Chalmers/GU
Mathematics

EXAM

TMA946/MAN280
APPLIED OPTIMIZATION

Date:	02–03–11
Time:	afternoon
Aids:	Text memory-less calculator
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Niclas Andréasson (0740-459022)
Result announced:	02–03–22
	Short answers are also given at the end the exam on the notice board for optimization in the MD building.

Exam instructions

When you solve the questions

State your methodology carefully.
Use generally valid methods.

Only write on one page of each sheet. Do not use a red pen.
Do not solve more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Uppgift 1

(Linear optimization)

Consider the LP problem to

$$
\begin{array}{lrcrcl}
\text{minimize } z = & 3x_1 & + & x_2 & & \\
\text{s.t.} & 3x_1 & + & 2x_2 & \geq & 1, \\
& 2x_1 & + & x_2 & \leq & 2, \\
& x_1 & , & x_2 & \geq & 0.
\end{array}
$$

(2p) a) Solve the problem in a *correct* manner with the help of the simplex method, that is, by using Phase I and Phase II.

(1p) b) Suppose that we replace the first element in the right-hand side (the number 1) with a parameter $d \in \Re$. Express the optimal value of z as a function of d. *Hint:* It is acceptable to use a geometric solution for this question.

Some matrix inverses that might come in handy are

$$
\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix},
$$
$$
\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix},
$$
$$
\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
$$

Uppgift 2

(Linear optimization)

We have a problem of the form

$$
\begin{aligned}
\text{minimize } & c^{\mathrm{T}} x, \\
\text{subject to } & Ax \geq b, \\
& x \geq 0^n,
\end{aligned}
$$

where $A \in \Re^{m \times n}$, and A has full row rank.

Your friend Kalle says that by divine intervention, he has found an optimal solution to the problem. Kalle's solution has $m + k$ non-zero variables, so it can not be a basic solution.

(2p) a) How would you check if Kalle's solution is correct *without solving the problem using the simplex method, or using any other method for solving an LP problem*? You may suppose that the dual optimal solution is unique. Describe your methodology carefully.

(1p) b) Suppose that you were able to check that Kalle's solution is correct. To show off, you now want to find an optimal solution to the primal problem which is a basic solution. How would you do that *without solving the problem using the simplex method, or using any other method for solving an LP problem*? Describe your methodology carefully.

Uppgift 3

(Exterior penalty algorithm)

Consider the constrained minimization problem to

$$\begin{aligned} \text{minimize} \quad & f(x) := \frac{1}{2}(x_1^2 + |x_2|^\rho) - 2x_2, \\ \text{subject to} \quad & x_2 = 0, \end{aligned}$$

where $\rho > 1$ is a constant.

(1p) a) Formulate the KKT conditions for this problem. Find the KKT point(s) and the corresponding Lagrange multiplier(s). Are the KKT conditions sufficient for global optimality in this problem? *Hint:* Utilize that $(|x_2|^\rho)' = \rho x_2 |x_2|^{\rho-2}$.

(1p) b) Formulate an unconstrained minimization problem by moving the constraint into the objective function using the standard quadratic penalty funstion. Solve the obtained unconstrained problem for each penalty parameter $\mu > 0$, assuming $\rho = 2$. *Hint:* Use the fact that $|x_2|^2 = x_2^2$.

Do the optimal solutions x_μ^* to the penalized problems converge to the optimal solution to the original problem as μ tends to infinity?

(1p) c) Repeat the previous step assuming $\rho = 1$. Is it possible to apply gradient-based methods (e.g., the steepest descent method) to solve the penalized problem in this case?

Uppgift 4

(1p) a) Define the meaning of the term "convex function on $\Re^n$." Give, for functions that are once continuously differentiable, a characterization of the term which utilizes the gradient of the function (valid in C^1). Give, for functions that are twice continuously differentiable, a characterization of the term which utilizes the Hessian of the function (valid in C^2). Establish, at least for one of these, that the characterization is correct.

(1p) b) Consider the problem to

$$\underset{x \in \Re^n}{\text{minimize}} \quad f(x),$$

where f is a continuously differentiable (in C^1) and convex function, but not strictly convex, $n = 5000$, and we have a fair idea where an optimum is. We know that there is an optimum, since f models an energy functional, which is non-negative, and $f(x)$ tends to infinity with the norm of x. Recommend an algorithm for this problem; be critical and motivate your choice. If need be, ask yourself some additional questions about the problem, and give recommendations depending on the answers.

(1p) c) Suppose that we wish to solve a sequence of LP problems with varying right-hand sides:

$$\begin{aligned} \text{minimize} \quad & z = c^{\mathrm{T}} x, \\ \text{subject to} \quad & Ax \geq b^k, \\ & x \geq 0^n, \end{aligned}$$

for, $k = 1, \ldots, 10$, that is, for ten problems. Each of these are very large and take an hour of solution time in CPLEX, which is the best linear programming solver. How would you solve these problems in the most efficient manner?

(3p) Uppgift 5

(Modelling)

Your task is to construct a pressure tank to store 1000m^3 nitrogen gas (volume at one atmosphere's pressure). Pressure and volume are related by the well known formula

$$pv = c_1,$$

where p is the pressure, v is the volume and c_1 is a constant. In order for the tank to fit in a standard container, its total length must not be above 5.87m, and its total width not more than 2.34m. In order to handle high pressure, the tank is built as a cylinder, with halv spheres at both ends (see Figure 1).

Each half sphere then has an area of $2\pi r^2 \text{m}^2$ and a volume of $2/3\pi r^3 \text{m}^3$. The cylindrical part has an area of $2\pi r l \text{m}^2$ and a volume of $\pi r^2 l \text{m}^3$. The thickness of the tank must fulfil

$$prc_2 \leq d,$$

where d is the thickness of the material and c_2 is a constant.

Formulate the problem to minimize the amount of material that is needed to build the tank, so that it fits into the container and fulfils the requirements for being able to contain the required volume.

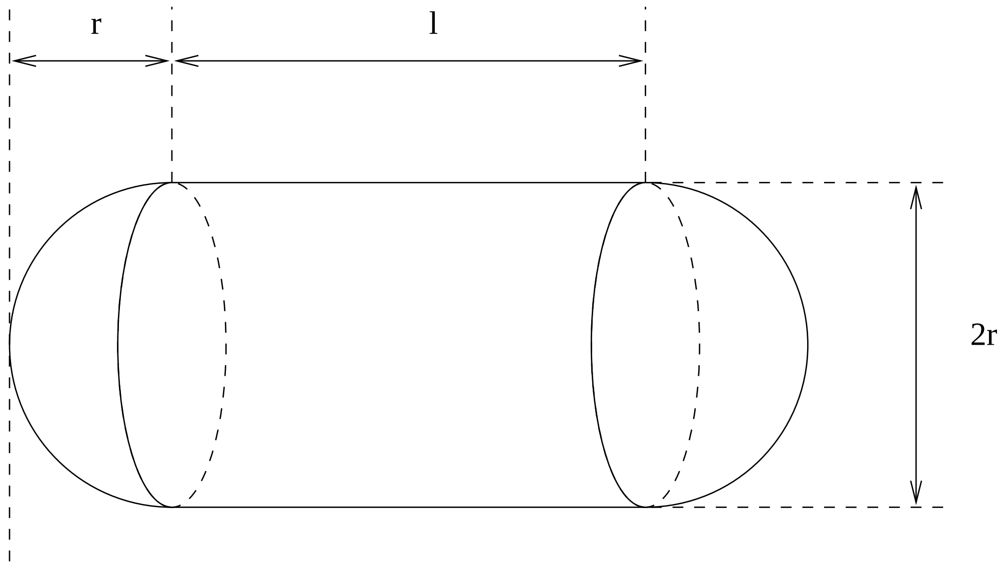

Figure 1: Image of a pressure tank

When you calculate the volume of the material, you may assume that it is given by the tank area times the thickness of the material.

Uppgift 6

(Global optimality for convex optimization and LP)

Consider the convex problem to

$$
\begin{aligned}
\text{minimize} \quad & f(x), & & (1)\\
\text{subject to} \quad & g_i(x) \geq 0, \qquad i = 1, \dots, m, & & (2)\\
& x \in X, & & (3)
\end{aligned}
$$

where $f : \Re^n \mapsto \Re$ is convex, $g_i : \Re^n \mapsto \Re$, $i = 1, \dots, m$, are concave och X is convex.

(1p) a) Define the Lagrangian dual problem when the constraint (2) is Lagrangian relaxed. Formulate and prove the Weak Duality Theorem.

(1p) b) For the primal and the dual problem, give the global optimality conditions. Establish that if they are fulfilled at a primal–dual pair of vectors, then they solve, respectively, the primal and Lagrangian dual problems, and strong duality holds.

(1p) c) Suppose now that $f(x) = c^{\mathrm{T}}x$, $g_i(x) = a_i^{\mathrm{T}}x - b_i$, $i = 1, \dots, m$, and $X = \{ x \in \Re^n \mid x \geq 0^n \}$, that is, we consider an LP problem. Give its LP dual and show that it is equivalent to the Lagrangian dual problem in a). Also give the primal–dual optimality conditions for linear programming, and show that they are the same as the global optimality conditions in b).

Uppgift 7

(Refined optimality conditions for unconstrained programming)

Consider a problem of unconstrained minimization of a *twice continuously differentiable* function f of *one variable*.

There is a "gap" between the first and the second order necessary, and the second order sufficient conditions for this problem. For example, both the first and the second order necessary conditions are satisfied for the function $f_1(x) = x^3$ at the point $x = 0$, which is not a local minimum of this function. On the other hand, the second order sufficient conditions do not hold for the function $f_2(x) = x^4$ at the point $x = 0$, which is a strict local minimum of this function.

If the function f is *infinitely many times* differentiable, then it is possible to analyze higher order derivatives and make the gap between the necessary and the sufficient conditions much smaller.

(1p) a) Show that the following condition is necessary for the point x^* to be a local minimum of the function f: either all derivatives of f at x^* are 0 or, for some *even* integer $n \geq 2$,

$$f'(x^*) = 0, \ldots, f^{(n-1)}(x^*) = 0, \text{ and } f^{(n)}(x^*) > 0.$$

Hint: Utilize that the opposite claim leads to a contradiction.

(1p) b) Show that the following conditions are sufficient for the point x^* to be a *strict* local minimum of the function f: for some *even* integer $n \geq 2$,

$$f'(x^*) = 0, \ldots, f^{(n-1)}(x^*) = 0, \text{ and } f^{(n)}(x^*) > 0.$$

(1p) c) Show that still there is a gap between the refined necessary and sufficient conditions. *Hint:* consider the functions

$$f_1(x) = \begin{cases} -e^{-1/x^2}, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ e^{-1/x^2}, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad f_2(x) = \begin{cases} 0, & \text{if } x = 0, \\ e^{-1/x^2}, & \text{otherwise,} \end{cases}$$

which both are infinitely many times differentiable.

Good luck!