Chapter 9: IBVP in 1D

We study heat and wave equations in 1D, and FEM to approx. the solution (mainly cG(1) method).

9.1 Heat equation in 1D

$$
\begin{array}{ll}
\text{DE} & \dot{u}(x,t) - u''(x,t) = f(x,t), \quad 0<x<1, \ t>0 \\
\text{BC} & u(0,t) = u'(1,t) = 0 \quad t>0 \qquad (1) \\
\text{IC} & u(x,0) = u_0(x) \quad 0<x<1 .
\end{array}
$$

Denote: $\dot{u} = u_t = \frac{\partial u}{\partial t}$, $u' = u_x = \frac{\partial u}{\partial x}$, $u'' = u_{xx} = \frac{\partial^2 u}{\partial x^2}$

$u(x,t)$ is the temperature at point x and time t.

$u(x,0)$ is the initial temperature at $t=0$

$u(0,t)$ means fixed temperature 0 at the boundary point $x=0$

$u'(1,t)$ means isolated boundary at $x=1$ (no heat flux occurs).

$f(x,t)$ is the heat source.

9.1.1 Stability estimates

Thm 9.1 The IBVP (1) satisfies the stability estimates

$$\|u(\cdot,t)\| \leq \|u_0\| + \int_0^t \|f(\cdot,s)\| ds \qquad (2)$$

$$\|u'(\cdot,t)\|^2 \leq \|u_0'\|^2 + \int_0^t \|f(\cdot,s)\|^2 ds \qquad (3)$$

$u_0, u_0' \in L_2(0,1)$, $f \in L_1([0,t], L_2(0,1)) \cap L_2([0,t], L_2(0,1))$.

$$\|v(\cdot,s)\| = \|v(\cdot,s)\|_{L_2(0,1)} = \sqrt{\int_0^1 |v(x,s)|^2 dx}$$

Proof. First we prove (2).

~~Proof~~. Multiply the DE in (1) by u and integrate over $(0,1)$

$$\int_0^1 \underbrace{\dot{u}u}_{=\frac{1}{2}\frac{d}{dt}u^2} dx - \int_0^1 u'' u\, dx = \int_0^1 f u\, dx$$

$$\xRightarrow[I.P]{2} \int_0^1 \frac{1}{2}\frac{d}{dt} u^2(x,t)\, dx - \left[u'(x,t)u(x,t)\right]_{x=0}^{x=1} + \int_0^1 u'u'\, dx = \int_0^1 f u\, dx$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\underbrace{\int_0^1 u^2(x,t)\,dx}_{=\|u(\cdot,t)\|^2} - \underbrace{u'(1,t)}_{=0}u(1,t) + u'(0,t)\underbrace{u(0,t)}_{=0} + \int_0^1 u'u'\,dx = \int_0^1 f u\, dx$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2 + \|u'\|^2 = \int_0^1 f u\, dx$$

$$\Rightarrow \quad \|u(\cdot,t)\|\frac{d}{dt}\|u(\cdot,t)\| + \|u'\|^2 = \int_0^1 f u\, dx \overset{C-S}{\leqslant} \|f\|\|u\|$$

$$\xRightarrow{\|u'\|^2 \geqslant 0} \quad \cancel{\|u(\cdot,t)\|}\frac{d}{dt}\|u(\cdot,t)\| \leqslant \|f\|\cancel{\|u\|}$$

$$\Rightarrow \frac{d}{dt}\|u(\cdot,t)\| \leqslant \|f\| \xRightarrow{\int_0^t} \|u(\cdot,t)\| - \|u(\cdot,0)\| \leqslant \int_0^t \|f(\cdot,s)\|\,ds$$

$$\overset{2}{\Rightarrow} \|u(\cdot,t)\| \leqslant \|u_0\| + \int_0^t \|f(\cdot,s)\|\,ds \qquad \boxed{(2)}$$

To prove (3) we multiply the DE in (1) by $\dot{u}$ and integrate over $(0,1)$

$$\int_0^1 \dot{u}\dot{u}\,dx - \int_0^1 u''\dot{u}\,dx = \int_0^1 f\dot{u}\,dx$$

$$\xRightarrow{I.P} \|\dot{u}\|^2 - \left[u'(x,t)\dot{u}(x,t)\right]_{x=0}^{1} + \int_0^1 \underbrace{u'\dot{u}'}_{\frac{1}{2}\frac{d}{dt}(u')^2}\,dx = \int_0^1 f\dot{u}\,dx$$

$$\Rightarrow \|\dot{u}\|^2 - \underbrace{u'(1,t)}_{=0}\dot{u}(1,t) + u'(0,t)\underbrace{\dot{u}(0,t)}_{=0} + \frac{1}{2}\int_0^1 \frac{d}{dt}(u')^2\,dx = \int_0^1 f\dot{u}\,dx$$

$$\Rightarrow \|\dot{u}\|^2 + \frac{1}{2}\frac{d}{dt}\|u'\|^2 = \int_0^1 f\dot{u}\,dx \leqslant \|f\|\|\dot{u}\| \underset{\substack{(?)\\ 2ab\leqslant a^2+b^2}}{\leqslant} \frac{1}{2}\left(\|f\|^2 + \|\dot{u}\|^2\right)$$

$$\Rightarrow \frac{1}{2}\|\dot{u}\|^2 + \frac{1}{2}\frac{d}{dt}\|u'\|^2 \leqslant \frac{1}{2}\|f\|^2$$

$$\|\dot u\|^2 \geq 0 \Longrightarrow \frac{d}{dt}\|u'\|^2 \leq \|f\|^2$$

$$\xrightarrow{\int_0^t} \|u'(\cdot,t)\|^2 - \|\underbrace{u'(\cdot,0)}_{=u_0'}\|^2 \leq \int_0^t \|f(\cdot,s)\|^2 ds$$

$$\Longrightarrow \|u'(\cdot,t)\|^2 \leq \|u_0'\|^2 + \int_0^t \|f(\cdot,s)\|^2 ds. \quad \boxed{(3)}$$

Stability estimates for homogeneous heat eq.

Thm 9.2 Consider the IBVP (heat eq.)

$$\begin{array}{ll} \text{DE} \\ \text{BC} \\ \text{IC} \end{array} \begin{cases} \dot u(x,t) - u''(x,t) = 0, & 0<x<1, \quad t>0 \\ u(0,t) = u'(1,t) = 0 & \quad t>0 \\ u(x,0) = u_0(x) & 0<x<1 \end{cases} \qquad (4)$$

We have the stability estimates:

(a) $\frac{d}{dt}\|u\|^2 + 2\|u'\|^2 = 0$ (b) $\|u(\cdot,t)\| \leq e^{-t}\|u_0\|$.

Proof

(a) Multiply the DE in (4) by u, then integrate over $(0,1)$ and I.P.,

$$0 = \int_0^1 \dot u u\, dx - \int_0^1 u'' u\, dx = \int_0^1 \frac{1}{2}\frac{d}{dt} u^2 dt - \underbrace{u'(1,t)}_{=0} u(1,t) + u'(0,t)\underbrace{u(0,t)}_{=0} + \int_0^1 (u')^2 dx$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|u\|^2 + \|u'\|^2 = 0.$$

(b) By Poincaré ineq. with $C_L = L = 1$, we have $\|u\| \leq \|u'\|$.

So
$$\frac{d}{dt}\|u\|^2 + 2\|u\|^2 \leq \frac{d}{dt}\|u\|^2 + 2\|u'\|^2 \overset{(a)}{=} 0$$

$$\xRightarrow[\text{integrating factor}]{\times e^{2t}} \frac{d}{dt}\|u\|^2 e^{2t} + 2\|u\|^2 e^{2t} \leq 0$$

$$\Rightarrow \frac{d}{dt}\left(\|u\|^2 e^{2t}\right) \leq 0 \xrightarrow{\int_0^t} \|u(\cdot,t)\|^2 e^{2t} - \|u(\cdot,0)\|^2 e^0 \leq 0$$

$$\Rightarrow \|u(\cdot,t)\|^2 e^{2t} \leq \|u_0\|^2 \Rightarrow \|u(\cdot,t)\|^2 \leq e^{-2t}\|u_0\|^2$$

$$\Rightarrow \|u(\cdot,t)\| \leq e^{-t}\|u_0\|. \quad \square$$

Remark 9.3: See the book (I will not ask)

Thm 9.3 (We ignore)