3.7.1 A Dirichlet problem

Here we show equivalence between a BVP and its variational formulation.

We consider the BVP (with homogeneous Dirichlet B.C.),

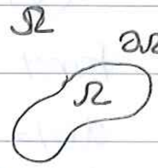
$$\text{(BVP)}\begin{cases} -(a(x)u'(x))' = f(x), & 0<x<1 \\ u(0)=u(1)=0 \end{cases}$$

Recall the function space (this is an example of a Sobolev space),

$$H_0^1(0,1) = \left\{ v \,\middle|\, \int_0^1 \{v^2 + (v')^2\}\,dx < \infty,\ v(0)=v(1)=0 \right\}.$$

For more general bounded domain $\Omega \subset \mathbb{R}^d$ ($d=1,2,3$),

$$H_0^1(\Omega) = \left\{ v \,\middle|\, \int_\Omega \{v^2 + (v')^2\}\,dx < \infty,\ v=0 \text{ on } \underbrace{\partial\Omega}_{\text{boundary of } \Omega} \right\}$$



VF: to find the VF for (BVP), we multiply the DE by a test function $v \in H_0^1(0,1)$, then integrate over $(0,1)$ and integrate by parts,

$$-\int_0^1 (au')'v\,dx = \int_0^1 fv\,dx$$

$$\Rightarrow \left[-au'v\right]_{x=0}^{x=1} + \int_0^1 au'v'\,dx = \int_0^1 fv\,dx$$

$$\Longrightarrow -a(1)u'(1)\underbrace{v(1)}_{=0} + a(0)u'(0)\underbrace{v(0)}_{=0} + \int_0^1 au'v'\,dx = \int_0^1 fv\,dx$$

$$\xrightarrow[v(0)=v(1)=0]{v\in H_0^1} \int_0^1 au'v'\,dx = \int_0^1 fv\,dx$$

Hence the VF is:

$$\text{(VF)}\begin{cases} \text{Find } u \in H_0^1(0,1) \text{ such that} \\ \int_0^1 au'v'\,dx = \int_0^1 fv\,dx, \quad \forall v \in H_0^1(0,1) \end{cases}$$

Now, we prove that

$$(BVP) \Longleftrightarrow \big((VF) \text{ \& } u \text{ is twice differentiable}\big)$$

Thm 3.10 The following two properties are equivalent:

(i) u satisfies (BVP)

(ii) u is twice differentiable & satisfies (VF)

Proof:

(i) $\Longrightarrow$ (ii) : We have already proved this, that is

$$(BVP) \Longrightarrow (VF)$$

Note that when u satisfies (BVP), then u is twice differentiable.

(ii) $\Longrightarrow$ (i) : So u is twice differentiable and is a solution to (VF). We show that u satisfies (BVP).

First since u satisfies (VF), so $u \in H_0^1(0,1)$ and $u(0)=u(1)=0$, that is the B.C. in (BVP).

Now, in (VF) we integrate by parts,

$$\int_0^1 a u' v' dx = \int_0^1 f v dx$$

$$\underset{I.P}{\overset{?}{\Longrightarrow}} -\int_0^1 (au')' v dx + [au'v]_{x=0}^{x=1} = \int_0^1 f v dx$$

$$\Longrightarrow -\int_0^1 (au')' v dx + a(1)u'(1)\underbrace{v(1)}_{=0} - a(0)u'(0)\underbrace{v(0)}_{=0} = \int_0^1 f v dx$$

$$\overset{?}{\Longrightarrow} \int_0^1 \{-(au')' - f\} v dx = 0, \quad \forall v \in H_0^1(0,1)$$

We should show this implies

$$-(au')' - f = 0, \quad \forall x \in (0,1) \qquad (1)$$

Suppose not. Then there exists a point $\xi \in (0,1)$ such that

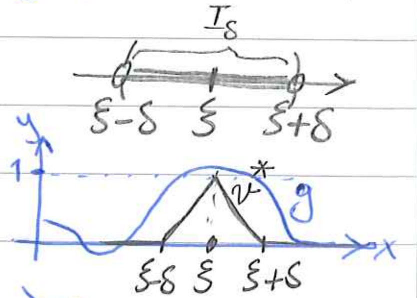
$$-(a(\xi)u'(\xi))' - f(\xi) \neq 0 \qquad (2)$$

Without loss of generality, we assume

$$-(a(\xi)u'(\xi))' - f(\xi) > 0$$

Denote $g(x) = -(a(x)u'(x))' - f(x)$, and since $a(x)$, $u'(x)$ and $f(x)$ are continuous so $g(x)$ is also continuous on $(0,1)$.
$g(\xi) > 0$ implies that in a δ-neighborhood of ξ,

$$g(x) = -(a(x)u'(x))' - f(x) > 0, \quad \forall x \in I_\delta = (\xi-\delta, \xi+\delta) \subset (0,1)$$



Choose a test function $v^*(x)$, such as a hat function with $v^*(\xi) = 1$ and support I_δ.
Then $v^* \in H_0^1$ and

$$\int_0^1 \{-(au')' - f\} v^* dx = \int_{\xi-\delta}^{\xi+\delta} \underbrace{g}_{>0} \underbrace{v^*}_{>0} dx > 0$$

that contradicts ①. Hence ② is not true, that is

$$-(a(x)u'(x))' - f(x) = 0, \quad x \in (0,1)$$

that is u satisfies (BVP). $\square$

Corollary 3.1

(i) If $f \in C(0,1)$ and $a \in C^1(0,1)$, then (BVP) and (VF) have the same solution.

(ii) If $a(x)$ is discontinuous (but bounded) and $f \in L_2(0,1)$, then (BVP) is not well-defined but (VF) is still well-defined.
So (VF) covers a larger set of functions.

(iii) In (VF) we need $u \in C^1(0,1)$, while (BVP) is formulated for $u \in C^2(0,1)$.

3.6 Some basic inequalities

Here we only state and prove Poincaré inequality in 1D

Thm 3.7 (Poincaré inequality). Assume that $\Omega \subset \mathbb{R}^d$ and $u \in H_0^1(\Omega)$. Then

$$\|u\| \leq C_\Omega \|\nabla u\|$$

for a constant C_Ω that is independent of u, but dependent on Ω.
($\|\cdot\|$ is the L_2-norm $\|\cdot\|_{L_2(\Omega)}$)

Proof. We prove it for 1D, that is $\Omega = (0,L) \subset \mathbb{R}$ and $u \in H_0^1(0,L)$. We show that

$$\|u\| \leq C_L \|u'\|$$

for some constant C_L.
For $x \in [0,L]$ we write (note that $u \in H_0^1(0,L)$, so $u(0)=0$),

$$u(x) = \int_0^x u'(y)\,dy \leq \int_0^x |u'(y)|\,dy \leq \int_0^L |u'(y)|\,dy = \int_0^L |u'(y)|\cdot 1\,dy$$

$$\overset{C\text{-}S}{\leq} \left(\int_0^L |u'(y)|^2 dy\right)^{1/2} \Big(\underbrace{\int_0^L 1^2 dy}_{=L}\Big)^{1/2} = \sqrt{L}\,\|u'\|$$

Hence

$$u(x)^2 \leq L\,\|u'\|^2$$

$$\Rightarrow \int_0^L u(x)^2 dx \leq \int_0^L \underbrace{L\|u'\|^2}_{\text{constant}} dx = L^2 \|u'\|^2$$

$$\Rightarrow \|u\|^2 \leq L^2 \|u'\|^2 \Rightarrow \|u\| \leq \underbrace{L}_{=C_L} \|u'\|. \quad \square$$

Note: We only used $u(0)=0$, so Thm 3.7 holds also, if

$$u \in V = \left\{ v \,\middle|\, \int_0^L \{v^2 + (v')^2\}\,dx < \infty,\ v(0)=0 \right\}.$$

And not that $H_0^1(0,L) \subset V$.

Note: Due to Poincaré inequality

$$H_0^1(0,L) = \{v \mid \int_0^L \{v^2 + (v')^2\} dx < \infty,\ v(0) = v(L) = 0\}$$

is identically defined as

$$H_0^1(0,L) = \{v \mid \int_0^L (v')^2 dx < \infty,\ v(0) = v(L) = 0\}.$$

The (full) norm in $H_0^1(0,L)$ is

$$\|v\|_{H_0^1(0,L)} = \sqrt{\|v\|^2_{L_2(0,L)} + \|v'\|^2_{L_2(0,L)}}$$

By Poincaré inequality we can also use

$$\|v\|_{H_0^1(0,L)} = \|v'\|_{L_2(0,L)}$$

Notation: A weighted L_2-norm by a weight function $a(x) > 0$ $x \in (0,L)$ is defined by

$$\|v\|_a = \sqrt{\int_0^L a(x)\,|v(x)|^2\,dx}$$

The corresponding inner product can be defined by

$$(u,v)_a = \underbrace{(\sqrt{a}\,u, \sqrt{a}\,v)}_{L_2\text{-inner product}} = \int_0^L \sqrt{a(x)}\,u(x)\,\sqrt{a(x)}\,v(x)\,dx$$

$$= \int_0^L a(x)\,u(x)\,v(x)\,dx$$

7.3 Error estimates in the energy norm

Here we present the so called a priori error estimates, that gives information about the order and size of the error.
We use the energynorm.

Recall the BVP:

$$(BVP)\begin{cases} -(au')' = f, & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

VF:

$$(VF)\begin{cases} \text{Find } u \in H_0^1 \text{ such that} \\ \int_0^1 a u' v' dx = \int_0^1 f v dx, \quad \forall v \in H_0^1 \end{cases}$$

We may use $H_0^1 = \{ v \mid \int_0^1 (v')^2 dx < \infty, \; v(0) = v(1) = 0 \}$

FEM: Consider a partition

$$\mathcal{T}_h: \; 0 = x_0 < x_1 < \dots < x_m < x_{m+1} = 1$$

and the corresponding FE space

$$V_h^0 = \{ v \mid v \text{ is cont. p.w. linear on } \mathcal{T}_h, \; v(0) = v(1) = 0 \}$$

Then FEM is:

$$(FEM)\begin{cases} \text{Find } U \in V_h^0 \text{ such that} \\ \int_0^1 a U' \chi' dx = \int_0^1 f \chi dx, \quad \forall \chi \in V_h^0 \end{cases}$$

Note: Since $V_h^0 \subset H_0^1$, so (VF) holds also with $v = \chi \in V_h^0$, that is

$$\int_0^1 a u' \chi' dx = \int_0^1 f \chi \, dx.$$

Hence

$$\int_0^1 a(u' - U')\chi' dx = \int_0^1 a u' \chi' dx - \int_0^1 a U' \chi' dx = \int_0^1 f\chi dx - \int_0^1 f \chi dx = 0, \quad \forall \chi \in V_h^0$$

This is called the Galerkin orthogonality.

Recall and Notation:

Weighted L_2-inner product & Weighted L_2-norm

$$(u,v)_a = \int_0^1 a u v \, dx \quad , \quad \|v\|_a = \sqrt{(v,v)_a} = \sqrt{\int_0^1 a |v|^2 dx}$$

Energy inner product & energy norm

$$(u,v)_E = \int_0^1 a u' v' \, dx \quad , \quad \|v\|_E = \sqrt{(v,v)_E} = \sqrt{\int_0^1 a |v'|^2 dx}$$

Note that

$$\|v\|_E = \sqrt{\int_0^1 a |v'|^2 dx} = \sqrt{\int_0^1 \sqrt{a}\, v' \sqrt{a}\, v' dx} = \sqrt{(v',v')_a}$$

$$= \|v'\|_a$$

So, if $a \equiv 1$, then $\|v\|_1 = \|v\|_{L_2} = \|v\|$

$$\stackrel{?}{\Rightarrow} \quad \|v\|_E = \|v'\|$$

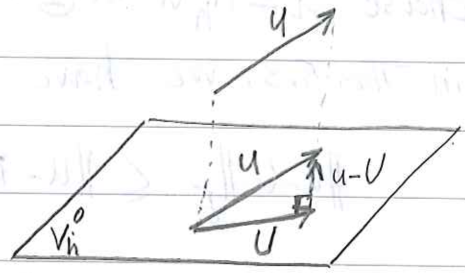
Recall that (BVP) and (VF) are equivalent.

Thm 7.1 Let u be the solution to (BVP) (or (VF)). and U be its FE approximation by (FEM). Then

$$\|u-U\|_E \leq \|u-\chi\|_E, \quad \forall \chi \in V_h^0. \qquad \text{I}$$

Note: This means that the FE solution U is the best approximation of u in V_h^0, in the energy norm.

Recall: The L_2-projection Pf is the best approx of f in $P^{(q)}$, in the L_2-norm



Proof. Take an arbitrary $\chi \in V_h^0$.

Then, in energy norm

$$\|u-U\|_E^2 = \int_0^1 a(u'-U')^2 dx$$

$$= \int_0^1 a(u'-U')(u'-U')dx = \int_0^1 a(u'-U')(u'-\chi'+\chi'-U')dx$$

$$= \int_0^1 a(u'-U')(u'-\chi')dx + \underbrace{\int_0^1 a(u'-U')(\underbrace{\chi'-U'}_{\in V_h^0})dx}_{=0 \text{ due to Galerkin orthogonality}}$$

$$= \int_0^1 \sqrt{a}(u'-U')\sqrt{a}(u'-\chi')dx$$

$$\overset{C-S}{\leq} \left(\int_0^1 a(u'-U')^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 a(u'-\chi')^2 dx\right)^{\frac{1}{2}}$$

$$= \|u-U\|_E \, \|u-\chi\|_E.$$

Hence $\|u-U\|_E^{\cancel{2}} \leq \cancel{\|u-U\|_E} \, \|u-\chi\|_E$

$$\Rightarrow \|u-U\|_E \leq \|u-\chi\|_E.$$

Since $\chi \in V_h^0$ was arbitrary, so the proof is now complete. □

Thm 7.2 (a priori error estimate) Let u and U be the solutions of (BVP) and (FEM), respectively. Then there exists an interpolation constant C_i, depending only on $a(x)$, such that

$$\|u-U\|_E \leqslant C_i \|hu''\|_a$$

Proof. Since $\pi_h u \in V_h^0$ (Cont. p.w. linear interpolant), we may choose $\chi = \pi_h u$ in (I). Then using the second error estimate in Thm 5.3, we have

$$\|u-U\|_E \leqslant \|u-\pi_h u\|_E = \|(u-\pi_h u)'\|_a$$

$$\leqslant C_i \|hu''\|_a . \quad \square$$

Note: 1. If $a(x) \equiv 1$, $\|\cdot\|_a = \|\cdot\|$ is the L_2-norm:

$$\|u-U\|_E = \|(u-U)'\| \leqslant C_i \|hu''\|$$

2. If $\mathcal{T}_h$ is a uniform partition, then the mesh function h is a constant, and we have

$$\|u-U\|_E \leqslant C_i h \|u''\|_a$$

3. If $\mathcal{T}_h$ is not uniform, we can define $h_{max} = \max\limits_{x\in(0,1)} h(x)$, and

$$\|u-U\|_E \leqslant C_i h_{max} \|u''\|_a$$

4. So if $a \equiv 1$, and $\mathcal{T}_h$ is uniform:

$$\|u-U\|_E = \|(u-U)'\| \leqslant C_i h \|u''\| .$$

5. The error in energy norm is of order one, $O(h)$.

Chapter 8: Scalar IVP

We study the IVP

$$\begin{cases} \text{DE} & \dot{u}(t) + a(t)u(t) = f(t), \quad 0 < t \le T \\ \text{IV} & u(0) = u_0 \end{cases} \qquad (1)$$

Here: $f(t)$ is the source (load) term.

$a(t)$ is bounded $\begin{cases} \text{if } a(t) \geq 0 \longrightarrow \text{parabolic IVP} \\ \text{if } a(t) \geq \alpha > 0 \longrightarrow \text{dissipative IVP} \end{cases}$

8.1 Solution formula and stability

Thm 8.1 The solution of (1) is

$$u(t) = u_0 e^{-A(t)} + \int_0^t e^{-(A(t)-A(s))} f(s)\,ds, \qquad (2)$$

where $A(t) = \int_0^t a(s)\,ds$.

($e^{A(t)}$ is the integrating factor, and note that $\dot{A}(t) = a(t)$ and $A(0) = 0$)

Proof. Multiply the DE by $e^{A(t)}$

$$\dot{u}(t)e^{A(t)} + \underbrace{a(t)e^{A(t)}}_{=\dot{A}(t)e^{A(t)}} u(t) = e^{A(t)} f(t)$$

$$\Rightarrow \frac{d}{dt}\left(u(t)e^{A(t)}\right) = e^{A(t)} f(t)$$

$$\int_0^t \Rightarrow u(t)e^{A(t)} - \underbrace{u(0)}_{=u_0}\underbrace{e^{A(0)}}_{=e^0=1} = \int_0^t e^{A(s)} f(s)\,ds$$

$$\Rightarrow u(t)e^{A(t)} = u_0 + \int_0^t e^{A(s)} f(s)\,ds \xrightarrow{\times e^{-A(t)}} u(t) = u_0 e^{-A(t)} + \int_0^t \overbrace{e^{-A(t)+A(s)}}^{e^{-(A(t)-A(s))}} f(s)\,ds.$$

Note: (2) is called variation of constants formula. □

Thm 8.2 (Stability estimates). Let $u=u(t)$ be the solution of 1.

We have:

(i) If $a(t)\geq\alpha>0$, then $\rightarrow$ dissipative problem

$$|u(t)|\leq e^{-\alpha t}|u_0|+\frac{1}{\alpha}(1-e^{-\alpha t})\max_{0\leq s\leq t}|f(s)|.$$

(ii) If $a(t)\geq 0$, then

parabolic problem $$|u(t)|\leq|u_0|+\underbrace{\int_0^t|f(s)|ds}_{=\|f\|_{L_1(0,t)}}$$

Proof

(i) $a(t)\geq\alpha>0$, so

$$A(t)=\int_0^t a(s)ds\geq\int_0^t\alpha ds=\alpha t\Rightarrow -A(t)\leq-\alpha t\Rightarrow e^{-A(t)}\leq e^{-\alpha t}$$

$$A(t)-A(s)=\int_0^t a(r)dr-\int_0^s a(r)dr=\int_s^t a(r)dr\geq\int_s^t\alpha\,dr=\alpha(t-s)$$

$$\Rightarrow -(A(t)-A(s))\leq-\alpha(t-s)$$

$$\Rightarrow e^{-(A(t)-A(s))}\leq e^{-\alpha(t-s)}$$

Hence, from 2, we have

$$|u(t)|\leq|u_0|e^{-A(t)}+\int_0^t e^{-(A(t)-A(s))}|f(s)|ds$$

$$\Rightarrow |u(t)|\leq|u_0|e^{-\alpha t}+\int_0^t e^{-\alpha(t-s)}|f(s)|ds \longleftarrow 3$$

$$\Rightarrow |u(t)|\leq|u_0|e^{-\alpha t}+\max_{0\leq s\leq t}|f(s)|\underbrace{\int_0^t e^{\alpha(s-t)}ds}_{=\frac{1}{\alpha}e^{\alpha(s-t)}\Big]_{s=0}^{s=t}=\frac{1}{\alpha}(1-e^{-\alpha t})}$$

$$\Rightarrow |u(t)|\leq|u_0|e^{-\alpha t}+\frac{1}{\alpha}(1-e^{-\alpha t})\max_{0\leq s\leq t}|f(s)|$$

(ii) Set $\alpha=0$ in 3: $|u(t)|\leq|u_0|+\int_0^t|f(s)|ds.$ $\square$

See Remark 8.1.

Ex. Find the solution of the following IVP.

(a) $\begin{cases} \mathring{u}(t) + 2u(t) = t, & 0 \leq t \leq 3 \\ u(0) = \frac{3}{4} \end{cases}$ (b) $\begin{cases} \mathring{u} = t, & 0 \leq t \leq 3 \\ u(0) = \frac{3}{4} \end{cases}$

(a)

$A(t) = \int_0^t 2\,ds = 2t$

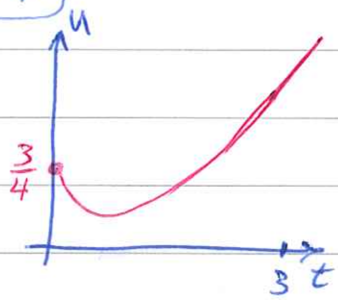
$\Rightarrow u(t) = u_0 e^{-2t} + \int_0^t e^{-(2t-2s)} s\,ds$

$$\int_0^t e^{2(s-t)} s\,ds = \left\{ \begin{matrix} u = s & \to & du = ds \\ dv = e^{2(s-t)} ds & & v = \frac{1}{2} e^{2(s-t)} \end{matrix} \right\}$$

$$= \frac{1}{2} s e^{2(s-t)} \Big]_{s=0}^{s=t} - \frac{1}{2} \int_0^t e^{2(s-t)} ds$$

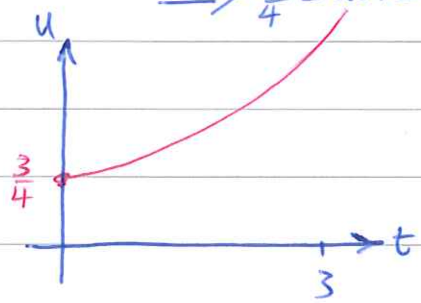
$$= \frac{1}{2}(t - 0) - \frac{1}{4} e^{2(s-t)} \Big]_{s=0}^{s=t} = \frac{1}{2} t - \frac{1}{4}(1 - e^{-2t})$$

$$\Rightarrow u(t) = (u_0 + \tfrac{1}{4}) e^{-2t} + \tfrac{1}{2} t - \tfrac{1}{4} \overset{u_0 = \frac{3}{4}}{=\!=} \boxed{e^{-2t} + \tfrac{t}{2} - \tfrac{1}{4}}$$



(b) $\mathring{u}(t) = t \Rightarrow u(t) = \frac{t^2}{2} + C$

$\xRightarrow{u(0) = \frac{3}{4}} \frac{3}{4} = u(0) = C \Rightarrow \boxed{u(t) = \frac{t^2}{2} + \frac{3}{4}}$



8.2 Finite Difference Methods for IVP

From the definition of derivative

$$\mathring{u}(t) = \lim_{\Delta t \to 0} \frac{u(t+\Delta t) - u(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{u(t) - u(t-\Delta t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{u(t+\frac{\Delta t}{2}) - u(t-\frac{\Delta t}{2})}{\Delta t}$$

We can approximate $\mathring{u}(t)$ by difference quotients:

$$\mathring{u}(t) \approx \frac{u(t+\Delta t) - u(t)}{\Delta t} \qquad (1)$$

$$\mathring{u}(t) \approx \frac{u(t) - u(t-\Delta t)}{\Delta t} \qquad (2)$$

$$\mathring{u}(t) \approx \frac{u(t+\frac{\Delta t}{2}) - u(t-\frac{\Delta t}{2})}{\Delta t} \qquad (3)$$

FDM for homogeneous IVP (with constant coefficient)

Consider the homogeneous IVP (r.h.s $\equiv 0$) with $a = \text{constant} > 0$,

$$\begin{cases} \mathring{u}(t) + a u(t) = 0, & 0 < t \le T \\ u(0) = u_0 \end{cases} \qquad (4)$$

We formulate three FDM. First we consider a partition for $[0, T]$. For simplicity a uniform partition:

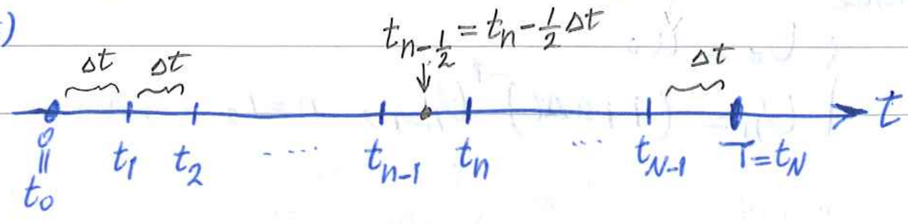
$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$$

uniform partition

with subintervals $I_n = (t_{n-1}, t_n]$, $n = 1, \ldots, N$, $\Delta t = |I_n| = t_n - t_{n-1}$

$t_j = t_0 + j(\Delta t) = j(\Delta t)$

$j = 0, 1, \ldots, N$



Forward (explicit) Euler method:

We stay at time level $t=t_{n-1}$ and use ① to approximate the solution at $t=t_{n-1}+\Delta t=t_n$. That is

$$\frac{u(t_n)-u(t_{n-1})}{\Delta t}+a\,u(t_{n-1})\approx 0$$

Denotting the approximate solution by U and $U_n\approx u(t_n)$, we have the recursive formula:

$$\frac{U_n-U_{n-1}}{\Delta t}+aU_{n-1}=0 \implies U_n=(1-a\Delta t)U_{n-1},\ n=1,\dots,N$$

Starting from $U_0=u(0)=u_0$,

$$\begin{cases} U_0=u_0 \\ U_n=(1-a\Delta t)U_{n-1}, \quad n=1,\dots,N \end{cases} \qquad (5)$$

Backward (implicit) Euler method:

We stay at time level $t=t_n$ (where we want to find the solution) and use ② to approximate the solution at $t=t_n$. That is ($t_{n-1}=t_n-\Delta t$)

$$\frac{u(t_n)-u(t_{n-1})}{\Delta t}+a\,u(t_n)\approx 0$$

Denotting $U\approx u$ and $U_n\approx u(t_n)$,

$$\frac{U_n-U_{n-1}}{\Delta t}+aU_n=0 \implies (1+a\Delta t)U_n=U_{n-1},\ n=1,\dots,N$$

Hence

$$\begin{cases} U_0=u_0 \\ U_n=(1+a\Delta t)^{-1}U_{n-1}, \quad n=1,\dots,N \end{cases} \qquad (6)$$

or write $\Rightarrow$ $U_n=\dfrac{1}{1+a\Delta t}U_{n-1}$

Crank-Nicolson method:

We stay at time level $t = t_{n-\frac{1}{2}} = t_n - \frac{1}{2}\Delta t = t_{n-1} + \frac{1}{2}\Delta t$

and use (3) and using the average value

$$u(t_{n-\frac{1}{2}}) \approx \frac{u(t_{n-1}) + u(t_n)}{2}$$

to approximate the solution at $t = t_n$. That is

$$\frac{u(t_n) - u(t_{n-1})}{\Delta t} + a\,\frac{u(t_{n-1}) + u(t_n)}{2} \approx 0$$

Denotting $U \approx u$ and $U_n \approx u(t_n)$,

$$\frac{U_n - U_{n-1}}{\Delta t} + a\,\frac{U_{n-1} + U_n}{2} = 0 \Rightarrow (1 + \tfrac{1}{2} a\Delta t)U_n = (1 - \tfrac{1}{2} a\Delta t)U_{n-1} \quad n = 1, \dots, N$$

Hence

$$\begin{cases} U_0 = u_0 \\ U_n = (1 + \frac{1}{2} a\Delta t)^{-1} (1 - \frac{1}{2} a\Delta t) U_{n-1}, \quad n = 1, \dots, N \end{cases} \quad (7)$$

$$\hookrightarrow U_n = \frac{1 - \frac{1}{2} a\Delta t}{1 + \frac{1}{2} a\Delta t}\, U_{n-1}.$$

Note: Iterating the algorithms we find U_n in terms of u_0:

Forward Euler method: (5)

$$U_n = (1 - a\Delta t)U_{n-1} = (1 - a\Delta t)\overbrace{(1 - a\Delta t)U_{n-2}}^{= U_{n-1}} = \dots = (1 - a\Delta t)^n u_0$$

Backward Euler method: (6)

$$U_n = (1 + a\Delta t)^{-n} u_0 = \frac{1}{(1 + a\Delta t)^n} u_0$$

Crank-Nicolson: (7)

$$U_n = \left(\frac{1 - \frac{1}{2} a\Delta t}{1 + \frac{1}{2} a\Delta t} \right)^n u_0$$

FDM for a general first order IVP

The FDM for

$$\begin{cases} \dot{u}(t) = F(t, u(t)), & 0 < t \leq T \\ u(0) = u_0 \end{cases} \quad ⑧$$

are

Forward Euler
$$\begin{cases} U_0 = u_0 \\ U_n = U_{n-1} + \Delta t\, F(t_{n-1}, U_{n-1}), & n = 1, \dots, N \end{cases}$$

Backward Euler
$$\begin{cases} U_0 = u_0 \\ U_n = U_{n-1} + \Delta t\, F(t_n, U_n), & n = 1, \dots, N \end{cases}$$

Crank Nicolson
$$\begin{cases} U_0 = u_0 \\ U_n = U_{n-1} + \frac{\Delta t}{2}\left\{ F(t_{n-1}, U_{n-1}) + F(t_n, U_n) \right\}, & n = 1, \dots, N \end{cases}$$

Note: We can obtain the above recursive methods, following the same idea for ⑤, ⑥ and ⑦. Indeed

Forward Euler: $\frac{U_n - U_{n-1}}{\Delta t} = F(t_{n-1}, U_{n-1}) \Rightarrow U_n = U_{n-1} + \Delta t\, F(t_{n-1}, U_{n-1})$

Backward Euler: $\frac{U_n - U_{n-1}}{\Delta t} = F(t_n, U_n) \Rightarrow U_n = U_{n-1} + \Delta t\, F(t_n, U_n)$

Crank-Nicolson:

$$\frac{U_n - U_{n-1}}{\Delta t} = \frac{F(t_{n-1}, U_{n-1}) + F(t_n, U_n)}{2}$$

$$\Rightarrow U_n = U_{n-1} + \frac{\Delta t}{2}\left\{ F(t_{n-1}, U_{n-1}) + F(t_n, U_n) \right\}$$

Note that to solve the non-homogeneous IVP (with constant coefficient)

$$\begin{cases} \dot{u}(t) + a u(t) = f(t), & 0 < t \leq T \\ u(0) = u_0 \end{cases} \quad (9)$$

We can write DE as

$$\dot{u}(t) = f(t) - a u(t) = F(t, u(t))$$

that is (8). Then the FDM are:

Forward Euler
$$\begin{cases} U_0 = u_0 \\ U_n = (1 - a\Delta t) U_{n-1} + \Delta t\, f(t_{n-1}), \quad n = 1, \dots, N \end{cases} \quad (10)$$

Backward Euler
$$\begin{cases} U_0 = u_0 \\ U_n = (1 + a\Delta t)^{-1} U_{n-1} + (1 + a\Delta t)^{-1} \Delta t f(t_n), \quad n = 1, \dots, N \end{cases} \quad (11)$$

$\Rightarrow$ or write $U_n = \dfrac{1}{1 + a\Delta t} U_{n-1} + \dfrac{\Delta t}{1 + a\Delta t} f(t_n)$

Crank Nicolson
$$\begin{cases} U_0 = u_0 \\ U_n = (1 + \frac{1}{2} a\Delta t)^{-1} (1 - \frac{1}{2} a\Delta t) U_{n-1} + \frac{1}{2} \Delta t \left\{ f(t_{n-1}) + f(t_n) \right\}, \quad n = 1, \dots, N \end{cases} \quad (12)$$

Note: Compare them ((10), (11) and (12)) with (5), (6) and (7).

Note: We obtained (10), (11), (12) as following:

Forward Euler:

$$\frac{U_n - U_{n-1}}{\Delta t} + a U_{n-1} = f(t_{n-1}) \Rightarrow U_n - U_{n-1} + a\Delta t\, U_{n-1} = \Delta t\, f(t_{n-1})$$

$$\Rightarrow U_n = (1 - a\Delta t) U_{n-1} + \Delta t\, f(t_{n-1})$$

Backward Euler:

$$\frac{U_n - U_{n-1}}{\Delta t} + a U_n = f(t_n) \Rightarrow U_n - U_{n-1} + a\Delta t\, U_n = \Delta t\, f(t_n)$$

$$\Rightarrow U_n = \frac{1}{1 + a\Delta t} U_{n-1} + \frac{\Delta t}{1 + a\Delta t} f(t_n)$$

Crank-Nicolson:

$$\frac{U_n - U_{n-1}}{\Delta t} + a\frac{U_{n-1}+U_n}{2} = \frac{f(t_{n-1})+f(t_n)}{2}$$

$$\Rightarrow U_n - U_{n-1} + \frac{1}{2}a\Delta t U_{n-1} + \frac{1}{2}a\Delta t U_n = \frac{1}{2}\Delta t\{f(t_{n-1})+f(t_n)\}$$

$$\Rightarrow U_n = \frac{1-\frac{1}{2}a\Delta t}{1+\frac{1}{2}a\Delta t}U_{n-1} + \frac{1}{2}\frac{\Delta t}{1+\frac{1}{2}a\Delta t}\{f(t_{n-1})+f(t_n)\}$$