbb{N}$ is a lin. sp.

For all $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \langle x_3, y_3\rangle \in V$ and all $\alpha, \beta \in \mathbb{R}$

(i) $\langle x_1, y_1\rangle + \alpha\langle x_2, y_2\rangle = \langle \underbrace{x_1+\alpha x_2}_{\in \mathbb{R}}, \underbrace{y_1+\alpha y_2}_{\in \mathbb{R}} \rangle \in V$

(ii) $\big(\langle x_1, y_1\rangle + \langle x_2, y_2\rangle\big) + \langle x_3, y_3\rangle = \langle x_1+x_2, y_1+y_2\rangle + \langle x_3, y_3\rangle$

$= \big\langle (x_1+x_2)+x_3, (y_1+y_2)+y_3 \big\rangle = \langle x_1+(x_2+x_3), y_1+(y_2+y_3)\rangle$

$= \langle x_1, y_1\rangle + \langle x_2+x_3, y_2+y_3\rangle = \langle x_1, y_1\rangle + \big(\langle x_2, y_2\rangle + \langle x_3, y_3\rangle\big)$

(iii) $0 = \langle 0, 0\rangle$

(iv) $\forall \langle x, y\rangle \in V$, $\exists \langle -x, -y\rangle \in V$, such that $\langle x, y\rangle + \langle -x, -y\rangle = \langle 0, 0\rangle$

⋮

Ex 2 $V = \{a_1 x + a_0 \mid a_1, a_0 \in \mathbb{R},\ a \leq x \leq b\}$

$= \{P(x) \mid P(x) \text{ is a polynomial in } x \text{ of degree} \leq 1,\ a \leq x \leq b\}$

$\overset{\text{denote}}{=} P^{(1)}_{[a,b]}$

for example: $3 \in P^{(1)}_{[a,b]}$, $-2x+5 \in P^{(1)}_{[a,b]}$, $\sqrt{2} \in P^{(1)}_{[a,b]}$

$x^2 - 1 \notin P^{(1)}_{[a,b]}$, $\sqrt{3}x - \frac{1}{3} \in P^{(1)}_{[a,b]}$

Note: one can show that $P^{(1)}_{[a,b]}$ is a linear space.

For all $p(x) = a_1 x + a_0$, $q(x) = b_1 x + b_0$, $r(x) = c_1 x + c_0$ and all $\alpha, \beta \in \mathbb{R}$:

(1) $p(x) + \alpha q(x) = (a_1 x + a_0) + \alpha(b_1 x + b_0) = (\underbrace{a_1 + \alpha b_1}_{\in \mathbb{R}})x + (\underbrace{a_0 + \alpha b_0}_{\in \mathbb{R}}) \in P^{(1)}_{[a,b]}$

(2)

⋮

Note: The space of all polynomials of degree $= 1$ on $[a,b]$ is not a linear space.

$V = \{a_1 x + a_0 \mid a_1, a_0 \in \mathbb{R},\ a_1 \neq 0,\ a \leq x \leq b\}$

Note that there is no $0 \in V$ such that $0 + p = p + 0 = p$

Ex 3 $P^{(3)}(a,b) =$ space of all polynomials of degree ≤ 3 on (a,b) is a linear space.

for example $p(x) = 4x^3 - \sqrt{2}x + \frac{1}{2}$, $x \in (a,b)$ is in $P^{(3)}_{(a,b)}$

$p(x) = 3x^2 + \frac{5}{7}x - \sqrt{3}$, $x \in (a,b)$

Ex 4 $P^{(q)}(\mathbb{R}) =$ space of all polynomials of degree $\leq q$ on $\mathbb{R}$

$= \{P(x) \mid P(x) \text{ is a polynomial of degree} \leq q,\ x \in \mathbb{R}\}$

$= \{a_q x^q + a_{q-1} x^{q-1} + \cdots + a_1 x + a_0 \mid a_q, \ldots, a_1, a_0 \in \mathbb{R},\ x \in \mathbb{R}\}$

Note: We can write $P^{(q)}(\mathbb{R}) = \{a_q t^q + a_{q-1} t^{q-1} + \cdots + a_1 t + a_0 \mid a_0, \ldots, a_q \in \mathbb{R},\ t \in \mathbb{R}\}$

For example: $P^{(2)}(a,b) = \{a_2 t^2 + a_1 t + a_0 \mid a_2, a_1, a_0 \in \mathbb{R},\ t \in (a,b)\}$

<u>Ex</u> $V = \{$all real-valued functions defined on $D\}$

$= \{ f : D \to \mathbb{R} \}$ where $D \subset \mathbb{R}$ is a linear space.

For any $f, g \in V, \alpha \in \mathbb{R}$; $(f+g)(x) = f(x) + g(x)$, $x \in D$

$(\alpha f)(x) = \alpha f(x)$, $x \in D$

So we defined sum of two functions, and the scalar product αf.
Now, we show that V is a linear (vector) space.
For all $f, g, h \in V$ and all $\alpha, \beta \in \mathbb{R}$:

(1) $(f+\alpha g)(x) = f(x) + (\alpha g)(x) = f(x) + \alpha g(x) \in \mathbb{R}$, $x \in D \Rightarrow f + \alpha g \in V$

(2) $[(f+g)+h](x) = (f+g)(x) + h(x) = (f(x)+g(x)) + h(x) = \underbrace{f(x) + (g(x)+h(x))}_{\in \mathbb{R}}$

$= f(x) + (g+h)(x) = [f + (g+h)](x)$, $x \in D$

$\Rightarrow (f+g)+h = f+(g+h)$

(3) function $0 \in V$ is defined (by $0(x) = \overset{\in \mathbb{R}}{0}$, $x \in D$).

(4) $\forall f \in V$, $\exists -f \in V$ defined by $(-f)(x) = \underbrace{-f(x)}_{\in \mathbb{R}}$, $x \in D$.

(5) $(f+g)(x) = f(x) + g(x) = \underbrace{g(x) + f(x)}_{\in \mathbb{R}} = (g+f)(x)$, $x \in D$

$\Rightarrow f+g = g+f$

(6) $((\alpha+\beta)u)(x) = (\alpha+\beta)u(x) = \underbrace{\alpha u(x) + \beta u(x)}_{\in \mathbb{R}} = (\alpha u + \beta u)(x)$, $x \in D$

$\Rightarrow (\alpha+\beta)u = \alpha u + \beta u$

(7) $(\alpha(u+v))(x) = \alpha(u+v)(x) = \alpha(u(x)+v(x)) = \overbrace{\alpha u(x) + \alpha v(x)}^{\in \mathbb{R}}$

$= (\alpha u + \alpha v)(x)$, $x \in D$ $\Rightarrow \alpha(u+v) = \alpha u + \alpha v$

(8) $(\alpha(\beta u))(x) = \alpha(\beta u)(x) = \alpha(\beta u(x)) = \underbrace{(\alpha\beta)u(x)}_{\in \mathbb{R}} = ((\alpha\beta)u)(x)$, $x \in D$

$\Rightarrow \alpha(\beta u) = (\alpha\beta)u$

(9) $(1u)(x) = \underbrace{1\,u(x)}_{\in \mathbb{R}} = u(x)$, $x \in D$ $\Rightarrow 1u = u$

Subspaces

Def A subset $U \subset V$, of a linear vector space V, is a subspace of V if

$$\alpha u + \beta v \in U, \quad \forall u, v \in U \text{ and } \forall \alpha, \beta \in \mathbb{R}$$

Ex5(a) Let V be a linear space, and $0 \in V$ be the zero vector.
$U = \{0\}$ is a subspace of V (it is called the **zero subspace**).

(b) V is a subspace of V.

$\{0\}$ and V are called the trivial subspaces of V.

E6 Let $V = P(\mathbb{R})$ = space of all polynomials with real coefficients.
Then, for any $n \in N = \{1, 2, \ldots\}$, $P^{(n)}(\mathbb{R})$ is a subspace of $P(\mathbb{R})$.

~~E7 Show tha~~ Let $n \in N$ be given. For any $p, q \in P^{(n)}(\mathbb{R})$ and any real numbers $\alpha, \beta \in \mathbb{R}$, we have

$$\alpha p(t) + \beta q(t) = \alpha(a_n t^n + \cdots + a_1 t + a_0) + \beta(b_n t^n + \cdots + b_1 t + b_0)$$

$$= \underbrace{(\alpha a_n + \beta b_n)}_{\in \mathbb{R}} t^n + \underbrace{(\alpha a_{n-1} + \beta b_{n-1})}_{\in \mathbb{R}} t^{n-1} + \cdots + \underbrace{(\alpha a_1 + \beta b_1)}_{\in \mathbb{R}} t + \underbrace{(\alpha a_0 + \beta b_0)}_{\in \mathbb{R}}$$

$$\in P^{(n)}(\mathbb{R}).$$

So $P^{(n)}(\mathbb{R})$ is a subspace of $P(\mathbb{R})$.

E7. Let $V = \mathbb{R}^3$ and $U = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$. Show that U is a subspace of V.

1. Obviously $U \subset V$.
2. Now, for any $u = (x_1, y_1, 0)$, $v = (x_2, y_2, 0) \in U$ and any real numbers $\alpha, \beta \in \mathbb{R}$, we have

$$\alpha u + \beta v = \alpha(x_1, y_1, 0) + \beta(x_2, y_2, 0) = (\overbrace{\alpha x_1 + \beta x_2}^{\in \mathbb{R}}, \overbrace{\alpha y_1 + \beta y_2}^{\in \mathbb{R}}, 0) \in U.$$

Def. A linear combination of vectors $v_1, \dots, v_n \in V$, is
$a_1 v_1 + \dots + a_n v_n$, for some real numbers $a_1, \dots, a_n \in \mathbb{R}$.

Ex8. Let $V = \mathbb{R}^3$ and $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Then for any $a, b \in \mathbb{R}$

$$a v_1 + b v_2 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} a+b \\ b \\ 0 \end{bmatrix}$$ is a linear combination of v_1 and v_2.

For example $3v_1 - \sqrt{2}\, v_2 = \begin{bmatrix} 3-\sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix}$ is a linear combination of v_1, v_2

Def. Let V be a linear space and vectors $v_1, \dots, v_p \in V$ are given. The space

$$\text{Span}\{v_1, \dots, v_p\} = \{a_1 v_1 + \dots + a_p v_p \mid a_1, \dots, a_p \in \mathbb{R}\}$$
$$= \{\text{all linear combinations of } v_1, \dots, v_p\}$$

is the spanned (generated) space by vectors $v_1, \dots, v_p$.

Note: one can show that $\text{Span}\{v_1, \dots, v_p\}$ is a subspace of V.

Ex9 Let V be a linear space and $v_1, v_2 \in V$ are given.
Show that $U = \text{Span}\{v_1, v_2\}$ is a subset of V.

We note that $U = \text{Span}\{v_1, v_2\} = \{a_1 v_1 + a_2 v_2 \mid a_1, a_2 \in \mathbb{R}\}$, so any $u \in U$ is in the form

$$u = a_1 v_1 + a_2 v_2, \text{ for some } a_1, a_2 \in \mathbb{R}$$

1. For any $u \in U$, we have $u = a_1 v_1 + a_2 v_2 \in V$.
   So $U \subset V$ (is a subset).

2. For any real numbers $\alpha, \beta \in \mathbb{R}$, we have any vectors $u, v \in U$ and

$$\alpha u + \beta v = \alpha(a_1 v_1 + a_2 v_2) + \beta(b_1 v_1 + b_2 v_2) = \underbrace{(\alpha a_1 + \beta b_1)}_{\in \mathbb{R}} v_1 + \underbrace{(\alpha a_2 + \beta b_2)}_{\in \mathbb{R}} v_2 \in U$$

Ex 10 Let $V = \mathbb{R}^4$ and $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Show the vectors in $U = \text{Span}\{v_1, v_2\}$.

If $u \in U$, then $u = a_1 v_1 + a_2 v_2$ for some $a_1, a_2 \in \mathbb{R}$.

$$a_1 v_1 + a_2 v_2 = \begin{bmatrix} a_1 - 2a_2 \\ -a_1 + a_2 \\ a_1 \\ a_2 \end{bmatrix} \Rightarrow U = \left\{ \begin{bmatrix} a_1 - 2a_2 \\ -a_1 + a_2 \\ a_1 \\ a_2 \end{bmatrix} \middle| \; a_1, a_2 \in \mathbb{R} \right\}$$

Ex 11 Let $V = \mathbb{R}^3$ and $v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3$.

For what value(s) of z will $y = \begin{bmatrix} -4 \\ 3 \\ z \end{bmatrix}$ is in $\text{Span}\{v_1, v_2, v_3\}$?

$y \in \text{Span}\{v_1, v_2, v_3\}$ if there are real numbers $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_1 v_1 + a_2 v_2 + a_3 v_3 = y$. That is, this linear system should have a (unique) solution:

nonzero

$$\begin{cases} a_1 + 5a_2 - 3a_3 = -4 \\ -a_1 - 4a_2 + a_3 = 3 \\ -2a_1 - 7a_2 \qquad = z \end{cases}$$

$$\overset{E_2 + E_1 \to E_2}{\underset{E_3 + 2E_1 \to E_3}{\Longrightarrow}} \begin{cases} a_1 + 5a_2 - 3a_3 = -4 \\ a_2 - 2a_3 = -1 \\ 3a_2 - 6a_3 = z - 8 \end{cases} \xrightarrow{E_3 - 3E_2 \to E_3} \begin{cases} a_1 + 5a_2 - 3a_3 = -4 \\ a_2 - 2a_3 = -1 \\ 0 + 0 = z - 5 \end{cases}$$

$$\Rightarrow z - 5 = 0 \Rightarrow \boxed{z = 5}$$

Ex For $t \in [a,b]$, $\text{Span}\{1, t, t^2\} = \{a_2 t^2 + a_1 t + a_0 \mid a_2, a_1, a_0 \in \mathbb{R}, t \in [a,b]\}$

$$= P^{(2)}[a,b]$$

So for example: $5t^2 - 3t + 5 = \underbrace{5}_{\in \mathbb{R}} t^2 + \underbrace{(-3)}_{\in \mathbb{R}} t + \underbrace{5}_{\in \mathbb{R}}(1) \in P^{(2)}_{[a,b]}$

$3(2t - t^2) + 4t^2 - 1 = t^2 + 6t - 1 = \underbrace{(1)}_{\in \mathbb{R}} t^2 + \underbrace{6}_{\in \mathbb{R}} t + \underbrace{(-1)}_{\in \mathbb{R}} 1 \in P^{(2)}_{[a,b]}$

Linearly independent sets and bases

bas

Linjärt oberoende

Assume V is a given linear space.

Def A set of vectors $\{v_1, \dots, v_p\}$ in V is linearly independent if the vector equation

$$a_1 v_1 + \dots + a_p v_p = 0 \qquad (1)$$

has only the trivial solution, $a_1 = \dots = a_p = 0$.

The set $\{v_1, \dots, v_p\}$ is linearly dependent, if (1) has a nontrivial solution. That is, there are some real numbers $a_1, \dots, a_p$ such that at least one of them is nonzero.

Ex Let $V = P^{(3)}_{[-1,2]}$. (a) The set $\{\underset{p_1(t)}{1}, \underset{p_2(t)}{t}, \underset{p_3(t)}{t^2}\}$ is linearly independent.

Consider the vector equation (linear combination): $a_1 p_1(t) + a_2 p_2(t) + a_3 p_3(t) = 0$, $t \in [-1, 2]$

$$a_1(1) + a_2 t + a_3 t^2 = 0$$

Then

$$\begin{aligned} t=0 &\rightarrow \begin{cases} a_1 = 0 \\ a_1 + a_2 + a_3 = 0 \\ a_1 - a_2 + a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 + a_3 = 0 \\ -a_2 + a_3 = 0 \end{cases} \xrightarrow{+} 2a_3 = 0 \Rightarrow a_3 = 0 \\ t=1 &\rightarrow \\ t=-1 &\rightarrow \end{aligned}$$

$$\Rightarrow -a_2 + a_3 = 0 \Rightarrow -a_2 + 0 = 0$$

$$\Rightarrow a_2 = 0$$

Hence $a_1 = a_2 = a_3 = 0$, that is $\{1, t, t^2\}$ is linearly dependent.

(b) The set $\{t, t-1\}$ is linearly independent.

Consider

$$a_1 t + a_2(t-1) = 0 \Rightarrow (a_1 + a_2)t - a_2 = 0$$

$$\begin{aligned} t=0 &\rightarrow \begin{cases} -a_2 = 0 \\ a_1 + a_2 - a_2 = 0 \end{cases} \Rightarrow \begin{cases} a_2 = 0 \\ a_1 = 0 \end{cases} \\ t=1 &\rightarrow \end{aligned}$$

(c) The set $\{1, t, 2+3t\}$ is linearly dependent.

Consider $a_1 p_1(t) + a_2 p_2(t) + a_3 p_3(t) = 0, \quad t \in [-1, 2]$

$\Rightarrow \quad a_1 + a_2 t + a_3(2+3t) = 0, \quad t \in [-1, 2]$

So

$$\begin{array}{l} t=0 \to \\ t=-1 \to \\ t=2 \to \end{array} \begin{cases} a_1 + 2a_3 = 0 \\ a_1 - a_2 - a_3 = 0 \\ a_1 + 2a_2 + 8a_3 = 0 \end{cases} \Longrightarrow \begin{cases} a_1 = -2a_3 \\ -a_2 - 3a_3 = 0 \\ 2a_2 + 6a_3 = 0 \end{cases} \Longrightarrow \begin{cases} a_2 = -3a_3 \\ a_2 = -3a_3 \end{cases}$$

take $a_3 = 1$ $\Longrightarrow$ $a_2 = -3$

$a_1 = -2a_3 = -2$

So $a_1 = -2,\ a_2 = -3,\ a_3 = 1$. That is the set is linearly dependent

Ex Show that $f_1(x) = \sin^2 x$, $f_2(x) = 1$ and $f_3(x) = \cos 2x$ are linearly dependent.

$$f_1(x) = \sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2}(1) - \frac{1}{2}(\cos 2x) = \frac{1}{2} f_2(x) - \frac{1}{2} f_3(x)$$

$$\Rightarrow \quad f_1(x) - \frac{1}{2} f_2(x) + \frac{1}{2} f_3(x) = 0, \quad x \in \mathbb{R}$$

Def. Let V be a linear (vector) space and $\{v_1, \dots, v_p\}$ be a set in V. We say that $\{v_1, \dots, v_p\}$ is a basis for V, if

(i) $\{v_1, \dots, v_p\}$ is linearly independent.

(ii) $\mathrm{Span}\{v_1, \dots, v_p\} = V$. (That is, every vector $v \in V$ can be written as a linear combination of $v_1, \dots, v_p$)

(We say that $\{v_1, \dots, v_p\}$ generates V)

($\forall v \in V$, $v = a_1 v_1 + \dots + a_p v_p$, for some $a_1, \dots, a_p \in \mathbb{R}$ and at least one of $a_1, \dots, a_p$ is nonzero.)

Def. If $\{v_1, \dots, v_p\}$ is a basis for V, we say that $\underset{\text{dimension}}{\dim}(V) = p$.

Ex $\{e_1, e_2, e_3\}$ is a basis for $V = \mathbb{R}^3$, where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(Note that $\{e_1, e_2, e_3\}$ is called the standard basis for $\mathbb{R}^3$)

(i) Consider

$$a_1 e_1 + a_2 e_2 + a_3 e_3 = 0$$

$$\Rightarrow \begin{cases} a_1 + 0 + 0 = 0 \\ 0 + a_2 + 0 = 0 \\ 0 + 0 + a_3 = 0 \end{cases} \Rightarrow \begin{matrix} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{matrix} \Rightarrow \text{linearly independent}$$

(ii) For every vector $v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \in \mathbb{R}^3$, we have

$$v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$$

$$\Rightarrow \text{Span}\{e_1, e_2, e_3\} = \mathbb{R}^3$$

Note: In $\mathbb{R}^n$ and

Ex $\{1, t, t^2\}$ is a basis for $P^{(2)}(\mathbb{R})$.

(i) Consider $a_1(1) + a_2 t + a_3 t^2 = 0, \quad t \in \mathbb{R}$

$$\Rightarrow \begin{matrix} t=0 \\ t=1 \\ t=2 \end{matrix} \begin{cases} a_1 = 0 \\ a_1 + a_2 + a_3 = 0 \\ a_1 + 2t + 4a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases} \Rightarrow \text{linearly indep.}$$

(ii) Any polynomial $p(t) = a_2 t^2 + a_1 t + a_0 \in P^{(2)}(\mathbb{R})$
can be written as

$$p(t) = a_2(t^2) + a_1(t) + a_0(1)$$

that is a linear combination of $t^2, t, 1$. So $P^{(2)}(\mathbb{R}) = \text{Span}\{1, t, t^2\}$

Note: In $V = \mathbb{R}^n$ and $V = P^{(n)}$ or any linear space with $\dim(V) = n$

We only need to check condition (i) or (ii) to find out if $\{v_1, \ldots, v_n\}$ is a basis for V.

Some other space of functions

$$C([a,b]) = \text{Space of continuous functions on } [a,b]$$
$$= \{ f : [a,b] \to \mathbb{R} \mid f \text{ continuous on } [a,b] \}$$

$$C^1([a,b]) = \text{Space of continuously differentiable functions on } [a,b]$$
$$= \{ f : [a,b] \to \mathbb{R} \mid f \text{ and } f' \text{ are continuous on } [a,b] \}$$

$$C^k([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f, f', \ldots, f^{(k)} \text{ are continuous on } [a,b] \}$$

Ex $\quad C^2((0,4)) = \{ f : (0,4) \to \mathbb{R} \mid f, f', f'' \text{ are continuous on } (0,4) \}$

Scalar product (inner product)

Let V be a linear space.

<u>Def</u> A scalar (inner) product on V, is a real valued operator on $V \times V$ (such as $\langle u,v\rangle : V\times V \longrightarrow \mathbb{R}$) such that, for all $u, v, w \in V$ and all $\alpha \in \mathbb{R}$

(i) $\langle u,v\rangle = \langle v,u\rangle$

(ii) $\langle u+\alpha v, w\rangle = \langle u,w\rangle + \alpha\langle v,w\rangle$

(iii) $\langle v,v\rangle \geq 0$

(iv) $\langle v,v\rangle = 0 \iff v=0$

<u>Def</u> A linear space V is called a scalar (inner) product space, if V is associated with a scalar product $\langle\cdot,\cdot\rangle$ defined on $V\times V$

<u>Ex.</u> $V=\mathbb{R}^2$. Then for any $u=(u_1,u_2)$, $v=(v_1,v_2) \in \mathbb{R}^2$, we have an (inner) scalar product

$$\langle u,v\rangle = \langle (u_1,u_2),(v_1,v_2)\rangle = u_1v_1 + u_2v_2$$

We usually used to write

$$u\cdot v = u_1v_1 + u_2v_2$$

Here we use $\langle u,v\rangle$ instead of $u\cdot v$.

Since : (i), (ii), (iii), (iv)

<u>Ex</u> $V=\mathbb{R}^n$. Then for all $u=(u_1,\dots,u_n)$, $v=(v_1,\dots,v_n) \in V$

$$\langle u,v\rangle = u_1v_1 + \dots + u_nv_n = \sum_{i=1}^{n} u_iv_i$$

<u>Ex</u> $V = P^{(n)}_{[a,b]}$. Then for every $p,q \in P^{(n)}_{[a,b]}$

$$\langle p,q\rangle = \int_a^b p(t)\,q(t)\,dt$$ is a scallar (inner) product

For all $p, q, r \in P^{(n)}_{[a,b]}$ and all $\alpha \in \mathbb{R}$:

(i) $\langle p, q \rangle = \int_a^b p(t) q(t)\, dt = \int_a^b q(t) p(t)\, dt = \langle q, p \rangle$

(ii) $\langle p + \alpha q, r \rangle = \int_a^b \big(p(t) + \alpha q(t)\big) r(t)\, dt =$

$= \langle p, r \rangle + \alpha \langle q, r \rangle$

(iii) $\langle p, p \rangle = \int_a^b \underbrace{(p(t))^2}_{\geq 0} dt \geq 0$

(iv) If $\langle p, p \rangle = \int_a^b (p(t))^2\, dt = 0 \overset{(p(t))^2 \geq 0}{\Longrightarrow} (p(t))^2 = 0$, for $t \in [a,b]$

$\Rightarrow p(t) = 0$, for $t \in [a,b]$

If $p = 0 \Longrightarrow \langle p, p \rangle = \int_a^b 0^2\, dt = 0$

Def Let V be a linear space with a scalar product. Two vectors $u, v \in V$ are called orthogonal if $\langle u, v \rangle = 0$. (We also denote $u \perp v$)

EX (a) $V = \mathbb{R}^2$, $u = (2, 0)$, $v = (0, -3)$ are orthogonal:

$\langle u, v \rangle = 2(0) + 0(-3) = 0$

$u = (-1, 2)$, $v = (4, 2)$ are orthogonal

$\langle u, v \rangle = (-1)(4) + (2)(2) = 0$

(b) $V = P^{(3)}_{[-1,1]}$ with $\langle p, q \rangle = \int_{-1}^1 p(x) q(x)\, dx$. Polynomials $p(x) = x$ and $q(x) = x^2$ are orthogonal:

$\langle p, q \rangle = \int_{-1}^1 x\, x^2\, dx = \frac{x^4}{4}\Big]_{-1}^1 = \frac{1}{4} - \frac{1}{4} = 0$

Def Let $V=\mathbb{R}^n$. The L_2-norm of $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ is defined by

$$\|x\|=\sqrt{\langle x,x\rangle}=\sqrt{(x_1)^2+(x_2)^2+\dots+(x_n)^2}. = \text{Eucledian length}$$

(It is also called Eucledian norm)

Def Let $u:[a,b]\to\mathbb{R}$ be a square integrable function on $[a,b]$, that is $\int_a^b |u(x)|^2\,dx<\infty$

then the L_2-norm of u is defined by

$$\|u\|=\sqrt{\int_a^b |u(x)|^2\,dx}.$$

We can define it for more general cases.

If $u:\Omega\to\mathbb{R}$ with $\Omega\subset\mathbb{R}^n$ be square integrable over Ω, then

$$\|u\|=\|u\|_{L_2(\Omega)}=\sqrt{\int_\Omega |u(x)|^2\,dx}$$

Def $L_2(\Omega)=\left\{u:\Omega\to\mathbb{R} \;\middle|\; \|u\|_{L_2(\Omega)}<\infty\right\}$

Ex (a) $u=\begin{bmatrix}1\\-2\\0\end{bmatrix} \Rightarrow \|u\|=\sqrt{(1)^2+(-2)^2+(0)^2}=\sqrt{5}$

(b) $u(x)=x^2,\ x\in[-1,2] \Rightarrow \|u\|_{L_2([-1,2])}=\sqrt{\int_{-1}^{2}|x^2|^2\,dx}$

$$=\sqrt{\int_{-1}^{2}x^4\,dx}=\sqrt{\frac{33}{5}}$$

(c) Let $\langle u,v\rangle=\int_\Omega uv\,dx$, then $\|u\|=\sqrt{\langle u,u\rangle}$.

~~Triangle inequality: Assume that the L_2-norm of u and v exist, then~~

$$\cancel{\|u+v\|\le\|u\|+\|v\|}$$

~~Note: this is similar to $|a+b|\le|a|+|b|$ for real numbers~~

Caucy-Schwarz' inequality: Assume that L_2-norm of u and v exists, then $|\langle u,v\rangle| \leq \|u\|\,\|v\|$.

Proof. For all $\lambda \in \mathbb{R}$ we have

$$0 \leq \|u-\lambda v\|^2 = \langle u-\lambda v, u-\lambda v\rangle = \langle u,u\rangle - \lambda\langle v,u\rangle - \lambda\langle u,v\rangle + \lambda^2\langle v,v\rangle$$

$$= \|u\|^2 - 2\lambda\langle u,v\rangle + \lambda^2\|v\|^2$$

This is a polynomial of degree 2 w.r.t. λ that is nonnegative. So we should have

$$\Delta = b^2 - 4ac = (-2\langle u,v\rangle)^2 - 4\|u\|^2\|v\|^2 \leq 0$$

$$\Rightarrow 4|\langle u,v\rangle|^2 \leq 4\|u\|^2\|v\|^2 \Rightarrow |\langle u,v\rangle| \leq \|u\|\,\|v\|.$$

Triangle inequality Assume that the L_2-norm of u and v exist. Then $\|u+v\| \leq \|u\| + \|v\|$

Proof. $$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \langle v,u\rangle + \langle u,v\rangle + \langle v,v\rangle$$

$$= \|u\|^2 + 2\langle u,v\rangle + \|v\|^2 \leq \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2$$

$$\overset{C\text{-}S}{\leq} \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2$$

$$\Rightarrow \|u+v\| \leq \|u\| + \|v\|$$

Note: We knew the triangle ineq. for real numbers: $|a+b| \leq |a| + |b|$.

3.2 Spaces of differentiable functions

Let $\Omega = (a,b)$ be an open interval. Then $\bar{\Omega} = \overline{(a,b)} = [a,b]$ is the clouser set of Ω.

Notes: $\overline{(a,b]} = [a,b]$, where

$$(a,b] = \{x \mid a < x \leq b\}$$

Let $\Omega = (a,b)$, then

$$C(\bar{\Omega}) = C([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is continuous on } [a,b] \}$$

$$C^1(\bar{\Omega}) = C^1([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f, f' \text{ are continuous on } [a,b] \}$$

$$C^2(\bar{\Omega}) = C^2([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f, f', f'' \text{ are cont. on } [a,b] \}$$

We can define **supremum-norm** by

$$\|u\|_{C(\bar{\Omega})} = \sup_{x \in \bar{\Omega}} |u(x)| \xrightarrow[\text{closed set}]{\bar{\Omega} \text{ is a}} \max_{x \in \bar{\Omega}} |u(x)|$$

$$\|u\|_{C^1(\bar{\Omega})} = \sup_{x \in \Omega} |u(x)| + \sup_{x \in \Omega} |u'(x)| = \max_{x \in \Omega} |u(x)| + \max_{x \in \Omega} |u'(x)|$$

$$\|u\|_{C([a,b])} = \max_{x \in [a,b]} |u(x)| \quad , \quad \|u\|_{C^1([a,b])} = \max_{x \in [a,b]} |u(x)| + \max_{x \in [a,b]} |u'(x)|$$

Ex. $f(x) = x^2 + \sin x$, $x \in [0,8]$

$$\|f\|_{C^1([0,8])} = \max_{x \in [0,8]} |x^2 + \sin x| + \max_{x \in [0,8]} |2x + \cos x| \leq |64+1| + |16+1| = 82$$

Note: We can generalize the spaces above for $\Omega \subset \mathbb{R}^n$, $n \geq 1$.

Note: If $u = u(x,t)$, $x \in \Omega$, $t \in [0,\infty)$, by $u \in C^1([0,\infty); C^2(\Omega))$

We mean that $\begin{cases} u, u_t \text{ are continuous for } t \in [0,\infty) \\ u, u_x, u_{xx} \text{ " " " } x \in \Omega \end{cases}$

3.3. Spaces of integrable functions

Let $\Omega = (a,b) \subset \mathbb{R}$ (We can consider a more general open set $\Omega \subset \mathbb{R}^n$, $n \geq 1$). Then, for ~~[illegible]~~ $p \geq 1$,

$$L_2(\Omega) = \{u : \Omega \to \mathbb{R} \,/\, \int_\Omega |u(x)|^2 dx < \infty\}$$

$$L_p(\Omega) = \{u : \Omega \to \mathbb{R} \,/\, \int_\Omega |u(x)|^p dx < \infty\}, \quad 1 \leq p < \infty$$

$$L_\infty(\Omega) = \{u : \Omega \to \mathbb{R} \,/\, \sup_{x \in \Omega} |u(x)| < \infty\}$$

We have the corresponding norms

$$\|u\|_{L_p(\Omega)} = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}, \quad 1 \leq p < \infty$$

$$\|u\|_{L_\infty(\Omega)} = \sup_{x \in \Omega} |u(x)|$$

Note: Only for $L_2(\Omega)$-norm we have a corresponding scalar product. That is

$$\langle u, v \rangle = \int_\Omega u(x) v(x) dx$$

Then $\|u\|_{L_2(\Omega)} = \sqrt{\langle u, u \rangle} = \left(\int_\Omega |u(x)|^2 dx\right)^{1/2}$

Ex. Let $u(x) = 2x + \sin x$, $x \in (0, \frac{\pi}{2}) = \Omega$

$$\|u\|_{L_1(\Omega)} = \int_0^{\frac{\pi}{2}} |u(x)| dx = \int_0^{\frac{\pi}{2}} |2x + \sin x| dx = \int_0^{\frac{\pi}{2}} (2x + \sin x) dx$$

$$= \left(x^2 - \cos x\right]_0^{\frac{\pi}{2}} = \left(\frac{\pi^2}{4} - 0\right) - (0 - 1) = \frac{\pi^2}{4} + 1$$

$$\|u\|_{L_2(\Omega)} = \left(\int_0^{\frac{\pi}{2}} |2x+\sin x|^2 dx\right)^{\frac{1}{2}} = \left(\int_0^{\frac{\pi}{2}} (4x^2+4x\sin x+\sin^2 x) dx\right)^{\frac{1}{2}}$$

$$= \left(\int_0^{\frac{\pi}{2}} 4x^2 dx + \int_0^{\frac{\pi}{2}} 4x\sin x\, dx + \int_0^{\frac{\pi}{2}} \sin^2 x\, dx\right)^{\frac{1}{2}}$$

$$= \left(\frac{\pi^3}{6} + 4 + \frac{\pi}{4}\right)^{\frac{1}{2}}$$

$$\int_0^{\frac{\pi}{2}} 4x^2 dx = \frac{4x^3}{3}\Big|_0^{\frac{\pi}{2}} = \frac{\pi^3}{6}$$

$$\int_0^{\frac{\pi}{2}} 4x\sin x\, dx = \left\{\begin{array}{ll} u=4x \longrightarrow & du = 4dx \\ dv = \sin x\, dx & v = -\cos x \end{array}\right\} = \underbrace{-4x\cos x\Big|_0^{\frac{\pi}{2}}}_{=0} + 4\int_0^{\frac{\pi}{2}} \cos x\, dx$$

$$= \underbrace{-4\frac{\pi}{2}\cos\frac{\pi}{2} - (-4(0)\cos 0)}_{=0} + 4\sin x\Big]_0^{\frac{\pi}{2}} = 4$$

$$\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx = \left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

Chapter 4: Polynomial Approximation in 1D

We have an introduction to approximate a function by a polynomials. And approximation of the solution to DE.
We use finite difference method (FDM) for time variable, and finite element method (FEM) for space variable.

4.1 Overture

- **Initial Value Problem (IVP):**

Model of population dynamics ($u=u(t)$ is the size of population at time t)

$$\begin{cases} \dot{u}(t)-\lambda u(t)=0, & 0<t<T \\ u(0)=u_0 \end{cases} \qquad (4.1.1)$$

Note that λ is a constant (usually positive).

Exact solution: $\dot{u}(t)=\lambda u(t)$

$$\frac{du}{dt}=\lambda u \Rightarrow \frac{du}{u}=\lambda dt \Rightarrow \ln|u|=\lambda t+c$$

$$\Rightarrow u(t)=e^{\lambda t+c}=e^{c}e^{\lambda t}=Ce^{\lambda t} \quad \text{general solution}$$

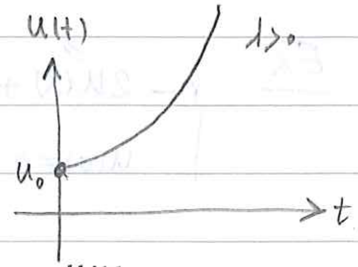
$$\xrightarrow{u(0)=u_0} u(0)=u_0=Ce^{0}=C \Rightarrow C=u_0$$

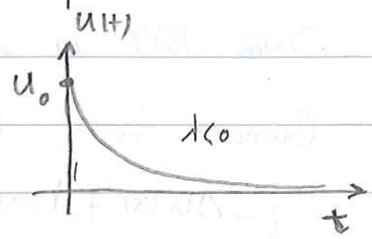
$$\Rightarrow \boxed{u(t)=u_0e^{\lambda t}} \quad \text{particular solution}$$

Note: If $\lambda=0 \Rightarrow u(t)=u_0, \quad 0\le t\le T$

If $\lambda>0 \Rightarrow \lim_{t\to\infty} u(t)=\infty$

if $\lambda<0 \Rightarrow \lim_{t\to\infty} u(t)=0$





PDEs in bounded domains

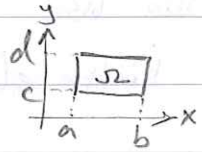
- **Boundary Value Problem (BVP):**

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.

For example: $\Omega = (a,b)$ in 1D ($\mathbb{R}$)

$\Omega = (a,b) \times (c,d)$ in 2D ($\mathbb{R}^2$)

$= \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$



Some BVP in 1D (so called two-point BVP);

$$\begin{cases} -u''(x) = f(x), & x \in \Omega = (a,b) \\ u(a) = 0,\ u(b) = 0 \end{cases}$$

homogeneous Dirichlet $\underbrace{\text{boundary condition}}_{\text{B.C.}}$

$$\begin{cases} -u''(x) = f(x), & x \in \Omega \\ u(a) = u_a,\ u(b) = u_b \end{cases}$$

$\underbrace{\text{nonhomogeneous}}$ Dirichlet B.C.

$u_a \neq 0$ or $u_b \neq 0$ (for example $u_a = 2$)

More general form: diffusion, convection, absorption

$$\begin{cases} -(a(x)\, u'(x))' + b(x)\, u'(x) + c(x)\, u(x) = f(x), & x \in \Omega \\ u(a) = u_a, \quad \lambda(b)\, u'(b) = u_b \end{cases}$$

Dirichlet B.C   Neumann B.C.

Where a, b, c are smooth functions (!?)

<u>Ex</u>

$$\begin{cases} -2u''(x) + 4u'(x) + x^2 u(x) = \sin x, & x \in \Omega = (0,1) \\ u(0) = 5, \quad 2u'(1) = 6 \end{cases}$$

Some BVP in 2D; ($\Omega \subset \mathbb{R}^2$, bounded and open and convex)

General form: ($X = (x,y)$)

$$\begin{cases} -\Delta u(X) + b(X) \cdot \nabla u(X) + c(X)\, u(X) = f(X), & X \in \Omega \\ u(X) = 0 & X \in \partial\Omega \end{cases}$$

homog. Dirichlet B.C.

Note that Δ is the Laplace operator

$$\Delta u(x,y) = u_{xx} + u_{yy}$$

$$\Delta u(x,y,z) = u_{xx} + u_{yy} + u_{zz}$$

$$\Delta u(x_1, \dots, x_n) = u_{x_1 x_1} + \dots + u_{x_n x_n}$$

$$b \cdot \nabla u = (b_1, \dots, b_n) \cdot (u_{x_1}, \dots, u_{x_n}) = b_1 u_{x_1} + \dots + b_n u_{x_n}$$

In 2D: $b \cdot \nabla u = (b_1, b_2) \cdot (u_x, u_y) = b_1 u_x + b_2 u_y$

- Initial-Boundary Value Problem (IBVP):

For example, the heat equation in 1D. That is, we look for a solution $u = u(x,t)$, such that

$$\begin{array}{ll} \text{D.E.} & u_t(x,t) - u_{xx}(x,t) = 0, \quad x \in \underbrace{\Omega}_{=(a,b)}, \ t > 0 \\ \text{B.C.} & u(a,t) = 0, \ u(b,t) = 0, \quad t > 0 \\ \text{I.C.} & u(x,0) = u_0(x) \quad x \in \Omega \end{array}$$

Note that the B.C. is homogeneous Dirichlet

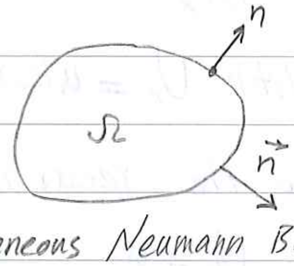
Another example

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = x \sin t, & x \in (0,L), \ t > 0 \\ u(0) = 0, \ u_x(L,t) = 2t, & t > 0 \\ u(x,0) = 3x & x \in (0,L) \end{cases}$$

Dirichlet B.C

Neumann B.C.

More general ($\Omega \subset \mathbb{R}^n$):



$$\begin{cases} u_t - \Delta u = f, & x \in \Omega, \ t > 0 \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \ t > 0 \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}$$

homogeneous Neumann B.C.

$$\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n}$$

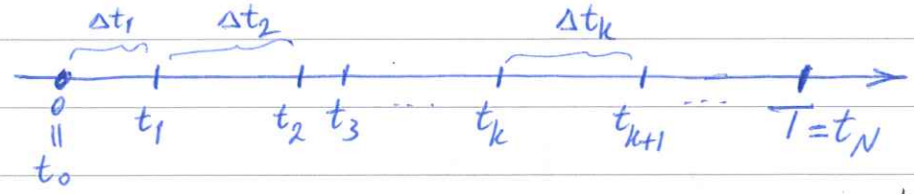
the unit normal vector

Numerical solutions of IVP (Forward Euler method)

We formulate the explicit (forward) Euler method, to find an approximate solution to the IVP (4.1.1)

$$\begin{cases} \dot{u}(t) = \lambda u(t), & 0<t<T \\ u(0) = u_0 \end{cases}$$

First we partition the time domain $[0,T]$ into N subintervals,



We know that

$$\dot{u}(t_k) = \lim_{\Delta t_k \to 0} \frac{u(t_k+\Delta t_k) - u(t_k)}{\Delta t_k} \overset{\Delta t_k \text{ small}}{\approx} \frac{u(\overbrace{t_k+\Delta t_k}^{=t_{k+1}}) - u(t_k)}{\Delta t_k}$$

So we have, from (4.1.1) with $t = t_k$:

$$\frac{u(t_{k+1}) - u(t_k)}{\Delta t_k} \approx \lambda u(t_k), \qquad \text{for } k = 0, 1, \dots, N-1$$

Now, if we call U to be the approximation of u, and denote

$U_k \approx u(t_k), \quad k = 0, 1, \dots, N$

then we have

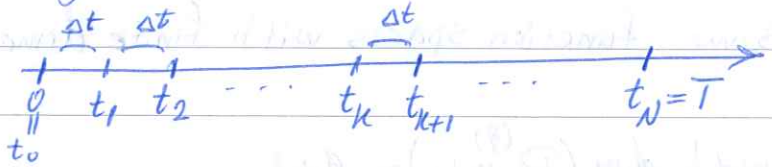
$$\frac{U_{k+1} - U_k}{\Delta t_k} = \lambda U_k \implies U_{k+1} = (1+\lambda \Delta t_k) U_k, \quad k = 0, 1, \dots, N-1$$

With $U_0 = u(t_0) = u(0) = u_0$.

So the recursion formula is

$$\begin{cases} U_0 = u_0 \\ U_{k+1} = (1+\lambda \Delta t_k) U_k, & k = 0, 1, \dots, N-1 \end{cases}$$

Note: For uniform partition, all subintervals have the same length Δt, and $t_k = t_0 + k\Delta t = k\Delta t$



In this case:

$$U_{k+1} = (1+k\Delta t)U_k = (1+k\Delta t)\underbrace{(1+k\Delta t)U_{k-1}}_{=U_k} = (1+k\Delta t)^2 U_{k-1}$$

$$= \cdots = (1+\lambda\Delta t)^{k+1} u_0$$

Note: For nonhomogeneous ODE (IVP)

$$\begin{cases} \dot{u}(t) - \lambda u(t) = f(t), & 0 < t < T \\ u(0) = u_0 \end{cases}$$

we similarly have, ~~assume uniform parti~~

$$\frac{U_{k+1} - U_k}{\Delta t_k} - \lambda U_k = f(t_k)$$

$$\Rightarrow U_{k+1} = (1+\lambda\Delta t_k)U_k + \Delta t_k f(t_k), \quad k = 0, 1, \ldots, N-1$$

Hence

$$\begin{cases} U_0 = u_0 \\ U_{k+1} = (1+\lambda\Delta t_k)U_k + \Delta t_k f(t_k), & k = 0, 1, \ldots, N-1 \end{cases}$$

Finite dimensional linear space of functions on an interval

We introduce some function spaces with finite dimension.

① $P^{(q)}(a,b)$, with $\dim(P^{(q)}(a,b)) = q+1$
A possible basis for $P^{(q)}(a,b)$ is $\{1, x, x^2, \dots, x^q\}$.
Note that they may not be orthogonal, and can be orthogonalized by the Gram-Schmidt procedure.

② Periodic orthogonal bases on $[0,L]$, such as trigonometric polynomials:

$$T^N = \left\{ f \;\middle|\; f(x) = \sum_{n=0}^{N} \left[a_n \cos\left(\frac{2n\pi}{L}x\right) + b_n \sin\left(\frac{2n\pi}{L}x\right) \right] \right\}$$

③ $P^{(q)}(a,b)$, with $\dim(P^{(q)}(a,b)) = q+1$, but with basis of Lagrange polynomials $\{\lambda_0(x), \lambda_1(x), \dots, \lambda_q(x)\} \subset P^{(q)}(a,b)$
Lagrange polynomials are continuous of degree $= q$, associated to $(q+1)$ distinct points $x_0 < x_1 < \dots < x_q$ in $[a,b]$, such that

$$\lambda_i(x_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}, \quad \lambda_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{q} \frac{x - x_j}{x_i - x_j}$$

$$\Rightarrow \lambda_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_q)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_q)}$$

One can show that $\{\lambda_i\}_{i=0}^{q}$ are linearly independent, and so

$$P^{(q)}(a,b) = \operatorname{Span}\{\lambda_0, \lambda_1, \dots, \lambda_q\}$$

Then, a polynomial $P \in P^{(q)}(a,b)$ with values $p_i = P(x_i)$ at the nodes $x = x_i$ $(i = 0, 1, \dots, q)$ is uniquely expressed by

$$P(x) = p_0\lambda_0(x) + p_1\lambda_1(x) + \dots + p_q\lambda_q(x) = \sum_{i=0}^{q} p_i\lambda_i(x)$$

<u>Ex.</u> For $\mathbb{P}^{(1)}(a,b)$, we choose $1+1=2$ distinct points (nodes)

$x_0 = a$, $x_1 = b$.

$$\lambda_0(x) = \frac{x-x_1}{x_0-x_1} = \frac{x-b}{a-b}, \quad \lambda_1(x) = \frac{x-x_0}{x_1-x_0} = \frac{x-a}{b-a}$$

Then any polynomial $P \in \mathbb{P}^{(1)}(a,b)$ can be written as

$$P(x) = P(a)\lambda_0(x) + P(b)\lambda_1(x) = P(a)\frac{x-b}{a-b} + P(b)\frac{x-a}{b-a}$$

For $\mathbb{P}^{(1)}(0,1)$, we have $x_0=0$, $x_1=1$, $\lambda_0(x) = \frac{x-1}{0-1} = 1-x$, $\lambda_1(x) = x$

Then any polynomial $P(x) = Ax+B \in \mathbb{P}^{(1)}(0,1)$ can be written

$$P(x) = Ax+B = \underbrace{P(0)}_{=B}\lambda_0(x) + \underbrace{P(1)}_{A+B}\lambda_1(x) = B(x-1)+(A+B)x$$

Note: We want to approximate a given continuous function $y=f(x)$, $x\in[a,b]$ by a polynomial $P(x) \in \mathbb{P}^{(q)}(a,b)$, such that at $(q+1)$ distinct points $x_0 < x_1 < \dots < x_q$ in $[a,b]$ we have

$$f(x_i) = P(x_i), \quad i=0,1,\dots,q$$

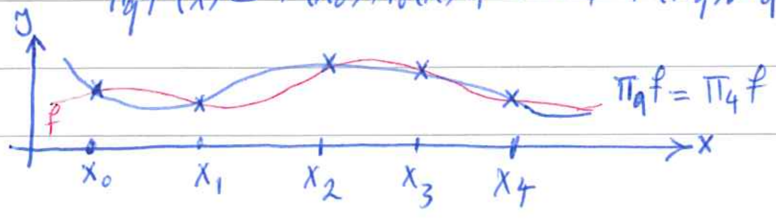
This is called interpolation. If we use Lagrange polynomials it is called Lagrange interpolation.

We will denote $P = \pi_q f$. So we find $\pi_q f \in \mathbb{P}^{(q)}(a,b)$ such that

$$f(x_i) = (\pi_q f)(x_i), \quad i=0,1,\dots,q.$$

We know that

$$\pi_q f(x) = f(x_0)\lambda_0(x) + \dots + f(x_q)\lambda_q(x), \quad x\in[a,b]$$

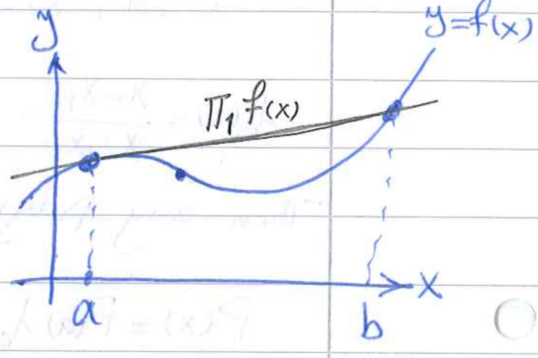


Ex Let $q=1$ and choose $x_0=a$, $x_1=b$.

$$\lambda_0(x)=\frac{x-x_1}{x_0-x_1}=\frac{x-b}{a-b}=\frac{b-x}{b-a}, \quad \lambda_1(x)=\frac{x-x_0}{x_1-x_0}=\frac{x-a}{b-a}$$

$$\Rightarrow \Pi_1 f(x)=f(x_0)\lambda_0(x)+f(x_1)\lambda_1(x)$$

$$=f(a)\cdot\frac{b-x}{b-a}+f(b)\frac{x-a}{b-a}$$



④ The space of continuous piecewise polynomials on a partition of an interal into a collection of subintervals.

Let

$$\mathcal{T}_h : 0 = x_0 < x_1 < \cdots < x_M < x_{M+1} = 1$$

be a partition of $[0,1]$ into $M+1$ subintervals (x_{j-1}, x_j), $j = 1, \ldots, M+1$, with step size $h_j = x_j - x_{j-1}$.

We introduce $V_h^{(q)}$ be the space of all continuous piecewise polynomials of degree $\leq q$ on $\mathcal{T}_h$. That is

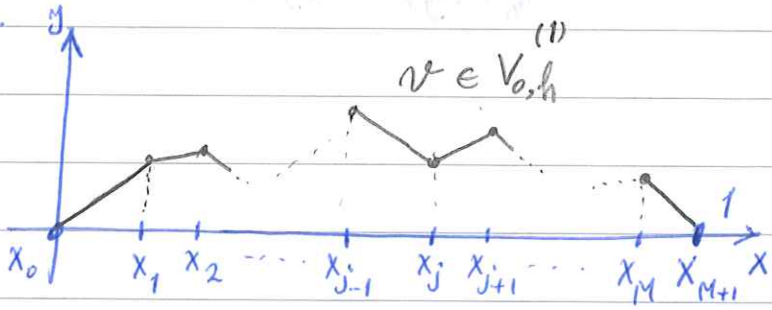
$$V_h^{(q)} = \left\{ v \;\middle|\; v(x) = \xi_0^{(j)} + \xi_1^{(j)} x + \cdots + \xi_q^{(j)} x^q, \text{ for } x \in (x_{j-1}, x_j),\ j = 1, \ldots, M+1 \right\}$$

$$= \left\{ v \;\middle|\; \underbrace{v(x) \in \mathcal{P}^{(q)}(x_{j-1}, x_j)}_{\text{for } x \in (x_{j-1}, x_j)},\ j = 1, \ldots, M+1 \right\}$$

We can also define

$$V_{0,h}^{(q)} = \left\{ v \;\middle|\; v \in V_h^{(q)},\ v(0) = v(1) = 0 \right\}$$

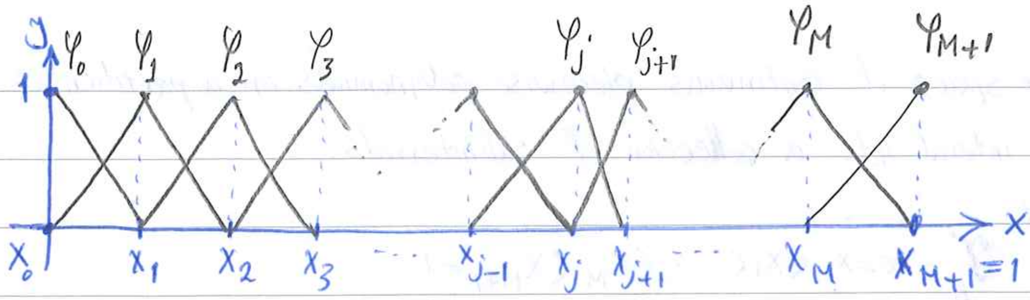
For example, consider $V_h^{(1)}$, the space of continuous piecewise linear polynomials on $\mathcal{T}_h$.



The standard basis for $V_h^{(1)}$ is by the hat functions $\{\varphi_j\}_{j=0}^{M+1}$

$$\varphi_j(x_i) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \qquad \varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h_j} & x_{j-1} \leq x \leq x_j \\[2ex] \dfrac{x - x_{j+1}}{-h_{j+1}} & x_j \leq x \leq x_{j+1} \\[2ex] 0 & x \notin [x_{j-1}, x_{j+1}] \end{cases}$$

$$\varphi_0(x) = \begin{cases} \dfrac{x - x_1}{-h_1} & x_0 \leq x \leq x_1 \\[2ex] 0 & x \in [x_0, x_1] \end{cases}, \qquad \varphi_{M+1}(x) = \begin{cases} \dfrac{x - x_M}{h_{M+1}} & x_M \leq x \leq x_{M+1} \\[2ex] 0 & x \notin [x_M, x_{M+1}] \end{cases}$$



We note that

$$V_h^{(1)} = \text{Span}\{\varphi_0, \varphi_1, \dots, \varphi_M, \varphi_{M+1}\}, \quad V_{0,h}^{(1)} = \text{Span}\{\varphi_1, \dots, \varphi_M\}$$

$$\Rightarrow \dim(V_h^{(1)}) = M+2, \quad \dim(V_{0,h}^{(1)}) = M$$

That is for any function $v \in V_h^{(1)}$ we have

$$v(x) = \xi_0 \varphi_0(x) + \xi_1 \varphi_1(x) + \dots + \xi_{M+1} \varphi_{M+1}(x) = \sum_{j=0}^{M+1} \xi_j \varphi_j(x)$$

Note: We can approximate a given continuous function f on $[a,b]$, by a continuous piecewise linear interpolant $\pi_h f \in V_h^{(1)}$

$$\pi_h f(x) = \sum_{j=0}^{N} f(x_j) \varphi_j(x), \quad x \in [a,b].$$

So

$$\pi_h f(x_j) = f(x_j), \quad j = 0, 1, \dots, N$$

The L_2-projection

For a given function $f(x)$ on (a,b), we define its L_2-projection $Pf \in \mathcal{P}^{(q)}_{(a,b)}$ by

$$\int_a^b f(x)\chi(x)\,dx = \int_a^b (Pf)(x)\chi(x)\,dx, \quad \text{for all } \chi \in \mathcal{P}^{(q)}_{(a,b)}$$

Note: If we denote the inner product

$$(f,g) = \int_a^b f(x)g(x)\,dx$$

We can write the definition of the L_2-projection as

$$(f,\chi) = (Pf,\chi), \quad \forall \chi \in \mathcal{P}^{(q)}_{(a,b)}$$

And we note that $(f-Pf,\chi)=0$, $\forall \chi \in \mathcal{P}^{(q)}_{(a,b)}$. That is, the error of L_2-projection $(e=f-Pf)$ is orthogonal to all polynomials in $\mathcal{P}^{(q)}_{(a,b)}$.

Note: Let's consider $\{1, x, \dots, x^q\}$ as a basis for $\mathcal{P}^{(q)}_{(a,b)}$. Then $Pf \in \mathcal{P}^{(q)}_{(a,b)}$ is the L_2-projection of f, if

$$(f, x^i) = (Pf, x^i), \quad \text{for } i=0,1,\dots,q.$$

This is used to find Pf.

Ex For a given continuous function on (a,b), find $Pf \in \mathcal{P}^{(0)}_{(a,b)}$?

We know that $\mathcal{P}^{(0)}_{(a,b)} = \text{Span}\{1\} = \{\text{Constant polynomials}\}$.

$$(f,1) = (Pf,1) \Rightarrow \int_a^b f(x)\,dx = \int_a^b \underbrace{(Pf)(x)}_{\in \mathcal{P}^{(0)}_{(a,b)}}\,dx = (Pf)(x)\int_a^b dx = (b-a)(Pf)(x)$$

$$\Rightarrow (Pf)(x) = \frac{1}{b-a}\int_a^b f(x)\,dx, \quad x \in (a,b)$$

Lemma 4.1 (i) Pf is unique (ii) Pf is the best approx of f in $P^{(q)}_{(a,b)}$ in the L_2-norm. That is,

$$\|f-Pf\|_{L_2(a,b)} \leq \|f-\chi\|_{L_2(a,b)}, \quad \forall \chi \in P^{(q)}_{(a,b)}$$

(i) Suppose that $P_1 f$ and $P_2 f$ are two polynomials in $P^{(q)}_{(a,b)}$ that are L_2-projection of f. That is

$$(f-P_1 f, \chi)=0, \quad (f-P_2 f, \chi)=0, \quad \forall \chi \in P^{(q)}_{(a,b)}$$

$$\Longrightarrow (P_2 f - P_1 f, \chi)=0, \quad \forall \chi \in P^{(q)}_{(a,b)}$$

Choosing $\chi = P_2 f - P_1 f$, we have

$$(P_2 f - P_1 f, P_2 f - P_1 f) = \cancel{\ }$$

$$\Longrightarrow P_2 f - P_1 f = 0 \Rightarrow P_2 f = P_1 f$$

(ii) For any $\chi \in P^{(q)}_{(a,b)}$, we have

$$(f-Pf, \chi)=0,$$

So

$$\|f-Pf\|^2_{L_2(a,b)} = (f-Pf, f-Pf) = (f-Pf, f-\chi+\chi-Pf)$$

$$= (f-Pf, f-\chi) + \underbrace{(f-Pf, \underbrace{\chi-Pf}_{\in P^{(q)}_{(a,b)}})}_{=0}$$

$$= (f-Pf, f-\chi)$$

Cauchy-Schwarz

$$\overset{C-S}{\leq} \|f-Pf\|_{L_2(a,b)} \|f-\chi\|_{L_2(a,b)}$$

$$\Longrightarrow \|f-Pf\|_{L_2(a,b)} \leq \|f-\chi\|_{L_2(a,b)}$$

4.2 Galerkin Method for IVP

To formulat a Galerkin method for an IVP

$$\begin{cases} \dot{u}(t) - \lambda u(t) = 0, & 0 < t < T \\ u(0) = u_0 \end{cases} \quad (1)$$

we should first write its variational formulation (weak form).

Variational formulation for IVP

We multiply the DE by a test function v (in a function space V) and integrate over $[0,T]$,

$$\int_0^T (\dot{u}(t) - \lambda u(t)) v(t)\, dt = 0, \qquad \forall \text{ test functions } v (\in V)$$

Note that considering the L_2-inner product

$$(v, w) = \int_0^T v(t) w(t)\, dt$$

We have

$$(\dot{u} - \lambda u, v) = 0, \qquad \forall v \in V \qquad (2)$$

That is $\quad (\dot{u} - \lambda u) \perp v, \quad \forall v \in V.$

Hence the variational form is: Find a function $u = u(t)$ such that

$$\begin{cases} u(0) = u_0 \text{ and} \\ \int_0^T (\dot{u}(t) - \lambda u(t)) v(t)\, dt = 0, \quad \forall v(t) \in V. \end{cases} \quad (3)$$

Note: We can take $V = C([0,T])$, or in more general form

$$V := H^1(0,T) := \{ f \mid \int_0^T f(t)^2 + (f'(t))^2\, dt < \infty \}$$

Def: Let u be the solution of the variational problem (3) and w be an approx. of u ($w \approx u$). Then

$$R(w(t)) = \dot{w}(t) - \lambda w(t)$$

is called the residual error of $w(t)$.

We note that

$$\mathcal{R}(u(t)) = \dot{u}(t) - \lambda u(t) = 0 \Rightarrow \mathcal{R}(u(t)) \perp v, \ \forall v \in V$$

but $\mathcal{R}(w(t))$ is not necessarily zero,

$$\mathcal{R}(w(t)) = \dot{w}(t) - \lambda w(t) \underset{\text{in general}}{\neq} 0$$

Galerkin method

We want to find an approximate solution $U(t)$ in a finite dimensional space, such as $P^{(q)}_{(0,T)} = \mathrm{Span}\{1, t, \dots, t^q\}$.

$$U(t) \in P^{(q)}_{(0,T)} \Rightarrow U(t) = \xi_0 + \xi_1 t + \xi_2 t^2 + \dots + \xi_q t^q$$

and we should determine the coefficients $\xi_0, \xi_1, \dots, \xi_q$.

Since

$$U(0) = u_0 = \xi_0 + 0 + \dots + 0 = \xi_0 \Rightarrow \xi_0 = u_0$$

So we ~~only~~ need to find q unknown coefficients $\xi_1, \dots, \xi_q$. And therefore we need q equations.

We introduce the function space

$$V_0 = P_0^{(q)}(0,T) = \{ v \in P^{(q)}_{(0,T)} \mid v(0) = 0 \} = \mathrm{Span}\{t, t^2, \dots, t^q\}$$

The the Galerkin ~~met~~ method is:

$$\begin{cases} \text{Find } U \in V = P^{(q)}_{(0,T)}, \text{ such that} \\ \displaystyle\int_0^T \mathcal{R}(U(t)) \chi(t)\, dt = \int_0^T \left(\dot{U}(t) - \lambda U(t) \right) \chi(t)\, dt = 0, \ \forall \chi \in V_0 = P_0^{(q)}(0,T) \qquad (4) \\ U(0) = u_0 \end{cases}$$

Let see how we can write the details to find U?

$U \in P^{(q)}$

$$\Rightarrow U(t) = \xi_0 + \xi_1 t + \dots + \xi_q t^q = \sum_{j=0}^{q} \xi_j t^j$$

$$\xrightarrow{U(0)=u_0} U(t) = u_0 + \sum_{j=1}^{q} \xi_j t^j$$

and $$\dot{U}(t) = \sum_{j=1}^{q} j \xi_j t^{j-1}$$

Also note that $\chi \in P_0^{(q)} = \text{Span}\{t, \dots, t^q\}$, so we can choose $\chi(t) = t^i$, $i = 1, \dots, q$ for the Galerkin formulation ④.

Hence, for simplicity take $T = 1$,

$$\int_0^1 \left(\sum_{j=1}^{q} j \xi_j t^{j-1} - \lambda u_0 - \lambda \sum_{j=1}^{q} \xi_j t^j \right) t^i \, dt = 0, \quad i = 1, 2, \dots, q$$

Moving the known values to the r.h.s;

$$\int_0^1 \left(\sum_{j=1}^{q} (j t^{i+j-1} - \lambda t^{i+j}) \right) \xi_j \, dt = \lambda u_0 \int_0^1 t^i \, dt, \quad i = 1, 2, \dots, q$$

$$\Rightarrow \sum_{j=1}^{q} \left(j \int_0^1 t^{i+j-1} dt - \lambda \int_0^1 t^{i+j} dt \right) \xi_j(t) = \lambda u_0 \int_0^1 t^i dt, \quad i = 1, \dots, q$$

$$\Rightarrow \sum_{j=1}^{q} \left(\frac{j}{i+j} - \frac{\lambda}{i+j+1} \right) \xi_j = \frac{\lambda}{i+1} u_0, \quad i = 1, 2, \dots, q$$

This is a linear system of q equations and q unknowns,

$$A \xi = b$$

where $A = (a_{ij})_{i,j=1}^{q}$, $\xi = (\xi_j)_{j=1}^{q}$, $b = (b_i)_{i=1}^{q}$

$$a_{ij} = \frac{j}{i+j} - \frac{\lambda}{i+j+1}$$

$$A = \begin{bmatrix} \frac{1}{2} - \frac{\lambda}{3} & \frac{2}{3} - \frac{\lambda}{4} & \cdots & \frac{q}{q+1} - \frac{\lambda}{q+2} \\ \frac{1}{3} - \frac{\lambda}{4} & \frac{2}{4} - \frac{\lambda}{5} & \cdots & \frac{q}{q+2} - \frac{\lambda}{q+3} \\ \vdots & & & \\ \frac{1}{q+1} - \frac{\lambda}{q+2} & \frac{2}{q+2} - \frac{\lambda}{q+3} & \cdots & \frac{q}{2q} - \frac{\lambda}{2q+1} \end{bmatrix}$$

4.3 Galerkin Finite Element Method for BVP

We apply Galerkin FEM to find an approximate solution for the (two-point) BVP:

$$\begin{cases} -(a(x)u'(x))' = f(x), & 0<x<1 \\ u(0)=0,\ u(1)=0 \end{cases} \quad (1)$$

For simplicity, let $a(x)=1$. So

$$\begin{cases} -u''(x) = f(x), & 0<x<1 \\ u(0)=0,\ u(1)=0 \end{cases} \quad (2)$$

First we write variational form (VF).
We define the function space

$$H_0^1(0,1) = \left\{ v \,\middle|\, \int_0^1 \{v^2(x) + (v'(x))^2\} dx < \infty,\ v(0)=v(1)=0 \right\}$$

Now multiply the DE by a test functio $v \in V^0 = H_0^1(0,1)$, and integrate over $(0,1)$ and then integrate by parts:

$$-\int_0^1 u''(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

$$\Rightarrow -[u'(x)v(x)]_0^1 + \int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

$$\Rightarrow -u'(1)\underbrace{v(1)}_{=0} + u'(0)\underbrace{v(0)}_{=0} + \int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

So the VF is:

$$\begin{cases} \text{Find } u \in V^0 \text{ such that} \\ \int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx, \quad \forall v \in V^0 \end{cases} \quad (3)$$

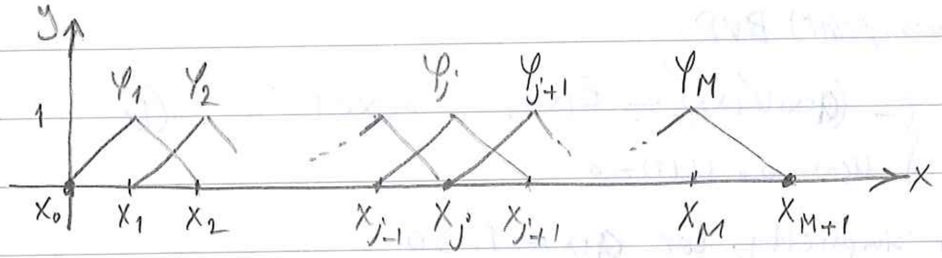
To apple FEM, we consider a partition of $(0,1)$,

$$\mathcal{T}_h:\ 0 = x_0 < x_1 < \cdots < x_M < x_{M+1} = 1$$

into subintervals $I_j = (x_{j-1}, x_j)$ with length $|I_j| = h_j = x_j - x_{j-1}$, $j=1,\ldots,M+1$

Then we define a corresponding finite dimensional function space

$$V_h^0 = \{ v \text{ is continuous piecewise linear on } \mathcal{T}_h,\ v(0) = v(1) = 0 \}$$



Note that $V_h^0 = \mathrm{Span}\{\varphi_1, \dots, \varphi_M\}$.

Now the FEM to solve ②, based on VF ③, is:

$$\begin{cases} \text{Find } U \in V_h^0 \text{ such that} \\ \int_0^1 U'(x)\chi'(x)\,dx = \int_0^1 f(x)\chi(x)\,dx, \quad \forall \chi \in V_h^0 \end{cases} \quad ④$$

Note that, since ④ is true for all $\chi \in V_h^0$, so it holds for every $\chi = \varphi_i$, $i = 1, \dots, M$.

Also note that $U \in V_h^0$, so

$$U(x) = \sum_{j=1}^{M} \xi_j \varphi_j(x) \Rightarrow U'(x) = \sum_{j=1}^{M} \xi_j \varphi_j'(x)$$

where the unknown coefficients $\xi_1, \dots, \xi_M$ should be found.

Substitute in ④, we have

$$\int_0^1 \Big(\sum_{j=1}^{M} \xi_j \varphi_j'(x)\Big)\varphi_i'(x)\,dx = \int_0^1 f(x)\varphi_i(x)\,dx, \quad i = 1, \dots, M$$

$$\Rightarrow \sum_{j=1}^{M} \Big(\int_0^1 \varphi_j'(x)\varphi_i'(x)\,dx\Big)\xi_j = \int_0^1 f(x)\varphi_i(x)\,dx, \quad i = 1, \dots, M$$

In matrix form:

$$A\xi = b$$

where

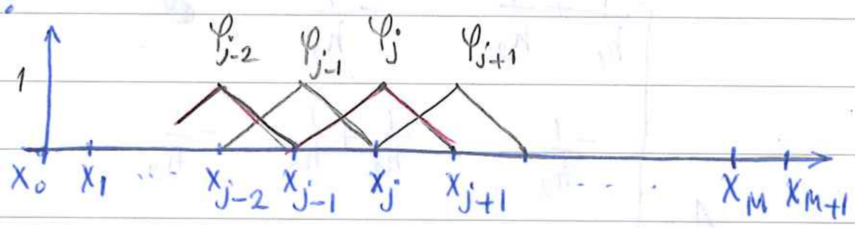
$$A = \{a_{ij}\}_{i,j=1}^{M} = \left\{ \int_0^1 \varphi_j' \varphi_i' \, dx \right\}_{i,j=1}^{M}$$

$$\xi = \{\xi_j\}_{j=1}^{M}, \quad b = \{b_i\}_{i=1}^{M} = \left\{ \int_0^1 f \varphi_i \, dx \right\}_{i=1}^{M}$$

Note that

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_i} & x_{i-1} \leq x \leq x_i \\ \dfrac{x - x_{i+1}}{-h_{i+1}} & x_i \leq x \leq x_{i+1} \\ 0 & \text{o.w.} \end{cases} \Rightarrow \varphi_i'(x) = \begin{cases} \dfrac{1}{h_i} & x_{i-1} < x < x_i \\ -\dfrac{1}{h_{i+1}} & x_i < x < x_{i+1} \\ 0 & \text{o.w} \end{cases}$$

Stiffness matrix A:



If $i-j>1$ or $j-i>1$ ($|i-j|>1$) then φ_i and φ_j have disjoint support, so

$$a_{ij} = \int_0^1 \varphi_i'(x) \varphi_j'(x) dx$$

$$= \int_{x_0}^{x_1} \overset{0}{\varphi_i'(x)} \overset{0}{\varphi_j'(x)} dx + \int_{x_1}^{x_2} \overset{0}{\varphi_i'} \overset{0}{\varphi_j'} dx + \cdots \int_{x_{i-1}}^{x_i} \overbrace{\varphi_i' \varphi_j'}^{=0} dx + \int_{x_i}^{x_{i+1}} \overbrace{\varphi_i' \varphi_j'}^{=0} dx$$

$$+ \cdots + \int_{x_M}^{x_{M+1}} \underset{0}{\varphi_i'} \underset{0}{\varphi_j'} dx = 0$$

For $i = j$,

$$a_{ii} = \int_0^1 \varphi_i' \varphi_i' dx = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h_i}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h_{i+1}}\right)^2 dx = \frac{\overbrace{x_i - x_{i-1}}^{h_i}}{h_i^2} + \frac{\overbrace{x_{i+1} - x_i}^{h_{i+1}}}{h_{i+1}^2}$$

$$= \frac{1}{h_i} + \frac{1}{h_{i+1}}$$

Now, it remains a_{ij} for $i=j+1$ and $i=j-1$,

$$a_{i,i+1}=\int_0^1 \varphi_i' \varphi_{i+1}' \, dx = \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h_{i+1}}\right)\left(\frac{1}{h_{i+1}}\right) dx = -\frac{x_{i+1}-x_i}{h_{i+1}^2} = -\frac{1}{h_{i+1}}$$

$$a_{i+1,i}=\int_0^1 \varphi_{i+1}' \varphi_i' \, dx = \int_0^1 \varphi_i' \varphi_{i+1}' \, dx = a_{i,i+1} = -\frac{1}{h_{i+1}}$$

Hence

$$\begin{cases} a_{ij} = 0 & |i-j|>1 \\ a_{ii} = \frac{1}{h_i}+\frac{1}{h_{i+1}} & i=1,\dots,M \\ a_{i-1,i} = a_{i,i-1} = -\frac{1}{h_i} & i=2,\dots,M \end{cases}$$

$$A=\begin{bmatrix} \frac{1}{h_1}+\frac{1}{h_2} & -\frac{1}{h_2} & 0 & \cdots & 0 \\ -\frac{1}{h_2} & \frac{1}{h_2}+\frac{1}{h_3} & -\frac{1}{h_3} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\frac{1}{h_M} \\ 0 & \cdots & 0 & -\frac{1}{h_M} & \frac{1}{h_M}+\frac{1}{h_{M+1}} \end{bmatrix}$$

For the load vector b,

$$b_i=\int_0^1 f\varphi_i \, dx = \int_{x_{i-1}}^{x_i} f(x) \frac{x-x_{i-1}}{h_i} dx + \int_{x_i}^{x_{i+1}} f(x) \frac{x-x_{i+1}}{-h_{i+1}} dx$$

Note: For uniform mesh: $h=h_j$, $j=1,\dots,M$

$$A=\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

Chapter 5: Interpolation, Numerical Integration in 1D

5.1. Preliminaries

Def For a given continuous function f on $I=[a,b]$, and $(q+1)$ distinct interpolation points $x_0 < x_1 < \dots < x_q$, a polynomial $\pi_q f \in \mathcal{P}^{(q)}_{(a,b)}$ is an interpolant for f, if

$$\pi_q f(x_j) = f(x_j), \quad j=0,1,\dots,q$$

EX (Linear interpolation) Consider a given cont. function f on $I=[0,1]$ and $q=1$. Find the interpolant $\pi_1 f \in \mathcal{P}^1_{(0,1)}$.

We find a polynomial $\pi_1 f \in \mathcal{P}^1_{(0,1)}$ such that

$$\pi_1 f(x_0) = f(x_0), \quad \pi_1 f(x_1) = f(x_1)$$

Take $x_0=0$, $x_1=1$.

We consider two basis for $\mathcal{P}^1(0,1)$

(a) $\mathcal{P}^{(1)}_{(0,1)} = \mathrm{Span}\{1, x\}$

$$\pi_1 f(x) = c_0 + c_1 x \Longrightarrow \begin{cases} \pi_1 f(x_0) = \pi_1 f(0) = c_0 + 0 = f(0) \\ \pi_1 f(x_1) = \pi_1 f(1) = c_0 + c_1 = f(1) \end{cases} \Rightarrow \begin{cases} c_0 = f(0) \\ c_0 + c_1 = f(1) \end{cases}$$

$$\Longrightarrow \begin{cases} c_0 = f(0) \\ c_1 = f(1) - f(0) \end{cases}$$

So $\boxed{\pi_1 f(x) = f(0) + (f(1) - f(0))x}$

(b) $\mathcal{P}^{(1)}_{(0,1)} = \mathrm{Span}\{1-x, x\}$

$$\pi_1 f(x) = c_0(1-x) + c_1 x \Longrightarrow \begin{cases} \pi_1 f(0) = c_0 + 0 = f(0) \\ \pi_1 f(1) = 0 + c_1 = f(1) \end{cases} \Rightarrow \begin{cases} c_0 = f(0) \\ c_1 = f(1) \end{cases}$$

So $\boxed{\pi_1 f(x) = f(0)(1-x) + f(1)x}$

Note: 1. Part (b) is the Lagrange interpolation $\lambda_0(x) = \dfrac{x-x_1}{x_0-x_1} = 1-x$

$$\lambda_1(x) = \frac{x-x_0}{x_1-x_0} = x$$

2. $\pi_1 f$ from (a) and (b) are equal. Since the interpolant is unique.

We recall the L_p-norms, $1 \leq p \leq \infty$,

$$\|f\|_{L_p(a,b)} = \left(\int_a^b |f(x)|^p dx\right)^{1/p}, \quad 1 \leq p < \infty$$

$$\|f\|_{L_\infty(a,b)} = \max_{x \in [a,b]} |f(x)|$$

and the corresponding L_p-spaces:

$$L_p(a,b) = \{ f \mid \|f\|_{L_p(a,b)} < \infty \}$$

So we can measure the difference of two function f, g by $\|f-g\|_{L_p(a,b)}$.

Now, we can measure the error of interpolant, that is $\|\pi_q f - f\|_{L_p(a,b)}$.
Here, we consider $q=1$, i.e., $\|\pi_1 f - f\|_{L_p(a,b)}$ for linear interpolant

First, we state and prove some error estimates in $L_\infty(a,b)$-norm.

<u>Thm 5.1</u> Assume $f'' \in L_\infty(a,b)$. Then, for some constants C_i, $i=1,2,3$, that are independent of f and the size of $[a,b]$, ~~such that~~ we have

(1) $\|\pi_1 f - f\|_{L_\infty(a,b)} \leq C_1 (b-a)^2 \|f''\|_{L_\infty(a,b)}$

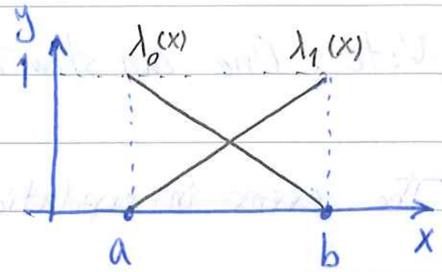
(2) $\|\pi_1 f - f\|_{L_\infty(a,b)} \leq C_2 (b-a) \|f'\|_{L_\infty(a,b)}$

(3) $\|(\pi_1 f)' - f'\|_{L_\infty(a,b)} \leq C_3 (b-a) \|f''\|_{L_\infty(a,b)}$

<u>Proof.</u> Take $x_0 = a$ and $x_1 = b$. We recall that every linear polynomial $p \in P^{(1)}_{(a,b)}$ can be written as

$$p(x) = p(x_0)\lambda_0(x) + p(x_1)\lambda_1(x)$$

Where $\lambda_0(x) = \frac{x-x_1}{x_0-x_1} = \frac{x-b}{a-b}$, $\lambda_1(x) = \frac{x-x_0}{x_1-x_0} = \frac{x-a}{b-a}$ are Lagrange poly.



So
$p(x)=1 \Longrightarrow 1 = \lambda_0(x)+\lambda_1(x)$
$p(x)=x \Longrightarrow x = x_0\lambda_0(x)+x_1\lambda_1(x)$

The linear Lagrange interpolant is

$$\pi_1 f(x) = f(x_0)\lambda_0(x) + f(x_1)\lambda_1(x)$$

By Taylor expansion of $f(x_0)$ and $f(x_1)$ about $x\in(a,b)$,

$$f(x_0) = f(x) + (x_0-x)f'(x) + \frac{(x_0-x)^2}{2} f''(\eta_0), \qquad \eta_0\in(x_0,x)$$

$$f(x_1) = f(x) + (x_1-x)f'(x) + \frac{(x_1-x)^2}{2} f''(\eta_1), \qquad \eta_1\in(x,x_1)$$

Hence

$$\pi_1 f(x) = \left\{f(x) + (x_0-x)f'(x) + \frac{(x_0-x)^2}{2}f''(\eta_0)\right\}\lambda_0(x)$$

$$+ \left\{f(x) + (x_1-x)f'(x) + \frac{(x_1-x)^2}{2}f''(\eta_1)\right\}\lambda_1(x)$$

$$= f(x)\big(\underbrace{\lambda_0(x)+\lambda_1(x)}_{=1}\big) + \big(\underbrace{(x_0-x)\lambda_0(x) + (x_1-x)\lambda_1(x)}_{=x_0\lambda_0(x)+x_1\lambda_1(x)-x(\lambda_0(x)+\lambda_1(x))=x-x=0}\big)f'(x)$$

$$+\frac{1}{2}(x_0-x)^2 f''(\eta_0)\lambda_0(x) + \frac{1}{2}(x_1-x)^2 f''(\eta_1)\lambda_1(x)$$

$$\Rightarrow \pi_1 f(x) - f(x) = \frac{1}{2}(x_0-x)^2 f''(\eta_0)\lambda_0(x) + \frac{1}{2}(x_1-x)^2 f''(\eta_1)\lambda_1(x), \quad x\in[a,b]$$

$$\Rightarrow |\pi_1 f(x) - f(x)| \le \frac{1}{2}\big|\underbrace{(x_0-x)^2}_{\le b-a}\big|\,|f''(\eta_0)|\,\big|\underbrace{\lambda_0(x)}_{\le 1}\big| + \frac{1}{2}\big|\underbrace{(x_1-x)^2}_{\le b-a}\big|\,|f''(\eta_1)|\,\big|\underbrace{\lambda_1(x)}_{\le 1}\big|$$

$$\le \frac{1}{2}(b-a)^2 \underbrace{\max_{x\in(a,b)}|f''(x)|}_{=\|f''\|_{L_\infty(a,b)}} + \frac{1}{2}(b-a)^2\max_{x\in(a,b)}|f''(x)|$$

$$= (b-a)^2\|f''\|_{L_\infty(a,b)}, \quad \forall x\in(a,b)$$

$$\Rightarrow \|\pi_1 f - f\|_{L_\infty(a,b)} \le (b-a)^2\|f''\|_{L_\infty(a,b)}, \qquad (C_1 = 1)$$

The other error estimates are proved similarly. □

Note: One can show that $C_1 = \frac{1}{8}$, which the optimal value of C_1.

The ~~error~~ interpolation error estimates in the L_p-norm ($p=1,2$) are:

Thm 5.2 Assume $f'' \in L_p(a,b)$, $p=1,2,\infty$. Then for some constants C_i, $i=1,2,3$, that are independent of f and $(b-a)$, we have

(1) $\|\pi_1 f - f\|_{L_p(a,b)} \leq C_1 (b-a)^2 \|f''\|_{L_p(a,b)}$

(2) $\|\pi_1 f - f\|_{L_p(a,b)} \leq C_2 (b-a) \|f'\|_{L_p(a,b)}$

(3) $\|(\pi_1 f)' - f'\|_{L_p(a,b)} \leq C_3 (b-a) \|f''\|_{L_p(a,b)}$

Linear

Vector space of piecewise linear functions

Let $I = [a,b]$ and consider the partition

$$\mathcal{T}_h: \quad a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$$

into subintervals $I_j = [x_{j-1}, x_j]$ of length $h_j = |I_j| = x_j - x_{j-1}$, $j = 1, \dots, N$.

Define

$$V_h = \{ v \mid v \text{ is cont. p.w. linear function on } \mathcal{T}_h \}.$$

Recalling the hat functions $\{\varphi_j\}_{j=0}^N$ with $\varphi_j(x_i) = \delta_{ij}$, we note that

$$\left. \sum_{j=0}^N v(x_j)\varphi_j(x) \right|_{x=x_i} = v(x_0)\underbrace{\varphi_0(x_i)}_{=0} + \cdots + v(x_{i-1})\underbrace{\varphi_{i-1}(x_i)}_{=0} + v(x_i)\underbrace{\varphi_i(x_i)}_{=1} + \cdots + v(x_N)\underbrace{\varphi_N(x_i)}_{=0}$$

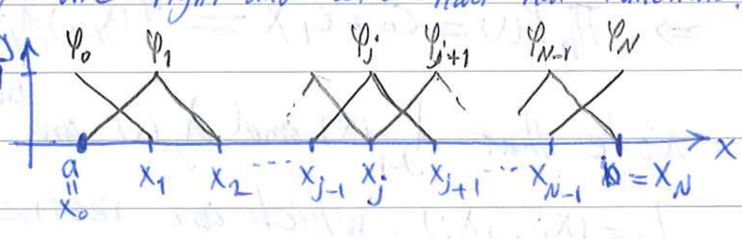
$$= v(x_i)$$

Hence, every cont. p.w. linear function $v \in V_h$ is written as

$$v(x) = \sum_{j=0}^N v(x_j)\varphi_j(x), \qquad x \in [a,b].$$

And therefore $\dim(V_h) = N+1$, i.e., $V_h = \operatorname{span}\{\varphi_0, \dots, \varphi_N\}$.

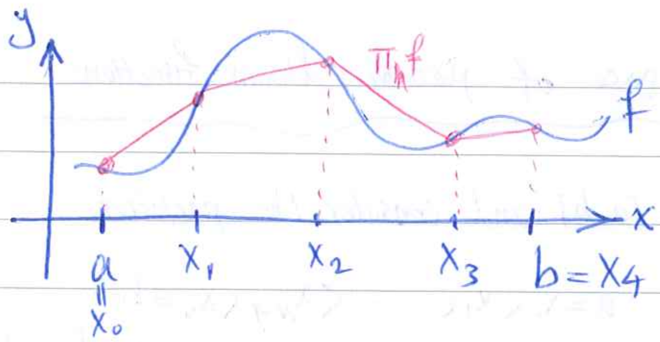
We note that φ_0 and φ_N are right and left half-hat functions.



Def For a given continuous function f on $[a,b]$ and a partition $\mathcal{T}_h$, we define the continuous piecewise linear interpolant of f by

$$\Pi_h f(x) = \sum_{j=0}^N f(x_j)\varphi_j(x), \qquad x \in [a,b]$$

Here h is the mesh size function $h(x) = h_j := x_j - x_{j-1}$, for $x \in I_j$, $j = 1, \dots, N$



Note: For each subinterval $I_j = (x_{j-1}, x_j)$, $j = 1, \dots, N$

(i) $\Pi_h f(x)$ is linear on $I_j \Rightarrow \Pi_h f(x) = c_0 + c_1 x$, $x \in I_j$

(ii) $\Pi_h f(x_{j-1}) = f(x_{j-1})$, $\Pi_h f(x_j) = f(x_j)$ or $\Pi_h f(x) = c_{j-1}\lambda_{j-1}(x) + c_j \lambda_j(x)$

So combining (i) and (ii) for each $j = 1, \dots, N$

$$\begin{cases} \Pi_h f(x_{j-1}) = c_0 + c_1 x_{j-1} = f(x_{j-1}) \\ \Pi_h f(x_j) = c_0 + c_1 x_j = f(x_j) \end{cases} \Rightarrow \begin{cases} c_1 = \dfrac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}} \\ c_0 = \dfrac{-x_{j-1} f(x_j) + x_j f(x_{j-1})}{x_j - x_{j-1}} \end{cases}$$

$$\Rightarrow \begin{cases} c_0 = f(x_{j-1}) \dfrac{x_j}{x_j - x_{j-1}} + f(x_j) \dfrac{-x_{j-1}}{x_j - x_{j-1}}. \\ c_1 x = f(x_{j-1}) \dfrac{-x}{x_j - x_{j-1}} + f(x_j) \dfrac{x}{x_j - x_{j-1}} \end{cases}$$

$j = 1, \dots, N$

$$\Rightarrow \Pi_h f(x) = c_0 + c_1 x = f(x_{j-1}) \lambda_{j-1}(x) + f(x_j) \lambda_j(x) = f(x_{j-1}) \frac{x - x_j}{x_{j-1} - x_j} + f(x_j) \frac{x - x_{j-1}}{x_j - x_{j-1}}$$

Note that $\lambda_{j-1}(x)$ and $\lambda_j(x)$ are linear Lagrange polynomials on $I_j = (x_{j-1}, x_j)$, which are restrictions of the hat functions $\varphi_{j-1}(x)$ and $\varphi_j(x)$ on $I_j = (x_{j-1}, x_j)$.

Thm 5.3 Let $f \in C^2(a,b)$ and $\Pi_h v$ be its cont. p.w. linear interpolant on the partition $\mathcal{T}_h$ of $[a,b]$. Then there are interpolation constants c_i, $i=1,2,3$, ~~that~~ are independent of f and $(b-a)$, such that, for $p=1,2,\infty$:

(1) $\|\Pi_h f - f\|_{L_p(a,b)} \le c_1 \|h^2 f''\|_{L_p(a,b)}$

(2) $\|\Pi_h f - f\|_{L_p(a,b)} \le c_2 \|h f'\|_{L_p(a,b)}$

(3) $\|(\Pi_h f)' - f'\|_{L_p(a,b)} \le c_3 \|h f''\|_{L_p(a,b)}$

Proof:

$$
\begin{aligned}
\|\Pi_h f - f\|^p_{L_p(a,b)} &= \int_a^b |\Pi_h f - f|^p \, dx \\
&= \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} |\Pi_h f(x) - f(x)|^p \, dx \\
&= \sum_{j=1}^{N} \|\Pi_h f - f\|^p_{L_p(I_j)} \le \sum_{j=1}^{N} c_1^p \|h_j^2 f''\|^p_{L_p(I_j)} \\
&= c_1^p \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} |h_j^2(x) f''(x)|^p \, dx \\
&= c_1^p \int_a^b |h^2(x) f''(x)|^p \, dx = c_1^p \|h^2 f''\|^p_{L_p(a,b)}
\end{aligned}
$$

$$\Longrightarrow \|\Pi_h f - f\|_{L_p(a,b)} \le c_1 \|h^2 f''\|_{L_p(a,b)}.$$

The other two error estimates are proved similarly. □