# Chapter 2 : Fourier Analysis

## 2.1 Periodic Functions

**Def** $f(x)$ is periodic if there is a constant $p>0$,

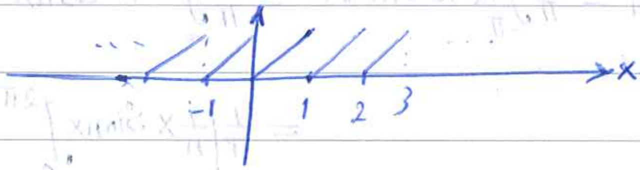
$$f(x+p)=f(x), \quad \forall x.$$

Ex. $f(x)=\cos x \rightarrow 2\pi, 4\pi, -6\pi, \ldots$

Def. prime period is the smallest $p>0$ (if exists).

$p=2\pi$ for $f(x)=\cos x$, $f(x)=\sin x$

Ex. $y=x$, $0<x<1$, $p=1$



**Lemma 1** Let $y=f(x)$ with period $p$, then

$$\int_a^{a+p} f(x)\,dx$$

does not depend on $a$.

Proof. Set

$$g(a)=\int_a^{a+p} f(x)\,dx \underset{}{\overset{\int_0^{a+p}=\int_0^a+\int_a^{a+p}}{=\!=\!=}} \int_0^{a+p} f(x) - \int_0^a f(x)\,dx$$

$$\Rightarrow g'(a)=\underbrace{f(a+p)}_{=f(a)}-f(a)=0 \quad \Rightarrow g(a)=\text{Constant}$$

so independent of $a$

## 2.2 Fourier Series

Consider a $2\pi$-periodic function $f(x)$.

We want to write $f(x)$ as a linear combination of

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots$$

$$f(x)=\frac{1}{2}a_0+(a_1\cos x+b_1\sin x)+(a_2\cos 2x+b_2\sin 2x)+\ldots$$

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

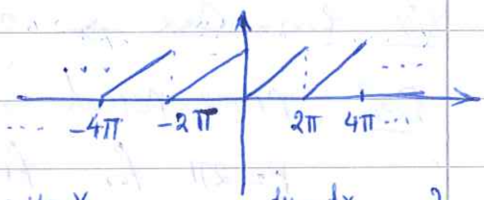
Where the Fourier coefficients are :

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx, \qquad n = 1, 2, \ldots$$

(*)

<u>Ex 11.</u> Find the Fourier series of $f(x) = x,\ 0 < x \leq 2\pi,\ p = 2\pi$




$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{f(x)}^{x}\, dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\, dx = \left\{\begin{array}{ll} u = x & du = dx \\ dv = \cos nx\,dx \rightarrow & v = \frac{1}{n}\sin nx \end{array}\right\}$$

$n = 1, \ldots$

$$= \underbrace{\frac{1}{\pi}\left[\frac{1}{n}x\sin nx\right]_0^{2\pi}}_{=0} - \frac{1}{n\pi}\int_0^{2\pi}\sin nx\, dx = \frac{1}{n^2\pi}\cos nx\Big]_0^{2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\, dx = \cdots = -\frac{2}{n}$$

$n = 1, \ldots$

Hence $$f(x) = \pi + \sum_{n=1}^{\infty}\left(-\frac{2}{n}\right)\sin nx = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

$$= \pi - 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right)$$

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## <u>2.2.2 Complex Representation of Fourier Series</u>

$$e^{inx} = \cos nx + i\sin nx$$
$$e^{-inx} = \cos nx - i\sin nx$$
$$\Longrightarrow \cos nx = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\frac{e^{inx} + e^{-inx}}{2} + b_n\frac{e^{inx} - e^{-inx}}{2i}\right)$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\frac{e^{inx} + e^{-inx}}{2} - ib_n\frac{e^{inx} - e^{-inx}}{2}\right)$$

$$= \underbrace{\frac{a_0}{2}}_{=c_0} + \sum_{n=1}^{\infty} \underbrace{\frac{a_n - ib_n}{2}}_{=c_n} e^{inx} + \sum_{n=1}^{\infty} \underbrace{\frac{a_n + ib_n}{2}}_{=c_{-n}} e^{-inx}$$

$$\Rightarrow f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=1}^{\infty} c_{-n} e^{-inx} = \sum_{-\infty}^{\infty} c_n e^{inx} \quad (2*)$$

Complex Fourier series

Note: $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n - ib_n}{2}$, $c_{-n} = \frac{a_n + ib_n}{2}$ $\Rightarrow \begin{cases} a_0 = 2c_0 \\ a_n = c_n + c_{-n} \\ b_n = i(c_n - c_{-n}) \end{cases}$

## 2.2.3 Derivation of the Euler Formulas

**Lemma 2** $\{e^{inx}\}_{-\infty}^{\infty}$ are orthogonal on $-\pi \leq x \leq \pi$. That is

$$\int_{-\pi}^{\pi} e^{inx} e^{-ikx}\, dx = \begin{cases} 0 & n \neq k \\ 2\pi & n = k \end{cases}$$

Proof

$n \neq k$ : $\displaystyle\int_{-\pi}^{\pi} e^{i(n-k)x}\, dx = \frac{e^{i(n-k)x}}{i(n-k)}\Big]_{x=-\pi}^{x=\pi} = \frac{\cos(n-k)x + i\sin(n-k)x}{i(n-k)}\Big]_{x=-\pi}^{\pi}$

$$= \frac{(-1)^{n-k} - (-1)^{n-k}}{i(n-k)} = 0$$

$n = k$: $\displaystyle\int_{-\pi}^{\pi} e^{0}\, dx = 2\pi.$ $\square$

Now, we obtain $c_n$ in (2*) and we prove (*):

$$(2*) \times e^{-ikx} \xrightarrow[\text{and } \int_{-\pi}^{\pi}]{} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx = \sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-k)x}\, dx$$

$$\overset{k \text{ is fixed}}{=} c_k (2\pi) = 2\pi c_k$$

$$\Rightarrow \boxed{c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx}, \quad k \in \mathbb{Z}$$

That is

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx \quad , \quad n\in\mathbb{Z} .$$

Then, for $n=1,\dots$

$$a_n = C_n + C_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,(\underbrace{e^{-inx}+e^{inx}}_{=2\cos nx})\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

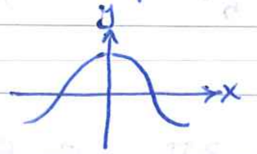
$$b_n = i(C_n - C_{-n}) = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)\,(\underbrace{e^{-inx}-e^{inx}}_{=-2i\sin nx})\,dx = \frac{-2i^2}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

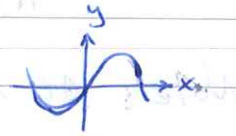
$$a_0 = 2C_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

## 2.2.4 Even and odd functions

Recall: function $f(x)$ is $\begin{cases} \text{even} & \text{if } f(x) = f(-x), \ \forall x \\ \text{odd} & \text{if } f(x) = -f(-x), \ \forall x \end{cases}$

Ex $f(x) = \cos x$


even
Symmetric w.r.t y axis

$f(x) = \sin x$


odd
Symmetric w.r.t the origin

Note: $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & f \text{ even} \\ 0 & f \text{ odd} \end{cases}$

Note: $\overset{+}{\text{even}} \times \overset{+}{\text{even}} = \text{even}$ $\qquad \overset{-}{\text{odd}} \times \overset{-}{\text{odd}} = \text{even}$

$\underset{+}{\text{even}} \times \underset{-}{\text{odd}} = \text{odd}$

**Lemma 3**

If $f(x)$ is even, then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{even}} \underbrace{\cos nx}_{\text{even}}}_{\text{even}}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{even}} \underbrace{\sin nx}_{\text{odd}}}_{\text{odd}}\,dx = 0, \quad n = 1, 2, \dots$$

If $f(x)$ is odd, then

$$a_n = 0, \ n = 0, 1, 2, \dots, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

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Hence

$f$ even $\Rightarrow$ $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$ (Cosine series)

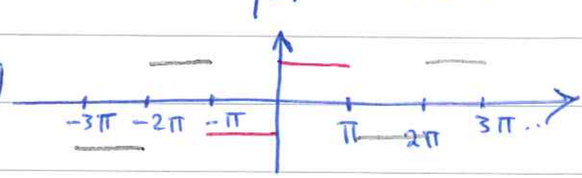
$f$ odd $\Rightarrow$ $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin nx$ (Sine series)

Ex 12 Find the Fourier series (the square wave): $f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$

$f$ is odd $\Rightarrow a_n = 0, \ n = 0, 1, \dots$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \underbrace{f(x)}_{=1} \sin nx\,dx = -\frac{2}{n\pi}\left(\underbrace{\cos n\pi}_{(-1)^n} - \cos 0\right)$$

$$= \frac{2}{\pi}\,\frac{1-(-1)^n}{n}$$



Hence

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin nx = \frac{2}{\pi}\left(2\sin x + \frac{2}{3}\sin 3x + \cdots\right)$$

$$= \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

Note: At $x = \frac{\pi}{2}$:

$$1 = f\left(\frac{\pi}{2}\right) = \frac{4}{\pi}\left(\sin\frac{\pi}{2} + \frac{\sin\frac{3\pi}{2}}{3} + \frac{\sin\frac{5\pi}{2}}{5} + \cdots\right)$$

$$\Longrightarrow \quad 1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

$$\Longrightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1}$$

## 2.2.5 Bessel Inequality and Riemann-Lebesgue lemma

**Thm 9** (Bessel ineq.) If $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$, then

$$\sum_{-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$$

Proof. Recall the Fourier series

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

Denote the $N^{th}$ partial sum of the Fourier series by

$$S_N^f(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

Now

$$0 \leq \left|f(x) - S_N^f(x)\right|^2 = \left|f(x) - \sum_{n=-N}^{N} c_n e^{inx}\right|^2$$

$$\overset{|z|^2 = z\bar{z}}{=} \left(f(x) - \sum_{n=-N}^{N} c_n e^{inx}\right)\overline{\left(f(x) - \sum_{n=-N}^{N} c_n e^{inx}\right)}$$

$$\overset{\overline{x+y}=\bar{x}+\bar{y},\ \overline{xy}=\bar{x}\bar{y}}{=} \left(f(x) - \sum_{n=-N}^{N} c_n e^{inx}\right)\left(\bar{f}(x) - \sum_{n=-N}^{N} \bar{c}_n e^{-inx}\right)$$

$$\overline{e^{inx}} = \overline{\cos nx + i\sin nx} = \cos nx - i \sin nx = e^{-inx}$$

$$= |f(x)|^2 - \sum_{n=-N}^{N} c_n \bar{f}(x) e^{inx} - \sum_{n=-N}^{N} \bar{c}_n f(x) e^{-inx} + \sum_{n=-N}^{N}\sum_{m=-N}^{N} c_n \bar{c}_m e^{inx} e^{-imx}$$

Hence, taking $\int_{-\pi}^{\pi}$,

$$0 \leq \int_{-\pi}^{\pi} \left|f(x) - \sum_{n=-N}^{N} c_n e^{inx}\right|^2 dx = \int_{-\pi}^{\pi} |f(x)|^2 - \sum_{n=-N}^{N} c_n \underbrace{\int_{-\pi}^{\pi} \bar{f}(x) e^{inx} dx}_{=\overline{\int_{-\pi}^{\pi} f(x) e^{-inx} dx} = 2\pi \bar{c}_n}$$

$$- \sum_{n=-N}^{N} \bar{c}_n \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$+ \sum_{n=-N}^{N}\sum_{m=-N}^{N} c_n \bar{c}_m \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx$$

$$\overset{\text{Lemma 2}}{=} \int_{-\pi}^{\pi} |f(x)|^2 dx - 2\pi \sum_{n=-N}^{N} \underbrace{c_n \bar{c}_n}_{=|c_n|^2} - 2\pi \sum_{n=-N}^{N} \bar{c}_n c_n + \sum_{n=-N}^{N} 2\pi c_n \bar{c}_n$$

$$= \int_{-\pi}^{\pi} |f(x)|^2 dx - 2\pi \sum_{n=-N}^{N} |c_n|^2$$

Hence:

$$0 \leq \int_{-\pi}^{\pi} |f(x)|^2 dx - 2\pi \sum_{n=-N}^{N} |c_n|^2$$

$$\Rightarrow \sum_{n=-N}^{N} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$$

$$\xRightarrow[\lim_{N\to\infty}]{} \sum_{-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx. \quad \square$$

~~Lemma~~

Recall: If $\sum_{n=1}^{\infty} x_n$ converges, then $\lim_{n\to\infty} x_n = 0$

Lemma 4 (Riemann-Lebesgue Lemma). If $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$, then

$\lim_{|n|\to\infty} c_n = 0$, $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$

Proof. From Bessel ineq

$$\sum_{-\infty}^{\infty} |c_n|^2 \leq \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx}_{<\infty} < \infty \Rightarrow \sum_{-\infty}^{\infty} |c_n|^2 \text{ converges}$$

$$\Rightarrow \sum_{-\infty}^{\infty} |c_n| \text{ converges}$$

$$\Rightarrow \lim_{\substack{n\to\infty \\ \text{or } n\to-\infty}} |c_n| = 0$$

$$\Rightarrow \lim_{|n|\to\infty} c_n = 0$$

Then $a_n = c_n + c_{-n} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n + \lim_{n\to\infty} c_{-n} = 0 + \lim_{n\to-\infty} c_n$

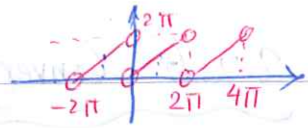
$$= 0 + 0 = 0$$

$$b_n = i(c_n - c_{-n}) \Rightarrow \lim_{n\to\infty} b_n = i\left(\lim_{n\to\infty} c_n - \lim_{n\to\infty} c_{-n}\right) = i(0-0) = 0. \quad \square$$

Note: Lemma 4 says that: If $f \in L_2(-\pi,\pi)$, then its Fourier coefficients tends to zero.

Note: So, if $a_n$ or $b_n$ or $c_n \not\to 0$, then Fourier series does not converge.

**Ex** Recall Ex 11: $f(x)=x,\ 0\leq x<2\pi,\ p=2\pi$




$$a_0=2\pi,\quad a_n=0,\quad b_n=-\frac{2}{n}$$

$$\int_{-\pi}^{\pi}|f(x)|^2\,dx=\int_{-\pi}^{0}f(x)^2\,dx+\int_{0}^{\pi}f(x)^2\,dx$$

$$=\int_{-\pi}^{0}(x+2\pi)^2\,dx+\int_{0}^{\pi}x^2\,dx=\frac{7\pi^3}{3}+\frac{\pi^3}{3}=\frac{8\pi^3}{3}$$

Or $\displaystyle\int_{-\pi}^{\pi}|f(x)|^2\,dx \xlongequal[f^2(x)\text{ is periodic}]{f(x+p)^2=f(x)^2} \int_{0}^{2\pi}|f(x)|^2\,dx=\int_{0}^{2\pi}x^2\,dx=\frac{8\pi^3}{3}$

Hence, as we expected (by Riemann-Lebesgue Lemma):

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}-\frac{2}{n}=0$$

See also Ex 12.

## 2.2.6 Converges of Fourier Series

Recall: For a given series $\sum_{n=1}^{\infty} x_n$, we can define the partial sum

$$S_N = \sum_{n=1}^{N} x_n = x_1 + \cdots + x_N$$

$$\sum_{n=1}^{\infty} x_n \text{ converges to } X \iff \sum_{n=1}^{\infty} x_n = X \iff \lim_{N\to\infty} S_N = X$$

Thm 10 (Dirichlet Theorem)

If $f$ is $2\pi$-periodic, $f, f'$ are p.w. continuous $\Rightarrow$ then $\lim_{N\to\infty} S_N^f(x_0) = \frac{f(x_0-)+f(x_0+)}{2}$, $\forall x_0$

Also when $f$ is continuous at $x_0$

$$\lim_{N\to\infty} S_N^f(x_0) = f(x_0).$$

Note: $f(x_0-) = \lim_{x\to x_0^-} f(x)$, $f(x_0+) = \lim_{x\to x_0^+} f(x)$

Note: For simplicity let $f$ be continuous $\forall x$. Then

$$\lim_{N\to\infty} S_N^f(x) = f(x), \quad \forall x$$

$$\Rightarrow \lim_{N\to\infty} \left(f(x) - S_N^f(x)\right) = 0, \quad \forall x$$

$$\Rightarrow \lim_{N\to\infty} \left|f(x) - S_N^f(x)\right|^2 = 0, \quad \forall x$$

$$\Rightarrow \lim_{N\to\infty} \int_{-\pi}^{\pi} \left|f(x) - S_N^f(x)\right|^2 dx = 0$$

Hence, recalling the proof of Thm 9 (Bessel ineq.),

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left|f(x) - S_N^f(x)\right|^2 dx = \lim_{N\to\infty} \left( \int_{-\pi}^{\pi} |f(x)|^2 dx - 2\pi \sum_{n=-N}^{N} |c_n|^2 \right) = 0$$

## Thm 11 (Parseval identity)

Having the assumption of Thm 10, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \sum_{-\infty}^{\infty}|c_n|^2$$

or

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = 2\sum_{-\infty}^{\infty}|c_n|^2 = 2|c_0|^2 + 2\sum_{n=1}^{\infty}|c_n|^2 + 2\sum_{n=1}^{\infty}|c_{-n}|^2$$

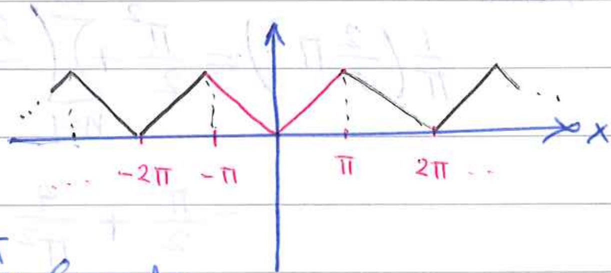
$$\left\{\begin{array}{l} c_0 = \frac{a_0}{2} \\ c_n = \frac{a_n - ib_n}{2} \\ c_{-n} = \frac{a_n + ib_n}{2} \end{array}\right\} \quad = \frac{|a_0|^2}{2} + 2\sum_{n=1}^{\infty}\frac{|a_n|^2+|b_n|^2}{4} + 2\sum_{n=1}^{\infty}\frac{|a_n|^2+|b_n|^2}{4}$$

$$= \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}\left(|a_n|^2+|b_n|^2\right).$$

**Ex 13.** Compute the Fourier series of $f(x)=|x|, -\pi\leq x\leq\pi, p=2\pi$ and then compute $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots$.

~~using the Fourier series expansion of f(x)=|x|, -π≤x≤π, p=2π~~

$f$ is even, so $b_n=0$ and




$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

Hence

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$$

For $n=1,2,\ldots$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \left\{\begin{array}{ll} u=x & \longrightarrow\ du = x \\ dv=\cos nx\,dx & \quad v=\frac{1}{n}\sin nx \end{array}\right\}$$

$$= \frac{2}{\pi n}\left[x\sin nx\right]_0^{\pi} - \frac{2}{\pi n}\int_0^{\pi}\sin nx\,dx = \frac{2}{\pi n^2}(\cos n\pi - 1)$$

$$= \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

So

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left(\frac{(-1)^1-1}{1^2}\cos x + \frac{(-1)^2-1}{2^2}\cos 2x + \frac{(-1)^3-1}{3^2}\cos 3x + \cdots\right)$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left(-2\cos x - \frac{2}{3^2}\cos 3x - \frac{2}{5^2}\cos 5x - \cdots\right)$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

Now, by Parseval Identity

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right)$$

So $\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\int_0^{\pi} |f(x)|^2 dx = 2\int_0^{\pi} x^2 dx = \frac{2}{3}\pi^3$

Hence

$$\frac{1}{\pi}\left(\frac{2}{3}\pi^3\right) = \frac{\pi^2}{2} + \sum_{n=1}^{\infty}\left(\frac{2((-1)^n-1)}{\pi n^2}\right)^2$$

$$= \frac{\pi^2}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{((-1)^n-1)^2}{n^4}$$

$$= \frac{\pi^2}{2} + \frac{4}{\pi^2}\left(\frac{4}{1^4} + \frac{4}{3^4} + \frac{4}{5^4} + \cdots\right)$$

$$= \frac{\pi^2}{2} + \frac{16}{\pi^2}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right)$$

$$\Rightarrow \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^2}{16}\left(\frac{2\pi^2}{3} - \frac{\pi^2}{2}\right) = \frac{\pi^4}{96}$$