Chalmers/GU
Mathematics

EXAM

TMA946/MAN280
APPLIED OPTIMIZATION

Date:	02–05–29
Time:	morning
Aids:	Text memory-less calculator
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Anton Evgrafov (0740-459022)
Result announced:	02–06–12
	Short answers are also given at the end the exam on the notice board for optimization in the MD building.

Exam instructions

When you solve the questions

State your methodology carefully.
Use generally valid methods.

Only write on one page of each sheet. Do not use a red pen.
Do not solve more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Uppgift 1

(Linear programming)

Consider the LP problem

$$
\begin{array}{lrcrcl}
\text{maximize } z = & 2x_1 & + & x_2, & & \\
\text{s.t.} & x_1 & + & 2x_2 & \geq & 1, \\
& -2x_1 & + & x_2 & \leq & 1, \\
& x_1 & , & x_2 & \geq & 0.
\end{array}
$$

Observe that the problem is one of maximization.

(2p) a) Solve the problem using Phase I and II of the simplex method.

(1p) b) Given the answer to the problem in a), what is the dual optimal solution?

(3p)

Uppgift 2

(LP duality, the set covering problem)

Consider the LP problem

$$
\begin{aligned}
\text{minimize } & c^{\mathrm{T}} x \\
\text{s.t. } & Ax \geq \mathbf{1} \\
& x \geq 0
\end{aligned}
$$

where each element in A can only have values 0 or 1, $c_i > 0$ for all i and $\mathbf{1}$ denotes a vector of ones.

Sometimes it is not necessary to solve this problem to proven optimality; to save time we may settle for a solution within 5% of the optimal value. In order to know how far from the optimal value we are, we need a lower bound on the optimal value. Formulate a method to create such a bound from a given primal feasible basis. As we wish to use the bound to terminate the solution, we need the bound to be fairly sharp: it is for instance not sufficient to set all x to 0. In addition to being sharp, the bound must not take too long to compute, as we wish to find it regularly to terminate the solution process. We may for instance not solve the dual problem from scratch. (It is however a good idea to use ideas from LP duality!)

(A motivation: We assume that we are to build a number of fire stations in a city, making it possible to reach each house with a truck within 10 minutes. We assume that we have N houses and M possible sites for stations. We let a_{ij} be 1 if it is possible to reach house i with a truck from station j within 10 minutes, and 0 otherwise. We let x_j be one if we build a station at site j, and 0 otherwise. We may now formulate the problem of finding the cheapest building pattern as

$$\begin{aligned} \text{minimize } & c^{\mathrm{T}}x \\ \text{s.t. } & Ax \geq \mathbf{1} \\ & x \in \{0,1\} \end{aligned}$$

A problem of this form is called a "set covering" problem, something encoundered within scheduling among other areas. When one solves this kind of problem, one frequently needs to solve its LP relaxation, which is exactly the problem considered above; the condition that $x \leq 1$ is redundant in the LP relaxation.)

Uppgift 3

(Nonlinear optimization)

We consider a problem of the type

$$\begin{aligned} \text{maximize } & f(x), \\ \text{s.t. } & g(x) \leq 0^m, \\ & h(x) = 0^\ell, \end{aligned}$$

where f, g_i, h_j are real valued, twice continuously differentiable functions.

Suppose that you wish to solve this problem using your favourite method.

(2p) a) Suppose that you get the print-out

```
Search terminated.  The problem is infeasible.
```

So, there exist no feasible solutions to this problem. Under some circumstances it is acceptable to find near-feasible solutions, ones that violate one or several constraints slightly, especially if the constraints denote some type of capacity; in such cases one can imagine letting a violation of a constraint have the effect of an increased cost, incurred by the need to buy extra capacity. (Such constraints are called *soft*.) For the above problem we are interested in formulating such a problem.

Suppose that the inequality constraints are soft. Give a new model which allows for the inequalities to be violated, at additional costs which are proportional to the degree of violation. (The model should not reward a solution that is strictly feasible.)

(1p) b) Suppose instead that we are using, as our favourite method, an SQP algorithm, for example the increasingly popular *filter-SQP* method. Running it on our problem, suppose we get the print-out

```
Search successfully completed.  Optimal solution stored on file.
```

Give the weakest possible conditions on the properties of the problem under which the solution obtained is guaranteed to be *globally* optimal.

Uppgift 4

(Nonlinear programming)

Consider the following least squares problem:

$$\min_{x \in \Re^n} \sum_{i=1}^{N} \big(f_i(x)\big)^2.$$

This problem can sometimes be efficiently solved by a so-called incremental gradient method:

1. Choose the initial guess x_0 and the step length $\alpha > 0$.
2. For $k = 0, 1, 2, \dots$
   (a) let $x_k^{(0)} = x_k$,
   (b) for $i = 0, 1, \dots N-1$, let $x_k^{(i+1)} = x_k^{(i)} - 2\alpha f_{i+1}(x_k^{(i)}) \nabla f_{i+1}(x_k^{(i)})$,
   (c) let $x_{k+1} = x_k^{(N)}$.

You will study the behaviour of the incremental gradient method on the example of a one-dimensional linear least squares problem (note: here, $N = 2$):

$$\min_{x \in \Re} (x+1)^2 + (x-1)^2. \tag{1}$$

(1p) a) What is the optimal solution to the problem (1)? Choose $\alpha = 1/2$, $x_0 = 2$ and make 2 steps of incremental gradient algorithm. Does it converge to the optimal solution?

(2p) b) For ease of notation, we denote $y_k := x_k^{(1)}$. Then, $y_k = x_k - 2\alpha f_1(x_k) \nabla \alpha f_1(x_k)$, $x_{k+1} = y_k - 2\alpha f_2(y_k) \nabla \alpha f_2(y_k)$. Assume that α is chosen in the interval $(0, 1)$. Let $\lim_{k\to\infty}(x_k, y_k) = (x^*, y^*)$. Calculate the limit value $(x^*, y^*) = (x^*(\alpha), y^*(\alpha))$. (Hint: if $(x_k, y_k) \to (x^*, y^*)$, $g(x_k, y_k) = 0$ and g is a continuous function, then $g(x^*, y^*) = 0$.) What can you say about $\lim_{\alpha\to 0} x^*(\alpha)$?

(3p) Uppgift 5

(Integer programming)

A manufacturer of printers has a chain of retailers; for those, the company wishes to set up a contract with maintenence companies. There are four retailers in the area, and it is possible to set up contracts with at most three maintenence companies. The retailers are estimated to need 5, 3, 3, and 4 man hours of service per day, respectively, and the maintenence companies can offer at most 10, 10, and 7 man hours of service per day, respectively. To sign a contract with the different maintenence companies cost 150, 160, and 100 k Skr, respectively. The cost (in thousands of Swedish crowns) to sign a maintenence company to a certain retailer is given in the table below.

Retailer	1	2	3	4
Maintenence shop 1	11	12	14	22
Maintenence shop 2	27	10	13	9
Maintenence shop 3	17	18	9	12

Some additional restrictions:

- At most two retailers can be assigned to the first maintenence shop.
- Each retailer may only utilize one repair shop.
- For setting up contracts there is a budget of 350 k Skr which must not be exceeded.

The goal is to satisfy the retailers' demand of service at least cost.

Formulate the above problem as a *linear integer programming* problem. *Do not solve it!*

Hint: Let the variables correspond to signing/not signing a contract with the different maintenence shops and to connecting them to the different retailers.

Uppgift 6

(Optimality conditions)

(1p) a) Consider the optimization problem:

$$\begin{aligned} &\min_{x \in \Re} x^2, \\ &\text{s.t.} \quad \sin(x) \leq -1. \end{aligned} \tag{1}$$

Find every locally and every globally optimal solution to this problem. Write down the KKT conditions. Are they necessary/sufficient for this problem?

(1p) b) The triple (x_0, α, λ) is said to satisfy Fritz-John (FJ) optimality conditions for the nonlinear programming problem

$$\begin{aligned} &\min_{x \in \Re^n} f(x), \\ &\text{s.t.} \quad g_i(x) \leq 0, \quad i = 1, \dots, m, \end{aligned} \tag{2}$$

defined by differentiable functions f, g_i, $i = 1, \dots, m$, if

$$\begin{aligned} & g_i(x_0) \leq 0, \quad i = 1, \dots, m, && \text{(primal feasibility)} \\ & (\alpha, \lambda) \geq 0, (\alpha, \lambda) \neq 0, && \\ & \alpha \nabla f(x_0) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x_0) = 0, && \text{(dual feasibility)} \\ & \lambda_i g_i(x_0) = 0, \quad i = 1, \dots, m, && \text{(complementarity slackness)} \end{aligned}$$

Do the locally/globally optimal solutions to the problem (1) satisfy the FJ optimality conditions?

(1p) c) Question the usefullness of the FJ optimality conditions by finding a point (x, y), which satisfies these conditions for the problem:

$$\begin{aligned} &\min_{(x,y)} y, \\ &\text{s.t.} \begin{cases} x^2 + y^2 \leq 1, \\ x^3 \geq y^4, \end{cases} \end{aligned}$$

but, nevertheless, is neither the local nor the global minimum for this problem.

Uppgift 7

(Relaxation through an augmented Lagrangian)

(1p) a) Consider the problem

$$\begin{aligned} f^* := \text{minimum } & f(x), \\ \text{s.t. } & x \in X, \end{aligned} \tag{P}$$

and

$$\begin{aligned} l^* := \text{minimum } & l(x), \\ \text{s.t. } & x \in G. \end{aligned} \tag{R}$$

If $X \subseteq G$ and $l(x) \leq f(x)$ for all $x \in X$ we say that (R) is a *relaxation* av (P). Conversely, (P) is then a *restrification* of (R).

Show that $l^* \leq f^*$.

(2p) b) Consider the problem of the form

$$\begin{aligned} f^* := \text{minimum } & f(x), \\ \text{s.t. } & g_i(x) = 0, \quad i = 1, \dots, m, \end{aligned}$$

where $x \in \Re^n$ and f and g_i, $i = 1, \dots, m$, are continuous functions on $\Re^n$. Let u_i, $i = 1, \dots, m$, be real multipliers (dual variables) for the constraints and let $P : \Re^m \mapsto \Re_+$ be a continuous *exterior penalty function*, that is, a function such that

$$P(y) \begin{cases} = 0, & \text{if } y = 0^m, \\ > 0, & \text{if } y \neq 0^m. \end{cases}$$

Consider the penalized problem

$$\theta^* = \underset{x \in \Re^n}{\text{minimum}} \; \theta(x) := f(x) + \sum_{i=1}^m u_i g_i(x) + \rho P(g(x)),$$

where $g(x)$ is the m-vector of g_i and where $\rho > 0$. Show that this problem is a relaxation of the original one.

Good luck!