5.2 Lagrange interpolation

Having $(q+1)$ distinct interpolation points $a=x_0<x_1<\dots<x_q=b$ in $[a,b]$, we have the Lagrange polynomials

$$\lambda_i(x)=\frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_q)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_q)}=\prod_{\substack{j=0\\ j\neq i}}^{q}\frac{x-x_j}{x_i-x_j}$$

and we have

$$\lambda_i(x_j)=\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

Every polynomial $P\in P^{(q)}_{(a,b)}$ is written

$$P(x)=\sum_{j=0}^{q}P(x_j)\,\lambda_j(x).$$

And for any continuous function $y=f(x)$, $x\in[a,b]$, the Lagrange interpolant is given by

$$\pi_q f(x)=\sum_{j=0}^{q}f(x_j)\lambda_j(x)$$

So, we note that

$$\pi_q f(x_i)=\sum_{j=0}^{q}f(x_j)\lambda_j(x_i)\overset{j=i}{=\!=}f(x_i),\quad i=0,1,\dots,q$$

Ex Interpolate $f(x)=x^3+1$ by Lagrange interpolation of degree ≤ 2, on $[0,2]$.

$x_0=0,\ x_1=1,\ x_2=2\ \to\ f(x_0)=1,\ f(x_1)=2,\ f(x_2)=9$

$$\lambda_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{1}{2}(x-1)(x-2),\quad \lambda_1(x)=-x(x-2)$$

$$\lambda_2(x)=\frac{1}{2}x(x-1)$$

$$\pi_2 f(x)=f(x_0)\lambda_0(x)+f(x_1)\lambda_1(x)+f(x_2)\lambda_2(x)=3x^2-2x+1$$

Thm 5.4 Assume $f \in C^{(q+1)}[a,b]$. Then

$$\left|\Pi_q f(x) - f(x)\right| \leq \frac{1}{(q+1)!}\left|\prod_{i=0}^{q}(x-x_i)\right| \max_{x\in[a,b]}\left|f^{(q+1)}(x)\right|.$$

EX For the previous example, find a bound for the error of the interpolant, for $x = \frac{1}{2}$ and $x \in [0,2]$.

$f(x) = x^3+1 \Rightarrow f^{(3)}(x) = 6$

$$\left|\Pi_2 f(x) - f(x)\right| \leq \frac{1}{(2+1)!}\left|\prod_{i=0}^{2}(x-x_i)\right| \underbrace{\max_{x\in[0,2]}\left|f^{(3)}(x)\right|}_{=6}$$

$$= \frac{1}{6}\left|(x-x_0)(x-x_1)(x-x_2)\right| \max_{x\in[0,2]}|6|$$

$$= \frac{1}{6}\left|x(x-1)(x-2)\right| 6$$

$$\Rightarrow \left|\Pi_2 f(\tfrac{1}{2}) - f(\tfrac{1}{2})\right| \leq \left|(\tfrac{1}{2})(-\tfrac{1}{2})(-\tfrac{3}{2})\right| = \frac{3}{8}$$

And $\left|\Pi_2 f(x) - f(x)\right| \leq \left|(x)(x-1)(x-2)\right|, \quad x \in [0,2]$

$$\leq |x||x-1||x-2| \leq 2(1)(2) = 4$$

$$g(x) = \left|x(x-1)(x-2)\right|, \; x\in[0,2] \Rightarrow \max_{x\in[0,2]} g(x) = 0$$

$$g(x) = \begin{cases} x(x-1)(x-2) = x^3-3x^2+2x & 0 \leq x \leq 1 \\ -x(x-1)(x-2) = -x^3+3x^2-2x & 1 \leq x \leq 2 \end{cases}$$

$$g'(x) = \begin{cases} 3x^2-6x+2 & 0 \leq x < 1 \\ -3x^2+6x-2 & 1 < x \leq 2 \end{cases} \xRightarrow{g'(x)=0} \begin{cases} x = 1-\frac{\sqrt{3}}{3} \approx 0.42265 \\ x = 1+\frac{\sqrt{3}}{3} \approx 1.57735 \end{cases}$$

Hence $$\max_{x\in[0,2]} g(x) = \max\left\{g(0), g(1-\tfrac{\sqrt{3}}{3}), g(1), g(1+\tfrac{\sqrt{3}}{3}), g(2)\right\}$$

$$= \max\left\{\underbrace{g(1-\tfrac{\sqrt{3}}{3})}_{\approx 0.3849}, \underbrace{g(1+\tfrac{\sqrt{3}}{3})}_{\approx 0.3849}\right\} \approx 0.3849$$

5.3 Numerical integration (Quadrature rule)

Sometimes we need to find an approximation to an integral $\int_a^b f(x)dx$.
Simply having an approx. of f, say $f \approx \Pi_q f$, we have

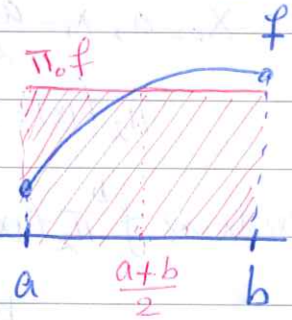
$$\int_a^b f(x)dx \approx \int_a^b \Pi_q f(x)dx$$

① $q=0$: Let $x_0 = \frac{a+b}{2}$, then $\Pi_0 f = f(x_0) = f\left(\frac{a+b}{2}\right)$

$$\int_a^b f(x)dx \approx \int_a^b \Pi_0 f(x)dx = \int_a^b f\left(\frac{a+b}{2}\right)dx = f\left(\frac{a+b}{2}\right)\int_a^b 1\,dx$$
$$= (b-a)f\left(\frac{a+b}{2}\right)$$

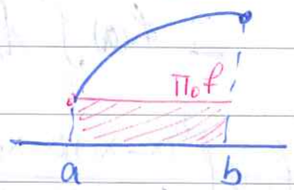
This is called (simple) midpoint rule.

$$\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right)$$



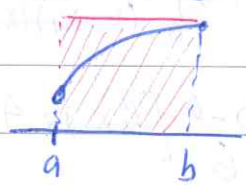
Note: If we choose $x_0 = a$

$$\int_a^b f(x)dx \approx (b-a)f(a)$$



If we choose $x_0 = b$

$$\int_a^b f(x)dx \approx (b-a)f(b)$$



These are called (simple) rectangle rule.

<u>Ex</u>. Approximate $\int_0^1 \sin x\,dx$ by simple midpoint rule.

$$\int_0^1 \sin x\,dx \approx (1-0)\sin\left(\frac{0+1}{2}\right) = \sin 0.5 = 0.4794255 \overset{2D}{\approx} \underline{0.48}$$

② $q=1$: $x_0=a,\ x_1=b$, then $\quad \Pi_1 f(x) = f(x_0)\lambda_0(x) + f(x_1)\lambda_1(x)$

$$= f(a)\,\frac{x-b}{a-b} + f(b)\,\frac{x-a}{b-a}$$

$$\int_a^b f(x)\,dx \approx \int_a^b \Pi_1 f(x)\,dx = \frac{f(a)}{a-b}\int_a^b (x-b)\,dx + \frac{f(b)}{b-a}\int_a^b (x-a)\,dx$$

$$= \frac{b-a}{2}\left(f(a)+f(b)\right)$$

This is called (simple) trapezoidal rule.

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left(f(a)+f(b)\right)$$

③ $q=2$: $x_0=a,\ x_1=\frac{a+b}{2},\ x_2=b$, then

$$\Pi_2 f(x) = f(x_0)\lambda_0(x) + f(x_1)\lambda_1(x) + f(x_2)\lambda_2(x)$$

$$\int_a^b f(x)\,dx \approx \int_a^b \Pi_2 f(x)\,dx$$

$$= f(a)\int_a^b \lambda_0(x)\,dx + f\left(\frac{a+b}{2}\right)\int_a^b \lambda_1(x)\,dx + f(b)\int_a^b \lambda_2(x)\,dx$$

$$= f(a)\int_a^b \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,dx + \cdots$$

$$= \frac{b-a}{6}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

This is called (simple) Simpson's rule.

5.3.1 Composite rules for uniform partitions

To evaluate $I=\int_a^b f(x)dx$, we

(1) Divide $[a,b]$ into N equidistanse subintervals (x_{k-1}, x_k), $k=1,\dots,N$

$$a=x_0<x_1<\cdots<x_{N-1}<x_N=b$$

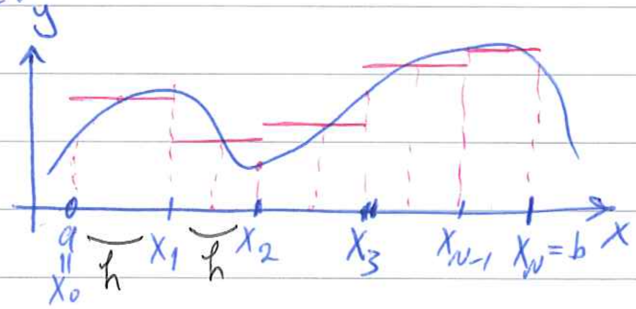
with length $h=|I_k|=\frac{b-a}{N}$.

(2) Write $\int_a^b f(x)dx=\int_{x_0}^{x_1} f(x)dx+\cdots+\int_{x_{N-1}}^{x_N} f(x)dx=\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k} f(x)dx$.

(3) Apply an integration rule (such as mid-point, trapezoid or Simpson) on each subintegral

(M) Composite midpoint rule.

$h=\frac{b-a}{N}$, $\bar{x}_k=\frac{x_{k-1}+x_k}{2}$, $k=1,\dots,N$



Then

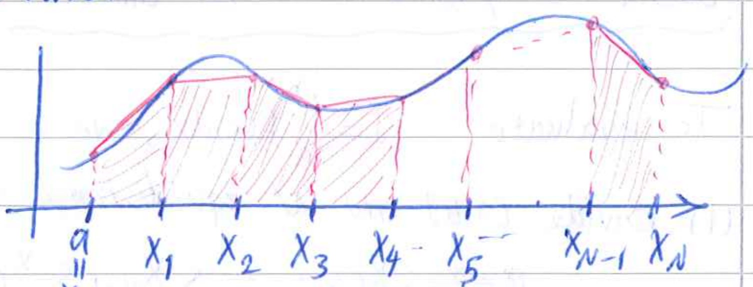
$$\int_a^b f(x)dx=\int_{x_0}^{x_N} f(x)dx=\int_{x_0}^{x_1} f(x)dx+\int_{x_1}^{x_2} f(x)dx+\cdots+\int_{x_{N-1}}^{x_N} f(x)dx$$

$$\approx \underbrace{(x_1-x_0)}_{=h}f(\bar{x}_1)+\underbrace{(x_2-x_1)}_{=h}f(\bar{x}_2)+\cdots+\underbrace{(x_N-x_{N-1})}_{=h}f(\bar{x}_N)$$

$$=h\sum_{k=1}^{N} f(\bar{x}_k)$$

(T) Composite trapezoidal rule.

$$h = \frac{b-a}{N}$$



$$\int_a^b f(x)dx = \int_{x_0}^{x_N} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{N-1}}^{x_N} f(x)\,dx$$

$$\approx \frac{\overbrace{x_1-x_0}^{h}}{2}\left(f(x_0)+f(x_1)\right) + \frac{\overbrace{x_2-x_1}^{h}}{2}\left(f(x_1)+f(x_2)\right) + \cdots$$

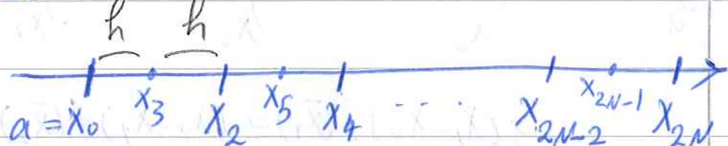
$$+ \frac{x_N - x_{N-1}}{2}\left(f(x_{N-1})+f(x_N)\right)$$

$$= \frac{h}{2}\left\{ f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + f(x_N) \right\}$$

$$= \frac{h}{2}\left\{ f(\underbrace{x_0}_{a}) + f(\underbrace{x_N}_{=b}) + 2\sum_{k=1}^{N-1} f(x_k) \right\}$$

(S) Composite Simpson's rule.

$$h = \frac{b-a}{2N}$$



$$\int_a^b f(x)dx = \int_{x_0}^{x_{2N}} f\,dx = \int_{x_0}^{x_2} f\,dx + \int_{x_2}^{x_4} f\,dx + \cdots + \int_{x_{2N-2}}^{2N} f\,dx$$

$$= \frac{\overbrace{x_2-x_0}^{2h}}{6}\left(f(x_0)+4f(x_1)+f(x_2)\right) + \frac{\overbrace{x_4-x_2}^{2h}}{6}\left(f(x_2)+4f(x_3)+f(x_4)\right)$$

$$+ \cdots + \frac{x_{2N-2}-x_{2N}}{6}\left(f(x_{2N-2})+4f(x_{2N-1})+f(x_{2N})\right)$$

$$= \frac{h}{3}\left\{f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\dots+2f(x_{2N-2}) + 4f(x_{2N-1})+f(x_{2N})\right\}$$

$$= \frac{h}{3}\left\{f(x_0)+f(x_{2N})+4\sum_{k=1}^{N} f(x_{2k-1})+2\sum_{k=1}^{N-1} f(x_{2k})\right\}$$

Error of Numerical Integration

Error in simple Midpoit rule

$$\left|\int_a^b f(x)dx-(b-a)f\left(\frac{a+b}{2}\right)\right| = \frac{(b-a)^3}{24}\left|f''(\eta)\right|, \quad \eta\in(a,b)$$

Error in simple trapezoid rule

$$\left|\int_a^b f(x)dx-\frac{b-a}{2}\left(f(a)+f(b)\right)\right| = \frac{(b-a)^3}{12}\left|f''(\eta)\right|, \quad \eta\in(a,b)$$

Error in simple Simpson's rule

$$\left|\int_a^b f(x)dx-\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right| = \frac{(b-a)^5}{2880}\left|f^{(4)}(\eta)\right|, \quad \eta\in(a,b)$$

Error in composite Midpoint rule

$$\left|\int_a^b f(x)dx-M_N\right| = \frac{h^2(b-a)}{24}\left|f''(\xi)\right|, \quad \xi\in(a,b)$$

Error in composite trapezoidal rule

$$\left|\int_a^b f(x)dx-T_N\right| = \frac{h^2(b-a)}{12}\left|f''(\xi)\right|, \quad \xi\in(a,b)$$

Error in composite Simpson's rule

$$\left|\int_a^b f(x)dx-S_N\right| = \frac{h^4(b-a)}{180}\left|f^{(4)}(\xi)\right|, \quad \xi\in(a,b)$$

Note: • Midpoint and trapezoid rules are exact for polynomials of degree ≤ 1, due to $f''(\eta)$ in the error.

If $f(x) = cx + d \Rightarrow f''(x) = 0 \Rightarrow$ error $= 0$

• Simpson's rule is exact for $f \in P^{(3)}(a,b)$, due to $f^{(4)}(\eta)$ in the error.

If $f \in P^{(3)}(a,b) \Rightarrow f^{(4)}(x) = 0 \Rightarrow$ error $= 0$

Ex: $f(x) = 2x^3 - 1, \quad x \in [0,1]$

$$\int_0^1 f(x)dx = \left(\frac{x^4}{4} - x\right]_0^1 = -\frac{1}{2}$$

Simpson's rule: $\int_0^1 f(x)dx \approx \frac{1-0}{6}\left(f(0) + 4f\left(\tfrac{1}{2}\right) + f(1)\right)$

$$= \frac{1}{6}\left(-1 + 4\left(2\frac{1}{2^3} - 1\right) + 1\right) = \frac{-1}{2}$$

$\Rightarrow$ error $= 0$

Chapter 7: Two-point BVP

7.2 The FEM

Formulate the FEM using continuous piecewise linear polynomials (is called cG(1) method) for two-point BVP

$$\begin{cases} -(a(x)u'(x))' = f(x), \quad 0 < x < 1 \\ u(0) = 0, \; u(1) = 0 \end{cases} \qquad (1)$$

Recall the function space

$$H_0^1 = H_0^1(0,1) = \left\{ v \;\middle|\; \int_0^1 \{v(x)^2 + (v'(x))^2\} dx < \infty, \; v(0)=0, \; v(1)=0 \right\}$$

Variational formulation (VF): Multiply the DE by a test function $v \in V^0 = H_0^1$, then integrate over $(0,1)$ and integrate by parts:

$$-\int_0^1 (au')' v \, dx = \int_0^1 fv \, dx$$

$$\xRightarrow{\text{I.P}} \left[-au'v\right]_{x=0}^{x=1} + \int_0^1 au'v' dx = \int_0^1 fv \, dx$$

$$\xRightarrow[v \in V]{} -a(1)u'(1)\underbrace{v(1)}_{=0} + a(0)u'(0)\underbrace{v(0)}_{=0} + \int_0^1 au'v' dx = \int_0^1 fv \, dx$$

$$\Rightarrow \int_0^1 au'v' dx = \int_0^1 fv \, dx$$

VF is:

$$\begin{cases} \text{Find } u = u(x) \in V^0 \text{ such that} \\ \int_0^1 au'v' dx = \int_0^1 fv \, dx, \quad \forall v \in V^0 \end{cases} \qquad (2)$$

FEM:

Consider a partition for $[0,1]$,

$$\mathcal{T}_h: \quad 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$$

into subintervals $I_j = (x_{j-1}, x_j)$ with length $h_j = x_j - x_{j-1}$, $j = 1, \ldots, m+1$.

Consider the corresponding FE space (finite dimensional space):

$$V_h^0 = \{ v \mid v \text{ is cont. p.w. linear on } \mathcal{T}_h,\ v(0)=0,\ v(1)=0 \}$$
$$= \operatorname{Span}\{\varphi_1, \ldots, \varphi_m\}$$

FEM is:

$$\begin{cases} \text{Find } U \in V_h^0 \text{ such that} \\ \displaystyle\int_0^1 a U' \chi' \, dx = \int_0^1 f \chi \, dx, \quad \forall \chi \in V_h^0 \end{cases} \qquad (3)$$

Note that, since FEM (3) holds $\forall \chi \in V_h^0$, so in particular it holds for every $\chi = \varphi_i$, $i = 1, \ldots, m$. That is

$$\begin{cases} \text{Find } U \in V_h^0 \text{ such that} \\ \displaystyle\int_0^1 a U' \varphi_i' \, dx = \int_0^1 f \varphi_i \, dx, \quad i = 1, \ldots, m \end{cases} \qquad (4)$$

Also note that $U \in V_h^0 \implies U(x) = \sum_{j=1}^{m} \xi_j \varphi_j(x)$ ← unknowns

$$= \xi_1 \varphi_1(x) + \cdots + \xi_m \varphi_m(x)$$

So $U'(x) = \left(\sum_{j=1}^{m} \xi_j \varphi_j(x) \right)' = \sum_{j=1}^{m} \xi_j \varphi_j'(x)$

Now substituting U in (4) we have

$$\int_0^1 a \left(\sum_{j=1}^{m} \xi_j \varphi_j' \right) \varphi_i' \, dx = \int_0^1 f \varphi_i \, dx, \quad i = 1, \ldots, m$$

$$\Rightarrow \sum_{j=1}^{m} \left(\int_0^1 a \varphi_j' \varphi_i' \, dx \right) \xi_j = \int_0^1 f \varphi_i \, dx, \quad i=1,\dots,m$$

In matrix form

$$A\xi = b$$

where

$$A = \{a_{ij}\}_{i,j=1}^{m} = \left\{ \int_0^1 a(x) \varphi_j'(x) \varphi_i'(x) \, dx \right\}_{i,j=1}^{m}, \quad \xi = \{\xi_j\}_{j=1}^{m}$$

$$b = \{b_i\}_{i=1}^{m} = \left\{ \int_0^1 f(x) \varphi_i(x) \, dx \right\}_{i=1}^{m}$$

7.4 FEM for Convection-diffusion-absorption BVP

Here we apply FEM to more general BVP and other type of boundary conditions.

Ex 7.1 Solve the following BVP by cG(1) finite element method.

$$\begin{cases} -u''(x)+4u(x)=0, & 0<x<1 \\ u(0)=\alpha, \quad u(1)=\beta & (\alpha\neq 0,\ \beta\neq 0) \end{cases} \quad (1)$$

Note that we have non-homogeneous Dirichlet B.C.

VF: Define

$$V=\{v \mid \int_0^1 \{v^2+(v')^2\}dx<\infty,\ v(0)=\alpha,\ v(1)=\beta\}$$

and

$$V^0=H_0^1=\{v \mid \int_0^1 \{v^2+(v')^2\}dx<\infty,\ v(0)=0,\ v(1)=0\}$$

Now, multiply the DE by a test function $v\in V^0$, then integrate over $(0,1)$ and integrate by parts,

$$-\int_0^1 u''v\,dx+4\int_0^1 uv\,dx=0$$

$$\underset{I.P.}{\Longrightarrow} [-u'v]_{x=0}^{x=1}+\int_0^1 u'v'dx+4\int_0^1 uv\,dx=0$$

$$\underset{v\in V^0}{\Longrightarrow} -u'(1)\underbrace{v(1)}_{=0}+u'(0)\underbrace{v(0)}_{=0}+\int_0^1 u'v'dx+4\int_0^1 uv\,dx=0$$

$$\Rightarrow \int_0^1 u'v'dx+4\int_0^1 uv\,dx=0$$

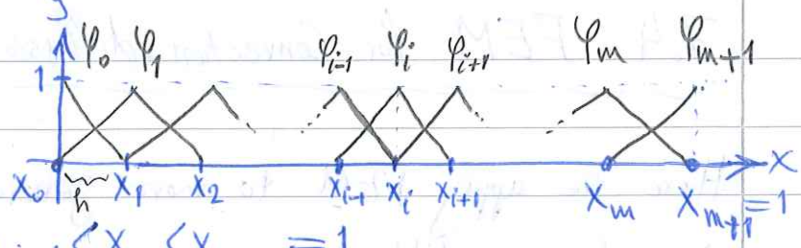
VF is:

$$\begin{cases} \text{Find } \overset{\text{trial space}}{u\in V} \text{ such that} \\ \int_0^1 u'v'dx+4\int_0^1 uv\,dx=0, \quad \forall v\in \overset{\text{test space}}{V^0} \end{cases} \quad (2)$$

Note that the trial space and the test space are different.

FEM:

Consider a uniform partition



$\mathcal{T}_h:\quad 0=x_0<x_1<\cdots<x_m<x_{m+1}=1$

with $I_j=(x_{j-1},x_j)$, $h=x_j-x_{j-1}$, $j=1,\ldots,m+1$. Note that

$$\varphi_0(x)=\begin{cases}\frac{x-x_1}{-h} & 0\leq x\leq x_1\\ 0 & \text{o.w.}\end{cases},\qquad \varphi_{m+1}(x)=\begin{cases}\frac{x-x_m}{h} & x_m\leq x\leq x_{m+1}=1\\ 0 & \text{o.w.}\end{cases}$$

Consider the FE spaces

$$V_h=\{v\mid v \text{ is cont. p.w. linear on } \mathcal{T}_h;\ v(0)=\alpha,\ v(1)=\beta\}$$

and

$$V_h^0=\{v\mid v \text{ is cont. p.w. linear on } \mathcal{T}_h,\ v(0)=0,\ v(1)=0\}$$

Note: $V_h=\text{Span}\{\varphi_0,\varphi_1,\ldots,\varphi_m,\varphi_{m+1}\}\subset V \Rightarrow \dim(V_h)=m+2$

$V_h^0=\text{Span}\{\varphi_1,\ldots,\varphi_m\}\subset V^0 \Longrightarrow \dim(V_h^0)=m$

Also note that $V_h\subset V$ is a subspace and $V_h^0\subset V^0$ is also a subspace.

FEM is:

$$\begin{cases}\text{Find } U\in V_h \text{ such that}\\ \int_0^1 U'\chi'\,dx+4\int_0^1 U\chi\,dx=0,\quad \forall \chi\in V_h^0\end{cases}\qquad (3)$$

In particular for $\chi=\varphi_i$, $i=1,\ldots,m$

$$\begin{cases}\text{Find } U\in V_h \text{ such that}\\ \int_0^1 U'\varphi_i'\,dx+4\int_0^1 U\varphi_i\,dx=0,\quad i=1,\ldots,m\end{cases}\qquad (4)$$

Note that

$$U\in V_h \Rightarrow U(x)=\sum_{j=0}^{m+1}\xi_j\varphi_j(x)=\xi_0\varphi_0(x)+\xi_1\varphi_1(x)+\cdots+\xi_m\varphi_m(x)+\xi_{m+1}\varphi_{m+1}(x)$$

Since $U(0)=U(x_0)=\alpha$ and $U(1)=U(x_{m+1})=\beta$, So

$$\alpha = U(x_0) = \xi_0 \underbrace{\varphi_0(x_0)}^{=1} + \xi_0 \underbrace{\varphi_1(x_0)}^{=0} + \dots + \xi_m \underbrace{\varphi_m(x_0)}^{=0} + \xi_{m+1} \underbrace{\varphi_{m+1}(x_0)}^{=0}$$
$$= \xi_0$$

$$\beta = U(x_{m+1}) = \xi_0 \overbrace{\varphi_0(x_{m+1})}^{=0} + \dots + \xi_m \overbrace{\varphi_m(x_{m+1})}^{=0} + \xi_{m+1} \underbrace{\varphi_{m+1}(x_{m+1})}_{=1}$$
$$= \xi_{m+1}$$

That is

$$U(x) = \alpha\varphi_0(x) + \left(\sum_{j=1}^{m} \xi_j \varphi_j(x)\right) + \beta\varphi_{m+1}(x)$$

and we need to find m unknowns $\xi_1, \dots, \xi_m$.

Substituting U in (4), for $i = 1, \dots, m$, we have

$$\int_0^1 \left\{\alpha\varphi_0'(x) + \left(\sum_{j=1}^{m} \xi_j \varphi_j'(x)\right) + \beta\varphi_{m+1}'(x)\right\}\varphi_i'(x)\,dx$$
$$+ 4\int_0^1 \left\{\alpha\varphi_0(x) + \left(\sum_{j=1}^{m} \xi_j \varphi_j(x)\right) + \beta\varphi_{m+1}(x)\right\}\varphi_i(x)\,dx = 0$$

$$\Rightarrow \sum_{j=1}^{m}\left(\int_0^1 \varphi_j'(x)\varphi_i'(x)\,dx\right)\xi_j + \alpha\int_0^1 \varphi_0'(x)\varphi_i'(x)\,dx + \beta\int_0^1 \varphi_{m+1}'(x)\varphi_i'(x)\,dx$$
$$+ \sum_{j=1}^{m}\left(4\int_0^1 \varphi_j(x)\varphi_i(x)\,dx\right)\xi_j + 4\alpha\int_0^1 \varphi_0(x)\varphi_i(x)\,dx + 4\beta\int_0^1 \varphi_{m+1}(x)\varphi_i(x)\,dx$$
$$= 0$$

We keep the unknowns at the left side, and the known values at the right side, for $i = 1, \dots, m$,

$$\sum_{j=1}^{m}\left\{\underbrace{\int_0^1 \varphi_j'(x)\varphi_i'(x)\,dx}_{=S_{ij}} + \underbrace{4\int_0^1 \varphi_j(x)\varphi_i(x)\,dx}_{=m_{ij}}\right\}\xi_j$$

$$\left.\begin{aligned} &= -\left\{\int_0^1 \varphi_0'(x)\varphi_i'(x)\,dx + 4\int_0^1 \varphi_0(x)\varphi_i(x)\,dx\right\}\alpha \\ &\quad -\left\{\int_0^1 \varphi_{m+1}'(x)\varphi_i'(x)\,dx + 4\int_0^1 \varphi_{m+1}(x)\varphi_i(x)\,dx\right\}\beta \end{aligned}\right\} = b_i \qquad (5)$$

In matrix form

$$\underbrace{(S+4M)}_{=A_{m\times m}}\xi = b$$

Where S is the stiffness matrix $S=\{s_{ij}\}_{i,j=1}^{m}=\left\{\int_0^1 \varphi_j'\varphi_i'\,dx\right\}_{i,j=1}^{m}$, that is,

$$S=\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}_{m\times m}$$

M is the mass matrix : $M=\{m_{ij}\}_{i,j=1}^{m}=\left\{\int_0^1 \varphi_j\varphi_i\,dx\right\}_{i,j=1}^{m}$

$\xi=\{\xi_j\}_{j=1}^{m}$ is the vector of unknown coefficients,

and $b=\{b_i\}_{i=1}^{m}$ is the load vector with

$$b_1=\left\{-\int_0^1 \varphi_0'\varphi_1'\,dx-4\int_0^1\varphi_0\varphi_1\,dx\right\}\alpha=\left\{-\int_0^{x_1}\varphi_0'\varphi_1'\,dx-4\int_0^{x_1}\varphi_0\varphi_1\,dx\right\}\alpha$$

$$=\left\{-\int_0^{x_1}\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right)dx-4\int_0^{x_1}\left(\frac{x-x_1}{-h}\right)\left(\frac{x-x_0}{h}\right)dx\right\}\alpha$$

$$=\left(\frac{1}{h}-\frac{2h}{3}\right)\alpha$$

$$b_i=\cdots=b_{m-1}=0$$

$$b_m=\left\{-\int_0^1 \varphi_{m+1}'\varphi_m'\,dx-4\int_0^1\varphi_{m+1}\varphi_m\,dx\right\}\beta$$

$$=\left\{-\int_{x_m}^{x_{m+1}}\varphi_{m+1}'\varphi_m'\,dx-4\int_{x_m}^{x_{m+1}}\varphi_{m+1}\varphi_m\,dx\right\}\beta$$

$$=\left\{-\int_{x_m}^{x_{m+1}}\left(\frac{1}{h}\right)\left(-\frac{1}{h}\right)dx-4\int_{x_m}^{x_{m+1}}\left(\frac{x-x_m}{h}\right)\left(\frac{x-x_{m+1}}{-h}\right)dx\right\}\beta$$

$$\Longrightarrow b_m = \left(\frac{1}{h} - \frac{2h}{3}\right)\beta$$

Similar to the stiffness matrix, we compute the entries of the mass matrix M.

$$m_{ij} = m_{ji} = \int_0^1 \varphi_j \varphi_i \, dx = \begin{cases} 0 & |i-j|>1 \\ \int_0^1 \varphi_i^2 \, dx & i=j \\ \int_0^1 \varphi_i \varphi_{i+1} \, dx & i=j+1 \end{cases}$$

$$m_{ii} = \int_0^1 \varphi_i^2(x)\, dx = \int_{x_{i-1}}^{x_i} \left(\frac{x-x_{i-1}}{h}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(\frac{x-x_{i+1}}{-h}\right)^2 dx = \frac{2}{3}h, \quad i=1,\dots,m$$

$$m_{i,i+1} = m_{i+1,i} = \int_0^1 \varphi_i(x)\varphi_{i+1}(x)\, dx = \int_{x_i}^{x_{i+1}} \left(\frac{x-x_{i+1}}{-h}\right)\left(\frac{x-x_i}{h}\right) dx = \frac{1}{6}h, \quad i=1,\dots,m-1$$

Hence

$$M = h\begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 & \cdots & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{6} \\ 0 & \cdots & 0 & \frac{1}{6} & \frac{2}{3} \end{bmatrix}_{m\times m} = \frac{h}{6}\begin{bmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 4 \end{bmatrix}_{m\times m}$$

See the next pages for some calculation.

Equations in 5

$$
\left\{
\begin{aligned}
&\left(\int_0^1 \varphi_1'\varphi_1' dx + 4\int_0^1 \varphi_1\varphi_1 dx\right)\xi_1 + \left(\int_0^1 \varphi_2'\varphi_1' dx + 4\int_0^1 \varphi_2\varphi_1 dx\right)\xi_2 + \cdots + \left(\int_0^1 \varphi_m'\varphi_1' dx + 4\int_0^1 \varphi_m\varphi_1 dx\right)\xi_m = \left(-\int_0^1 \varphi_0'\varphi_1' dx - 4\int_0^1 \varphi_0\varphi_1 dx\right)\alpha \\
&\qquad + \Big(\underbrace{-\int_0^1 \varphi_{m+1}'\varphi_1' dx}_{=0} \underbrace{- 4\int_0^1 \varphi_{m+1}\varphi_1 dx}_{=0}\Big)\beta \\
&\left(\int_0^1 \varphi_1'\varphi_2' dx + 4\int_0^1 \varphi_1\varphi_2 dx\right)\xi_1 + \left(\int_0^1 \varphi_2'\varphi_2' dx + 4\int_0^1 \varphi_2\varphi_2 dx\right)\xi_2 + \cdots + \left(\int_0^1 \varphi_m'\varphi_2' dx + 4\int_0^1 \varphi_m\varphi_2 dx\right)\xi_m = \Big(\underbrace{-\int_0^1 \varphi_0'\varphi_2' dx}_{=0} \underbrace{- 4\int_0^1 \varphi_0\varphi_2 dx}_{=0}\Big)\alpha \\
&\qquad + \Big(\underbrace{-\int_0^1 \varphi_{m+1}'\varphi_2' dx}_{=0} \underbrace{- 4\int_0^1 \varphi_{m+1}\varphi_2 dx}_{=0}\Big)\beta \\
&\qquad\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
&\left(\int_0^1 \varphi_1'\varphi_{m-1}' dx + 4\int_0^1 \varphi_1\varphi_{m-1} dx\right)\xi_1 + \left(\int_0^1 \varphi_2'\varphi_{m-1}' dx + 4\int_0^1 \varphi_2\varphi_{m-1} dx\right)\xi_2 + \cdots + \left(\int_0^1 \varphi_m'\varphi_{m-1}' dx + 4\int_0^1 \varphi_m\varphi_{m-1} dx\right)\xi_m = \Big(\overbrace{-\int_0^1 \varphi_0'\varphi_{m-1}' dx}^{=0} \overbrace{- 4\int_0^1 \varphi_0\varphi_{m-1} dx}^{=0}\Big)\alpha \\
&\qquad + \Big(\underbrace{-\int_0^1 \varphi_{m+1}'\varphi_{m-1}' dx}_{=0} \underbrace{- 4\int_0^1 \varphi_{m+1}\varphi_{m-1} dx}_{=0}\Big)\beta \\
&\left(\int_0^1 \varphi_1'\varphi_m' dx + 4\int_0^1 \varphi_1\varphi_m dx\right)\xi_1 + \left(\int_0^1 \varphi_2'\varphi_m' dx + 4\int_0^1 \varphi_2\varphi_m dx\right)\xi_2 + \cdots + \left(\int_0^1 \varphi_m'\varphi_m' dx + 4\int_0^1 \varphi_m\varphi_m dx\right)\xi_m = \Big(\underbrace{-\int_0^1 \varphi_0'\varphi_m' dx}_{=0} \underbrace{- 4\int_0^1 \varphi_0\varphi_m dx}_{=0}\Big)\alpha \\
&\qquad + \left(-\int_0^1 \varphi_{m+1}'\varphi_m' dx - 4\int_0^1 \varphi_{m+1}\varphi_m dx\right)\beta
\end{aligned}
\right.
$$

$$\int_0^{x_1}(x-x_1)(x-\overset{0}{\overset{\shortparallel}{x_0}})\,dx = \int_0^h x(x-h)\,dx = \int_0^h (x^2-hx)\,dx = \left(\frac{x^3}{3}-\frac{hx^2}{2}\right]_0^h = \frac{-h^3}{6}$$

$$\int_{x_m}^{x_{m+1}}(x-x_m)(x-x_{m+1})\,dx = \left\{\begin{matrix} t = x-x_m \\ dt = dx \end{matrix}\right\} = \int_0^h t(t-h)\,dt = -\frac{h^3}{6}$$

$$\int_{x_{i-1}}^{x_i}(x-x_{i-1})^2\,dx = \frac{(x-x_{i-1})^3}{3}\Bigg]_{x=x_{i-1}}^{x=x_i} = \frac{(x_i-x_{i-1})^3}{3} - 0 = \frac{h^3}{3}$$

$$\int_{x_i}^{x_{i+1}}(x-x_i)(x-x_{i+1})\,dx = \left\{\begin{matrix} t = x-x_i \\ dt = dx \end{matrix}\right\} = \int_0^h t(t-h)\,dt = -\frac{h^3}{6}$$

$$b_1 = -\left\{\int_0^1 \varphi_0'\varphi_1'\,dx + 4\int_0^1 \varphi_0\varphi_1\,dx\right\}\alpha - \left\{\underbrace{\int_0^1 \varphi_{m+1}'\varphi_1'\,dx}_{=0} + 4\underbrace{\int_0^1 \varphi_{m+1}\varphi_1\,dx}_{=0}\right\}\beta$$

$$= -\left\{\int_0^1 \varphi_0'\varphi_1'\,dx + 4\int_0^1 \varphi_0\varphi_1\,dx\right\}\alpha$$

Remark 7.4. Simpson's formula is exact for polynomials of degree ≤ 3. (See page 126 in the book)

We know that $\varphi_i(x)$ is a polynomial of order 1, so $\varphi_i(x)\varphi_{i+1}(x)$ is of order 2. Thus, we can evaluate the entries of the mass matrix by the Simpson's rule:

$$m_{i,i+1} = \int_0^1 \varphi_i(x)\varphi_{i+1}(x)\,dx = \int_{x_i}^{x_{i+1}} \varphi_i(x)\varphi_{i+1}(x)\,dx$$

$$= \frac{x_{i+1}-x_i}{6}\Bigg[\varphi_i(x_i)\underbrace{\varphi_{i+1}(x_i)}_{=0} + 4\varphi_i\left(\frac{x_i+x_{i+1}}{2}\right)\varphi_{i+1}\left(\frac{x_i+x_{i+1}}{2}\right) + \underbrace{\varphi_i(x_{i+1})}_{=0}\varphi_{i+1}(x_{i+1})\Bigg]$$

$$= \frac{h}{6}(4)\varphi_i\left(\frac{x_i+x_{i+1}}{2}\right)\varphi_{i+1}\left(\frac{x_i+x_{i+1}}{2}\right) = \frac{h}{6}(4)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{h}{6}$$

In a similar way for m_{ii}.

Ex 7.2 Solve the following convection-diffusion problem by cG(1) finite element method.

$$\begin{cases} -\varepsilon u''(x) + p u'(x) = r, & 0<x<1 \\ u(0)=0, \quad u'(1)=\beta & (\beta \neq 0) \end{cases} \quad (6)$$

Note: homogeneous Dirichlet B.C. at $x=0$ and non-homogeneous Neumann B.C. at $x=1$.

For simplicity: ε, p are positive real numbers, and $r \in \mathbb{R}$

VF: Define

$$V = \left\{ v \,\middle|\, \int_0^1 \{v^2 + (v')^2\} dx < \infty, \; v(0)=0 \right\}$$

Now, multiply the DE by a test function $v \in V$, then integrate over $(0,1)$ and integrate by parts,

$$-\varepsilon \int_0^1 u'' v \, dx + p \int_0^1 u' v \, dx = r \int_0^1 v \, dx$$

$$\xRightarrow{\text{I.P.}} \left[-\varepsilon u' v\right]_{x=0}^{x=1} + \varepsilon \int_0^1 u' v' dx + p \int_0^1 u' v \, dx = r \int_0^1 v \, dx$$

$$\underset{v \in V}{\Longrightarrow} -\varepsilon \underbrace{u'(1)}_{=\beta} v(1) + \varepsilon u'(0) \underbrace{v(0)}_{=0} + \varepsilon \int_0^1 u' v' dx + p \int_0^1 u' v \, dx = r \int_0^1 v \, dx$$

$$\Rightarrow \varepsilon \int_0^1 u' v' dx + p \int_0^1 u' v \, dx = r \int_0^1 v \, dx + \varepsilon \beta v(1)$$

VF is:

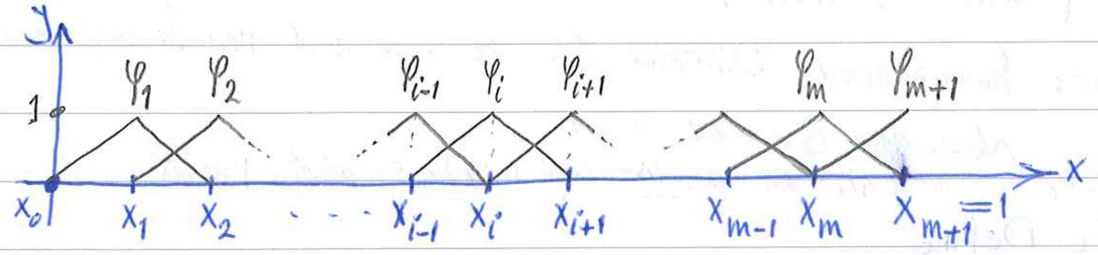
$$\begin{cases} \text{Find } u \in V \text{ such that} \\ \varepsilon \int_0^1 u' v' dx + p \int_0^1 u' v \, dx = r \int_0^1 v \, dx + \varepsilon \beta v(1), \quad \forall v \in V \end{cases} \quad (7)$$

Note that trial space = test space.

FEM: Consider a uniform partition

$$\mathcal{T}_h: \quad 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$$

with $I_j = (x_{j-1}, x_j)$, $h = x_j - x_{j-1}$, $j = 1, \ldots, m+1$



Note that we do not need φ_0, since we have homogeneous Dirichlet B.C. at $x=0$. But we need φ_{m+1}, since $u(1)$ is not given (we have Neumann B.C. at $x=1$).
Note that φ_{m+1} is a (left) half hat function.
Consider the FE space

$$V_h = \{ v \mid v \text{ is cont. p.w. linear on } \mathcal{T}_h,\ v(0)=0 \} \underset{\text{subspace}}{\subset} V$$

and note that

$$V_h = \operatorname{Span}\{\varphi_1, \ldots, \varphi_m, \varphi_{m+1}\} \Rightarrow \dim(V_h) = m+1$$

FEM is:

$$\left\{ \begin{array}{l} \text{Find } U \in V_h \text{ such that} \\ \varepsilon \int_0^1 U' \chi' \, dx + p \int_0^1 U' \chi \, dx = r \int_0^1 \chi \, dx + \varepsilon \beta \chi(1), \quad \forall \chi \in V_h \end{array} \right. \quad (8)$$

Since (8) holds $\forall \chi \in V_h$, so in particular it holds for every $\chi = \varphi_i$, $i = 1, \ldots, m+1$. That is

$$\left\{ \begin{array}{l} \text{Find } U \in V_h \text{ such that} \\ \varepsilon \int_0^1 U' \varphi_i' \, dx + p \int_0^1 U' \varphi_i \, dx = r \int_0^1 \varphi_i \, dx + \varepsilon \beta \varphi_i(1), \quad i = 1, 2, \ldots, m+1 \end{array} \right. \quad (9)$$

$$U \in V_h \Rightarrow U(x) = \sum_{j=1}^{m+1} \xi_j \varphi_j(x)$$
$$= \xi_1 \varphi_1(x) + \cdots + \xi_{m+1} \varphi_{m+1}(x)$$
$$\Rightarrow U'(x) = \sum_{j=1}^{m+1} \xi_j \varphi_j'(x)$$

We need to find the unknowns $\xi_1, \dots, \xi_m, \xi_{m+1}$.
So we substitute U in ⑧ (or in fact in ⑨), and we have

$$\varepsilon \int_0^1 \left(\sum_{j=1}^{m+1} \xi_j \varphi_j'(x) \right) \varphi_i'(x)\,dx + p \int_0^1 \left(\sum_{j=1}^{m+1} \xi_j \varphi_j'(x) \right) \varphi_i(x)\,dx$$
$$= r \int_0^1 \varphi_i(x)\,dx + \varepsilon \beta \varphi_i(1), \quad i = 1, \dots, m+1$$

$$\Rightarrow \sum_{j=1}^{m+1} \Big\{ \varepsilon \underbrace{\int_0^1 \varphi_j' \varphi_i'\,dx}_{=\tilde{S}_{ij}} + p \underbrace{\int_0^1 \varphi_j' \varphi_i\,dx}_{=c_{ij}} \Big\} \xi_j = \underbrace{r \int_0^1 \varphi_i\,dx + \varepsilon \beta \varphi_i(1)}_{=b_i}, \quad i = 1, \dots, m+1$$

In matrix form
$$(\varepsilon \tilde{S} + pC)\xi = b$$

Where $\tilde{S}$ is the stiffness matrix $\tilde{S} = \{\tilde{S}_{ij}\}_{i,j=1}^{m+1} = \left\{ \int_0^1 \varphi_j' \varphi_i'\,dx \right\}_{i,j=1}^{m+1}$, that is

$$\tilde{S} = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & 2 & -1 & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix}_{(m+1)\times(m+1)}$$

Note that the last diagonal element, corresponding to the half-hat function φ_{m+1}, is

$$\tilde{S}_{m+1,m+1} = \int_0^1 \varphi_{m+1}'(x) \varphi_{m+1}'(x)\,dx = \int_{x_m}^{x_{m+1}} (\varphi_{m+1}'(x))^2\,dx$$
$$= \int_{x_m}^{x_{m+1}} \left(\frac{1}{h}\right)^2 dx = \frac{1}{h^2} \int_{x_m}^{x_{m+1}} dx = \frac{1}{h^2} (\overbrace{x_{m+1} - x_m}^{=h}) = \frac{1}{h}$$

(Compare it with the stiffness matrix $S_{m\times m}$ in page 68 (in this lecture note)

The convection matrix $C = \{c_{ij}\}_{i,j=1}^{m+1} = \left\{\int_0^1 \varphi_j' \varphi_i \, dx\right\}_{i,j=1}^{m+1}$

Obviously, $c_{ij} = 0$, for $|i-j| > 1$. Then,

$$c_{ii} = \int_0^1 \varphi_i \varphi_i' \, dx = \int_{x_{i-1}}^{x_{i+1}} \varphi_i(x) \varphi_i'(x) \, dx = \int_{x_{i-1}}^{x_i} \varphi_i(x)\varphi_i'(x)\,dx + \int_{x_i}^{x_{i+1}} \varphi_i(x)\varphi_i'(x)\,dx$$

$$= \int_{x_{i-1}}^{x_i} \left(\frac{x-x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{1}{h}\right) dx + \int_{x_i}^{x_{i+1}} \left(\frac{x-x_{i+1}}{x_i - x_{i+1}}\right)\left(-\frac{1}{h}\right) dx$$

$$= \frac{1}{h^2}\int_{x_{i-1}}^{x_i} (x-x_{i-1})\,dx + \left(-\frac{1}{h}\right)^2 \int_{x_i}^{x_{i+1}} (x-x_{i+1})\,dx$$

$$= \frac{1}{h^2} \frac{(x-x_{i-1})^2}{2}\Bigg|_{x_{i-1}}^{x_i} + \frac{1}{h^2} \frac{(x-x_{i+1})^2}{2}\Bigg|_{x_i}^{x_{i+1}} = \frac{1}{h^2}\left(\frac{h^2}{2}\right) + \frac{1}{h^2}\left(\frac{-h^2}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

So $c_{ii} = 0$, for $i = 1, \dots, m$. But for $i = m+1$ we have

$$c_{m+1,m+1} = \int_0^1 \varphi_{m+1} \varphi_{m+1}' \, dx = \int_{x_m}^{x_{m+1}} \varphi_{m+1}(x)\varphi_{m+1}'(x)\,dx = \frac{1}{2}$$

Also

$$c_{i,i+1} = \int_0^1 \varphi_i \varphi_{i+1}' \, dx = \int_{x_i}^{x_{i+1}} \varphi_i(x)\varphi_{i+1}'(x)\,dx = \int_{x_i}^{x_{i+1}} \left(\frac{x-x_{i+1}}{x_i - x_{i+1}}\right)\left(\frac{1}{h}\right) dx$$

$$= -\frac{1}{h^2} \frac{(x-x_{i+1})^2}{2}\Bigg|_{x_i}^{x_{i+1}} = -\frac{1}{h^2}\left(0 - \frac{(-h)^2}{2}\right) = \frac{1}{2}$$

$$c_{i+1,i} = \int_0^1 \varphi_{i+1}\varphi_i' \, dx = \int_{x_i}^{x_{i+1}} \varphi_{i+1}(x)\varphi_i'(x)\,dx = \int_{x_i}^{x_{i+1}} \left(\frac{x-x_i}{x_{i+1}-x_i}\right)\left(-\frac{1}{h}\right) dx$$

$$= -\frac{1}{2}$$

Note that $c_{i,i+1} = -c_{i+1,i}$.

Hence, for the convection matrix C:

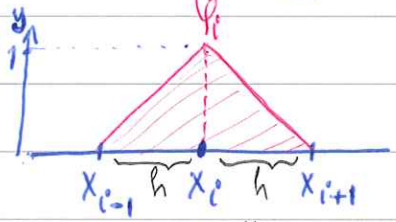
$$c_{ij} = \begin{cases} 0 & |i-j|>1 \\ 0 & i=j,\ i=1,\dots,m \\ \frac{1}{2} & i=j=m+1 \\ \frac{1}{2} & j=i+1,\ i=1,\dots,m \\ -\frac{1}{2} & j=i-1,\ i=1,\dots,m \end{cases}$$

$$C = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & 1 & \ddots & & \vdots \\ 0 & -1 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix}$$

For the load vector $b = \{b_i\}_{i=1}^{m+1} = \{r\int_0^1 \varphi_i\,dx + \varepsilon\beta\,\varphi_i(\underset{\substack{\uparrow \\ 1=x_{m+1}}}{1})\}_{i=1}^{m+1}$;

for $i=1,\dots,m$, we have

$$b_i = r\int_0^1 \varphi_i\,dx + \varepsilon\beta\underbrace{\varphi_i(x_{m+1})}_{=0} = rh$$

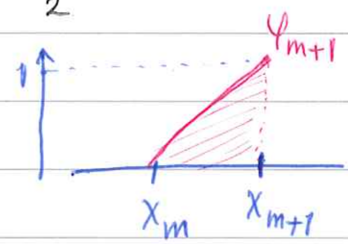


$$\int_0^1 \varphi_i\,dx = \int_{x_{i-1}}^{x_{i+1}} \varphi_i\,dA$$
$= $ area of the triangle
$$= \frac{(1)(2h)}{2} = h$$

and for $i=m+1$

$$b_{m+1} = r\underbrace{\int_0^1 \varphi_{m+1}\,dx}_{=\frac{h}{2}} + \varepsilon\beta\underbrace{\varphi_{m+1}(x_{m+1})}_{=1}$$

Hence, the load vector is



$$b = rh\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \frac{1}{2} \end{bmatrix} + \varepsilon\beta\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} rh \\ rh \\ \vdots \\ rh \\ r\frac{h}{2} + \varepsilon\beta \end{bmatrix}_{(m+1)\times 1}$$

<u>Note</u>: note that $\varepsilon\beta$ in b_{m+1} is the contribution from the boundary condition $u'(1) = \beta$.

Note: 1. The stiffness matrices S and $\tilde{S}$ ~~and~~ the mass matrix M are symmetric. But C is anti-symmetric

2. See Remark 7.5 and Remark 7.6 (page 176 in the book).