Convexity. Polyhedra. Separation.

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Summary of Chapter 3

In **Section 3.1** *convex sets* are defined.

Section 3.2 starts by definitions of *affine hull* and *convex hull*. From the definition of *convex hull* we define *polytope*. Further, the concept of *extreme point* is introduced. For the *polyhedron* it turns out that there exists an *algebraic characterization* of *extreme points*. From this characterization it follows that the number of *extreme points* of a *polyhedron* is finite. The *algebraic characterization* is also used to prove the important *Representation Theorem*. Then, the *Separation Theorem* is proved, and the *Separation Theorem* and the *Representation Theorem* are used in order to show that a set is a *polytope* if and only if it is a *bounded polyhedron*. *Cones* are defined and *polyhedral cones* and *finitely generated cones* are studied in particular. It is shown that a *cone* is *finitely generated* if and only if it is *polyhedral*. Hence, *finitely generated cones* are convex and closed. Finally, *Farkas' Lemma* is proved.

Section 3.3 deals with the convexity of functions. First, *convex (concave) functions* and *strictly convex (concave) functions* are defined. Proposition 3.40 (convexity of composite functions) is sometimes useful when proving that a function is convex. Theorem 3.44 deals with *convexity characterizations in C^1*. Part (a) of this theorem is the most important one. Similarly, in Theorem 3.45 *convexity characterizations in C^2* are given. These characterizations can be useful when proving that a function is convex or strictly convex. Further, *level sets* are defined and it is shown that *level sets* of *convex functions* are convex and closed. The section is closed by a definition of a *convex problem*.

Proofs: You should be able to prove the Representation Theorem 3.22, the Separation Theorem 3.26, and Proposition 3.48 (convex level sets from convex functions).

Övningsuppgifter

Konvexa optimeringsproblem har många goda egenskaper vilket vi kommer att se framöver. Därför är det viktigt att kunna avgöra om ett problem är konvext eller ej. Följande övningar skall illustrera hur man visar att ett givet problem är konvext eller ej.

Övning 1. Suppose that $g : \mathbb{R}^n \mapsto \mathbb{R}$ is convex and $\boldsymbol{d} \in \mathbb{R}^n$. Is the problem to

$$\begin{aligned} \text{maximize} \quad & -(x_1^2 + \cdots + x_n^2) \\ \text{subject to} \quad & -\frac{1}{\ln(-g(\boldsymbol{x}))} \geq 0, \\ & \boldsymbol{d}^{\mathrm{T}}\boldsymbol{x} = 2, \\ & g(\boldsymbol{x}) \leq -2, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

convex? □

Övning 2. Is the problem to

$$\begin{aligned} \text{maximize} \quad & x_1 \ln x_1 \\ \text{subject to} \quad & x_1^2 + x_2^2 \geq 1, \\ & \boldsymbol{x} \geq \mathbf{0}^2, \end{aligned}$$

convex? $\square$

Från representationssatsen har vi att varje punkt i en polyeder (med minst en extrempunkt) kan skrivas som summan av en konvexkombination av extrempunkter och en obegränsad riktning (element ur en polyedrisk kon). Vi visar först att alla polyedriska mängder är konvexa, sedan att det för en viss typ av polyedrar (de som förekommer i LP) alltid existerar minst en extrempunkt och slutligen illustrerar vi representationssatsen i två dimensioner.

Övning 3 (convexity of polyhedra)**.** Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{b}$ an $m \times 1$ vector. Show that the polyhedron

$$P = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \,\},$$

is a convex set. □

Övning 4 (existence of extreme points in LPs). Let $\boldsymbol{A}$ be an $m \times n$ matrix such that rank $\boldsymbol{A} = m$ and $\boldsymbol{b}$ an $m \times 1$ vector. Show that if the polyhedron

$$P = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}; \quad \boldsymbol{x} \geq \boldsymbol{0}^n \,\}$$

has a feasible solution, then it has an extreme point. $\square$

Övning 5 (illustration of the Representation Theorem). Let

$$Q = \{\, \boldsymbol{x} \in \mathbb{R}^2 \mid -2x_1 + x_2 \leq 1; \quad x_1 - x_2 \leq 1; \quad -x_1 - x_2 \leq -1 \,\},$$

$$C = \{\, \boldsymbol{x} \in \mathbb{R}^2 \mid -2x_1 + x_2 \leq 0; \quad x_1 - x_2 \leq 0; \quad -x_1 - x_2 \leq 0 \,\},$$

and P be the convex hull of the extreme points of Q. Show that the feasible point $\tilde{\boldsymbol{x}} = (1, 1)^{\mathrm{T}}$ can be written as

$$\tilde{\boldsymbol{x}} = \boldsymbol{p} + \boldsymbol{c},$$

where $\boldsymbol{p} \in P$ and $\boldsymbol{c} \in C$. $\square$

Separation är ett fundamentalt begrepp inom optimering och konvexanalys. I följande övning används separation.

Övning 6 ((iii):1.1.39). Show that each closed convex set A in $\mathbb{R}^n$ is the intersection of all the closed halfspaces in $\mathbb{R}^n$ containing A. □