Chalmers/GU
Mathematics

**EXAM**

# TMA946/MAN280
# APPLIED OPTIMIZATION

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| **Date:** | 02–08–26 |
| **Time:** | ML 7, afternoon |
| **Aids:** | Text memory-less calculator |
| **Number of questions:** | 7; passed on one question requires 2 points of 3.<br>Questions are *not* numbered by difficulty.<br>To pass requires 10 points and three passed questions. |
| **Examiner:** | Michael Patriksson |
| **Teacher on duty:** | Niclas Andréasson (0740-459022) |
| **Result announced:** | 02–09–09<br>Short answers are also given at the<br>end the exam on the notice board for optimization<br>in the MD building. |

## Exam instructions

**<u>When you solve the questions</u>**

*State your methodology carefully.*
*Use generally valid methods.*

*Only write on one page of each sheet. Do not use a red pen.*
*Do not solve more than one question per page.*

**<u>At the end of the exam</u>**

*Sort your solutions by the order of the questions.*
*Mark on the cover the questions you have answered.*
*Count the number of sheets you hand in and fill in the number on the cover.*

## Question 1

(Linear programming)

Consider the LP problem (observe that it is a max-problem) to

$$\begin{array}{llcccl}\text{maximize} & z = x_1 & + & 2x_2 & & \\ \text{subject to} & x_1 & - & x_2 & \geq & 1, \\ & x_1 & + & x_2 & \leq & 2, \\ & x_1 & , & x_2 & \geq & 0.\end{array}$$

**(2p)** a) Solve the problem by using, in a correct manner, the simplex method using both Phase I and Phase II.

**(1p)** b) Is the solution obtained a unique optimal solution? Motivate algebraically from your solution. A graphic motivation is not sufficient, as it only works in low dimensions.

A few matrix inverses that may come in handy:

$$\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} \qquad \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}^{-1} = \begin{bmatrix}0 & 1\\-1 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix} \qquad \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^{-1} = \begin{bmatrix}0 & 1\\1 & -1\end{bmatrix}$$

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## (3p) Question 2

(Modelling)

The Swedish company Stone & Gravel (shorthand: S&G) has made a contract with the Swedish National Road Administration (SNRA, "Vägverket"), according to which they are supposed to deliver a certain amount of small stone chips ("småflis") to two central warehouses, from which trucks will collect the material to spread on roads during Winter. The amount to be delivered per warehouse is $l_1, l_2$ tonnes, where the digit denotes the warehouse. The company S&G has two pits where they can collect gravel ("grus") and an open-air mine where they can collect blast stone ("sprängsten"), and also a small extra supply of stone chips at another location. At their facility, S&G has a sieve ("sikt") and a stone crusher ("stenkross"). The gravel from the pits are run through the sieve. For each tonne of gravel run through the sieve $a_1$ tonnes of stone chips are produced, $a_2$ tonnes of macadam, as well as $a_3$ tonnes of sand. For each tonne of blast stone that is run through the stone crusher $k_1$ tonnes of stone chips, $k_2$ tonnes of macadam and $k_3$ tonnes of sand are produced. In addition, you can run macadam through the stone crusher, to produce $l_1$ tonnes of stone chips and $l_3$ tonnes of sand per tonne of macadam. The sieve has a maximal capacity, according to which we can run through at most $t_1$ tonnes of any material during any one month. Similarly, the stone crusher can only handle $t_2$ tonnes during one month.

The total cost of quarrying ("bryta") and transporting blast stone to the facility is $p_1$ Skr per tonne; the corresponding cost for quarrying and transporting gravel is $p_2, p_3$ for the two pits. Macadam and sand are not worthless products; they can be sold for $d_2$ and $d_3$ Skr per tonne, respectively. No transportation costs arise for these products as they are collected by the customers. We suppose that we can sell any amount of macadam and sand produced. The cost of transporting stone chips from the facility to the warehouses is $g_1, g_2$ Skr per tonne. S&G has a supply of stone chips at another location; in this warehouse, there are $q$ tonnes of stone chips which can be transported to the SNRA warehouses for a cost of $h_1$ and $h_2$ Skr per tonne, respectively.

Formulate a linear programming model which will help S&G to fulfill the contract at the lowest cost (including the sales of macadam and sand). We request that You produce a picture showing how the material flows through the model, and that the variables declared are also shown in this picture.

Summary: Summary:

$l_1, l_2$ Amount of stone chips to be delivered to the SNRA warehouses (in tonnes).

$a_1, a_2, a_3$ Amount of stone chips, macadam and sand produced in the sieve from one tonne of gravel (in tonnes).

$k_1$,$k_2$,$k_3$ Amount of stone chips, macadam and sand produced by the stone crusher from one tonne of blast stone (in tonnes).

$l_1$,$l_3$ Amount of stone chips and sand produced by the stone crusher from one tonne of macadam (in tonnes).

$p_1$,$p_2$, $p_3$ Total cost (for quarrying and transporting) for one tonne of blast stone from the open-air mine and the gravel pits (in Skr).

$t_1, t_2$ Total capacity of the sieve and stone crusher (in tonnes).

$d_2$,$d_3$ Sales price per tonne of macadam and sand (in Skr).

$g_1, g_2$ Transportation cost per tonne of stone chips from the facility to the warehouses (in Skr).

$h_1$,$h_2$ Transportation cost per tonne of stone chips from the extra supply to the warehouses (in Skr).

$q$ Size of the extra supply.

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(3p) **Question 3**

(LP theory)

Consider the LP problem of the form

$$\begin{aligned} \text{minimize } & c^{\mathrm{T}}x \\ \text{subject to } & Ax \geq b \\ & x \geq 0 \end{aligned}$$

This problem is solved by the simplex method. Suppose that during a given step of the algorithm, there is only one possible incoming variable, $x_i$. Suppose further that the pivoting step is non-degenerate, so that after the pivot, the new solution will have a strictly lower function value.

Will the variable $x_i$ be non-zero in any optimal solution? (In other words, will the variable be a basic variable in every optimal basis?) Motivate Your answer well.

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**Question 4**

(Nonlinear programming)

(1p) a) Consider the parametric minimization problem

$$\min_{x_1,x_2} \frac{3}{2}(x_1^2 + x_2^2) + (1+a)x_1x_2 - (x_1 + x_2) + b, \tag{1}$$

where $a$ and $b$ are some unknown real-valued parameters.

Find all possible values of $a$ and $b$ such that the problem (1) possesses a unique globally optimal solution. Write down this solution (in terms of the parameters $a$ and $b$).

(1p) b) Consider the following nonlinear programming problem:

$$\min_{x \in \Re} x^{\frac{4}{3}}, \qquad (= \sqrt[3]{x^4}), \tag{2}$$

having $x^* = 0$ as a unique globally optimal solution.

Describe Newton's method (with a constant step length 1), then apply it to this problem, and show that it diverges for any starting point $x_0 \neq 0$ (no matter how close to zero we start). Explain why.

**(1p)** c) In numerical implementations of optimization algorithms, owing to various types of numerical errors, the computation of the gradient of the function $f : \Re^n \mapsto \Re$ usually results in $(I + \mathcal{E}(x))\nabla f(x)$, where $\mathcal{E}(x)$ is a matrix including the errors and $\nabla f(x)$ is the correct gradient value. (Both $I$ and $\mathcal{E}(x)$ are $n \times n$-matrices.)

Write down the conditions on the matrix $\mathcal{E}(x)$ under which we can guarantee that the computed direction $-(I + \mathcal{E}(x))\nabla f(x)$ is a descent direction, provided that $\nabla f(x) \neq 0^n$.

Note: the condition that You provide is not supposed to include the term $\nabla f(x)$; the condition does not depend on this entity at all.

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## Question 5

(Optimality conditions)

**(1p)** a) Consider the nonlinear programming problem with equality constraints:

$$\begin{aligned} &\min_{x \in \Re^n} \; f(x), \\ &\text{s.t} \begin{cases} h_1(x) &= 0, \\ \vdots \\ h_m(x) &= 0, \end{cases} \end{aligned} \tag{1}$$

where $f$, $h_1, \dots, h_m$ are once continuously differentiable functions.

Show that the problem (1) is equivalent to the following problem with one inequality constraint:

$$\begin{aligned} &\min_{x \in \Re^n} \; f(x), \\ &\text{s.t} \left\{ \sum_{i=1}^{m} \big(h_i(x)\big)^2 \leq 0. \right. \end{aligned} \tag{2}$$

Show (by giving a formal argument or an illustrative example) that the KKT conditions for the latter problem are never necessary for local optimality.

*Hint:* Which constraint qualifications (CQ) can be satisfied for the problem (1)?

Can any constraint qualifications be satisfied for the problem (2)? If yes, which ones?

**(2p)** b) Consider the unconstrained minimization problem

$$\min_{x \in \Re^n} \; \max\big\{f_1(x), f_2(x)\big\},$$

where $f_1 : \Re^n \to \Re$, $f_2 : \Re^n \to \Re$ are once continuously differentiable functions.

Show that if $x^*$ is a local minimum for this problem, then there exist $\mu_1, \mu_2 \in \Re$ such that

$$\mu_1 \geq 0, \mu_2 \geq 0, \quad \mu_1 \nabla f_1(x^*) + \mu_2 \nabla f_2(x^*) = 0, \quad \mu_1 + \mu_2 = 1,$$

and $\mu_i = 0$ if $f_i(x^*) < \max\{\, f_1(x^*), f_2(x^*) \,\}$, $i = 1, 2$.

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**(3p)** 
## Question 6

(Convexity)

Define the terms convex set and convex function. Show that the two definitions are (in some ways) equivalent by proving that for a function $f : \Re^n \mapsto \Re$ it is true that it is a convex function if, and only if, its *epigraph*,

$$\text{epi}\, f = \{\, (x, \alpha) \mid f(x) \leq \alpha \,\},$$

is a convex set.

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## Question 7

(Pseudo-convexity)

The class of *pseudo-convex* functions is important, since they have several of the important properties of convex functions. So, for example, the necessary and sufficient conditions for the global optimality of a feasible solution in a convex program are also valid for problems where the objective is only pseudo-convex. Since there are interesting application areas where we find cases of pseudo-convex minimization, where the objective functions are pseudo-convex but not convex, the importance of this result is immediate. The goal of this question is to investigate precisely this property.

**(1p)** a) A function $f : \Re^n \mapsto \Re$ which is once continuously differentiable is said to be pseudo-convex if

$$\nabla f(x)^{\mathrm{T}}(y - x) \geq 0 \implies f(y) \geq f(x), \qquad x, y \in \Re^n.$$

(The definition states that if $f$ is pointing uphill at $x$ towards $y$ then the value of $f$ is larger at $y$ than at $x$.) Show that a convex function is pseudo-convex. Also show that the converse is not true, that is, that there are pseudo-convex functions that are not convex. An example, even graphical, is sufficient.

**(2p)** b) For convex problems the following necessary and sufficient condition states precisely when a feasible solution $x^* \in X$ is a global minimum in the problem to

$$\begin{aligned} &\underset{x \in \Re^n}{\text{minimize}} \ f(x), \\ &\text{subject to } x \in X, \end{aligned}$$

where $X \subseteq \Re^n$ is a closed and convex set:

$$\nabla f(x^*)^{\mathrm{T}}(y - x^*) \geq 0, \qquad y \in X.$$

Show that this condition is both necessary and sufficient also for only pseudo-convex functions $f$.

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*Good luck!*