Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MAN280
APPLIED OPTIMIZATION

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Question 1

(the simplex method)

(2p) a) The problem in standard form is to

$$
\begin{array}{rrrrrrl}
\text{minimize} & x_1 + 2\,x_2 - x_3 & & & & & \\
\text{subject to} & x_1 + 2\,x_2 - x_3 + x_4 & & & & = 1, & \\
& 2\,x_1 - \quad x_2 & & & -x_5 & = 1, & \\
& x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5 & & & & \geq 0. &
\end{array}
$$

Introduce an artificial variable in the second constraint to get the phase I problem to

$$
\begin{array}{rl}
\text{minimize} \quad w = & a \\
\text{subject to} & x_1 + 2\,x_2 - x_3 + x_4 = 1, \\
& 2\,x_1 - x_2 - x_5 + a = 1, \\
& x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad a \geq 0.
\end{array}
$$

Start with the basis $\boldsymbol{x}_B = (x_4, a)^{\mathrm{T}}$. The simplex method then gives that x_1 is the entering variable and a the leaving. Hence we have found a feasible solution for which $a = 0$, which means that $\boldsymbol{x}_B = (x_4, x_1)^{\mathrm{T}}$ is a feasible solution to the phase II problem. The reduced costs of the nonbasic variables $\boldsymbol{x}_N = (x_2, x_3, x_5)^{\mathrm{T}}$ become

$$\boldsymbol{c}_N^{\mathrm{T}} - \boldsymbol{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} \boldsymbol{N} = (5/2, -1, 1/2)^{\mathrm{T}},$$

which means that x_3 is the entering variable. Further, we have that

$$\boldsymbol{B}^{-1}\boldsymbol{b} = (1/2, 1/2)^{\mathrm{T}},$$
$$\boldsymbol{B}^{-1}\boldsymbol{N}_2 = (-1, 0)^{\mathrm{T}}.$$

Hence it follows that the phase II problem is unbounded, and we can draw the conclusion that the original problem (P) is unbounded.

(1p) b) Since (P) is unbounded it follows from weak duality that its linear programming dual is infeasible.

(3p) Question 2

(application of the Levenberg–Marquardt algorithm)

With a unit step the Levenberg–Marquardt algorithm is, for a given $\boldsymbol{x}_k$, to generate $\boldsymbol{x}_{k+1}$ through the formula

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \nabla^2 f(\boldsymbol{x}_k + \gamma_k \boldsymbol{I}^n)^{-1} \nabla f(\boldsymbol{x}_k),$$

where $\gamma_k \geq 0$ is the shift used in iteration k.

For the given problem and starting point,

$$f(\boldsymbol{x}_0) = 0;\ \nabla f(\boldsymbol{x}_0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix};\ \nabla^2 f(\boldsymbol{x}_0) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

With the shift γ_0 the next iterate therefore is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/\gamma_0 \\ -1/(2+\gamma_0) \end{pmatrix}.$$

Inserting this into f yields that it is enough to set the value of γ_0 to something slightly larger than 1, while a choice of $\gamma_0 = 1$ would produce an undefined value of f (notice the presence of the logarithmic terms).

With $\gamma_0 = 2$ we obtain $\boldsymbol{x}_1 = (1/2, 5/4)^{\mathrm{T}}$ with $f(\boldsymbol{x}_1) \approx -0.76$.

(3p) Question 3

(on the SQP algorithm and the KKT conditions)

The result is based on a comparison between the KKT conditions of the original problem,

$$\begin{aligned} \text{minimize } & f(\boldsymbol{x}), && && \text{(1a)} \\ \text{subject to } & g_i(\boldsymbol{x}) \leq 0, && i = 1, \dots, m, && \text{(1b)} \\ & h_j(\boldsymbol{x}) = 0, && j = 1, \dots, \ell, && \text{(1c)} \end{aligned}$$

and those of the SQP subproblem,

$$\begin{aligned} \underset{p}{\text{minimize }} & \frac{1}{2}\boldsymbol{p}^{\mathrm{T}}\boldsymbol{B}_k\boldsymbol{p} + \nabla f(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p}, && && \text{(2a)} \\ \text{subject to } & g_i(\boldsymbol{x}_k) + \nabla g_i(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} \leq 0, && i = 1, \dots, m, && \text{(2b)} \\ & h_j(\boldsymbol{x}_k) + \nabla h_j(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} = 0, && j = 1, \dots, \ell. && \text{(2c)} \end{aligned}$$

We first note that the latter problem is a convex one (the matrix $\boldsymbol{B}_k$ was assumed to be positive semidefinite), and that the solution $\boldsymbol{p}_k$ is characterized by its KKT conditions, since the constraints are linear (so that Abadie's CQ is fulfilled). It remains to compare the two problems' KKT conditions. With $\boldsymbol{p}_k = \mathbf{0}^n$ they are in fact identical!

(3p) **Question 4**

(convexity)

We have the following convexity characterization:

$$f(y) \geq f(z) + f'(z)^{\mathrm{T}}(y - z).$$

The assertion follows by letting $y = g(x)$ and $z = \int_{\mathbb{R}} h(x)g(x)\,dx$, then multiply both the sides by $h(x)$, and finally integrate both sides over $\mathbb{R}$.

(3p) **Question 5**

(strong duality in linear programming)

See the notes for the proof.

Question 6

(Lagrangian duality)

(1p) a) We obtain that

$$q(\mu) = \begin{cases} 2\mu + 3\frac{1}{2}, & \text{if } \mu \leq 2, \\ -\frac{1}{2}(\mu - 1)^2 + 3\mu + 2, & \text{if } 2 \leq \mu \leq 6, \\ -\frac{1}{2}(\mu - 1)^2 - 4\frac{(2+\mu)}{8}^2 + 4\mu, & \text{if } \mu \geq 6. \end{cases}$$

(1p) b) $q(0) = 3\frac{1}{2}$; $q(\frac{5}{2}) = \frac{65}{8}$; $q(5) = 9$.

$f(2,2) = 16$; $f(1,3) = 31\frac{1}{2}$; $f(3,1) = 9\frac{1}{2}$.

Conclusion: $f^* \in [9, 9\frac{1}{2}]$.

(1p) c) From a) we obtain that

$$q(\mu) = \begin{cases} 2, & \text{if } \mu \leq 2, \\ -(\mu - 1) + 3, & \text{if } 2 \leq \mu \leq 6, \\ -(\mu - 1) - \frac{2+\mu}{8} + 4, & \text{if } \mu \geq 6. \end{cases}$$

$q'(0) = 2$; $q'(\frac{5}{2}) = \frac{3}{2}$; $q'(5) = -1$.

We note that the function q is concave and differentiable, and therefore its derivative is decreasing. According to the above, it must have a stationary point, hence the optimal solution, within the closed interval $[\frac{5}{2}, 5]$ which hence defines an interval wherein the optimum lies.

(3p)

Question 7

(modelling)

For $d = 1, \ldots, 6$ and $m = 1, \ldots, 7$, introduce the integer variables

$$x_{dm} = \text{number of mesh panels of type } m \text{ used between door } d-1 \text{ and } d,$$

where "door" 0 is the wall on the left-hand side and "door" 6 is the wall on the right-hand side. Further, let c_m and w_m, respectively, be the cost and the width, respectively, of mesh panel m for $m = 1, \ldots, 7$, and let $l_1, \ldots, l_5$ denote the lengths of the sections AB, BC, CD, DE, and EF. Then the following integer linear program solves the problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{m=1}^{7} \sum_{d=1}^{6} c_m x_{dm} \\ \text{subject to} \quad & \sum_{m=1}^{7} \sum_{d=1}^{k} w_m x_{dm} + 90k \leq \sum_{i=1}^{k} l_i, \quad k = 1, \ldots, 5, \\ & \sum_{m=1}^{7} \sum_{d=1}^{k+1} w_m x_{dm} + 90k \geq \sum_{i=1}^{k} l_i, \quad k = 1, \ldots, 4, \\ & \sum_{m=1}^{7} \sum_{d=1}^{k+1} w_m x_{dm} + 90k = \sum_{i=1}^{k} l_i, \quad k = 5, \\ & x_{dm} \in \mathbb{Z}_+^{6 \times 7}. \end{aligned}$$
