Chalmers/GU
Mathematics

**EXAM**

# TMA946/MAN280
# APPLIED OPTIMIZATION

| | |
|---|---|
| **Date:** | 03–03–10 |
| **Time:** | House V, morning |
| **Aids:** | Text memory-less calculator |
| **Number of questions:** | 7; passed on one question requires 2 points of 3. |
| | Questions are *not* numbered by difficulty. |
| | To pass requires 10 points and three passed questions. |
| **Examiner:** | Michael Patriksson |
| **Teacher on duty:** | Niclas Andréasson (0740-459022) |
| **Result announced:** | 03–03–24 |
| | Short answers are also given at the |
| | end the exam on the notice board for optimization |
| | in the MD building. |

## Exam instructions

**When you solve the questions**

*State your methodology carefully.*
*Use generally valid methods.*

*Only write on one page of each sheet. Do not use a red pen.*
*Do not solve more than one question per page.*

**At the end of the exam**

*Sort your solutions by the order of the questions.*
*Mark on the cover the questions you have answered.*
*Count the number of sheets you hand in and fill in the number on the cover.*

**(3p)**

## Question 1

(The Simplex method)

Solve the following LP by using Phase I and Phase II of the Simplex method.

$$
\begin{array}{llrrrrcl}
\text{minimize} & z = 2x_1 & & & & & & \\
\text{subject to} & & x_1 & & -x_3 & & = & 3, \\
& & x_1 & -x_2 & & -2x_4 & = & 1, \\
& & 2x_1 & & & +x_4 & \leq & 7, \\
& & x_1, & x_2, & x_3, & x_4 & \geq & 0.
\end{array}
$$

Is the optimal solution obtained unique? Motivate your answer; give an alternative optimal solution if the solution obtained is not unique.

Some matrix inverses that might come in handy are

$$
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0.5 & -0.5 & 0 \\ 1 & 0 & 0 \\ -2.5 & 0.5 & 1 \end{pmatrix},
$$
$$
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -0.2 & -0.4 \\ 0 & 0.2 & 0.4 \\ 0 & -0.4 & 0.2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix},
$$
$$
\begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & -1 & -2 \\ 1 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix}.
$$

---

## Question 2

(Linear programming theory)

**(2p)** a) Consider the LP-problem to find

$$
\begin{aligned}
z^*(b) := \text{minimum } & c^{\mathrm{T}} x \\
\text{subject to } & Ax = b, \\
& x \geq 0,
\end{aligned}
$$

given a right-hand side vector $b \in \Re^m$. As we vary the vector $b$, the problem will have different optimal solutions, $x^*(b)$, and different optimal values, $z^*(b)$. Let $b \in \Re^m$. Give conditions on the optimal solution $x^*(b)$ such that the gradient $\nabla_b z^*(b)$ of the optimal objective value exists, and state a formula for this gradient when these conditions are fullfilled.

**(1p)** b) Determine if the function $b \mapsto z^*(b)$ is convex, concave or neither.

---

## (3p) Question 3

(Modelling)

You are the CEO of ACME Concrete Inc., and it is your job to maximize the company's operating profit. As the name suggets, ACME Concrete Inc. manufactures concrete, made from two main components, clay and limestone. Limestone is bought from the local quarry, at a price of $q$ per ton, and clay is fetched in the company's own clay-pit for free. These raw materials must then be transported to the factory using the company's fleet of trucks, which may carry at most $b$ tons per month. Once at the factory, the clay must be dried, at a cost of $c$ per ton of clay *input* into the dryer, and the limestone crushed, at a cost of $d$ per ton crushed. When the clay is dried, it loses 30% of its weight, while the crushing of limestone does not incur any losses. The maximum drying capacity is $e$ tones of input per month, while the crushing process has no practical capacity limit. Clay and limestone are then mixed to form cement; in order to make cement, we must use between 1.7 and 2.1 tons of *dried* clay per ton of limestone. Once the cement is made it is sold to three different building sites. The amounts we sell are given as a function of the price. If $p$ is the price per ton that we set, then the amount bought by site $i$ is $\frac{k_i}{m_i+p^2}$, with $k_i$ and $m_i$ being different positive constants for different customers. In addition, there exists a maximum price of $r_i$ for each customer above which they will purchase nothing, as it will then be cheaper for them to buy their concrete from your competitor, Concrete-O-Rama. The cut-off prices are different, as the distance to the customers differ, and the customers pay for the transport.

Formulate the problem of maximizing the company's profits as a *mixed integer program with linear constraints*.

As solvers generally have problems with non-differentiable functions, you must make sure that the objective function is differentiable everywhere.

---

## Question 4

(Nonlinear programming duality)

Consider the nonlinear programming problem to find

$$\begin{aligned} f^* := \text{minimum} \quad & f(x), \\ \text{subject to} \quad & g(x) \geq 0^m, \\ & x \in X, \end{aligned}$$

where $f : \Re^n \mapsto \Re$ and $g : \Re^n \mapsto \Re^m$ are continuous functions, and $X \subseteq \Re^n$ is a closed set. We are interested in solving this problem through the use of Lagrangian duality. So, we define

$$\begin{aligned} \mathcal{L}_*(\lambda) := \text{minimum} \quad & \{\mathcal{L}(x, \lambda) := f(x) - \lambda^{\mathrm{T}} g(x)\}, \\ \text{subject to} \quad & x \in X, \end{aligned}$$

to be the Lagrangian dual function, and

$$\mathcal{L}_*^* := \begin{array}{ll} \text{maximum} & \mathcal{L}_*(\lambda), \\ \text{subject to} & \lambda \geq 0^m, \end{array}$$

to be the Lagrangian dual problem.

The *global optimality conditions* for the primal–dual pair of problems above are that the system

$$\{ x \in X \mid \mathcal{L}(x, \lambda) = \mathcal{L}_*(\lambda) \} \cap \{ x \in \Re^n \mid g(x) \geq 0^m \} \cap \{ (x, \lambda) \in \Re^n \times \Re^m \mid \lambda^{\mathrm{T}} g(x) = 0 \}$$

has a nonempty set of solutions $(x, \lambda)$ with $\lambda \geq 0^m$.

The following three questions shall be solved *independently* from each other.

**(1p)** a) Suppose that $f$ is convex and continuously differentiable, and that every function $g_i$ $(i = 1, \dots, m)$ is concave and continuously differentiable (so that the primal problem is a convex and differentiable one). Suppose further that $X = \Re^n$.

State the Karush–Kuhn–Tucker conditions for the primal problem, and show that they are equivalent to the above stated global optimality conditions.

**(1p)** b) Show that if the global optimality conditions are satisfied, then every solution pair $(x, \lambda)$ solves, respectively, the primal and dual problem, and that strong duality then holds [that is, $f^* = \mathcal{L}_*^*$ holds].

**(1p)** c) Suppose that $X$ is bounded. Let $\lambda^*$ be an optimal solution to the Lagrangian dual problem. Suppose that at $\lambda^*$, the Lagrangian dual function $\mathcal{L}_*$ is differentiable.

State the form of the gradient of $\mathcal{L}_*$ at $\lambda^*$, and state sufficient conditions on the original problem such that differentiability is ensured.

Establish that under the differentiability condition, every vector in $\{ x \in X \mid \mathcal{L}(x, \lambda^*) = \mathcal{L}_*(\lambda^*) \}$ is optimal in the primal problem, and that strong duality holds.

*Hint!* What are the necessary and sufficient conditions for $\lambda^*$ to be globally optimal in the dual problem, when you know that differentiability holds? State these conditions and compare them to the global optimality conditions.

(The conclusion is that when $\mathcal{L}_*$ is differentiable at the optimal solution, then it is much easier to generate an optimal primal solution from the Lagrangian subproblem than in the general case.)

## Question 5

(Linear programming theory)

Consider the standard form of the linear programming problem,

$$\begin{aligned} \text{minimize } & z := c^{\mathrm{T}} x \\ \text{subject to } & Ax = b, \\ & x \geq 0, \end{aligned}$$

where $A \in \Re^{m \times n}$, $x \in \Re^n$, $c \in \Re^n$, and $b \in \Re^m$. Further, we assume that $n > m$ and that $A$ has full row rank.

**(2p)** a) State and prove the Weak and Strong duality Theorems for this LP problem.

**(1p)** b) Suppose that the partition $\bar{x} = (x_B, x_N)$ defines a non-degenerate basic feasible solution for the LP problem.

The incoming criterion in the simplex method is a very special case of the generation of a feasible descent direction $p \in \Re^n$, in which a specified number of elements of $p$ are allowed to be non-zero. (In fact, this number is $m + 1$.) Show that this descent direction can be derived by constructing and solving a *restriction* of the original problem at $\bar{x}$, such that, in addition to $p$ being a feasible direction,

$$\sum_{j \in N} p_j \leq 1$$

must hold.

Show that, consequently to the above connection, the optimal value of the search-direction finding problem has a non-positive optimal objective value, and that it is negative *if and only if* $\bar{x}$ is *not* an optimal basis.

---

## Question 6

(Unconstrained optimization)

Let $A$ be a symmetric $n \times n$ matrix. For $x \in \Re^n$, $x \neq 0$, consider the function $\rho(x) := \frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} x}$, and the related optimization problem to

$$(P) \quad \underset{x \neq 0}{\text{minimize}} \ \rho(x).$$

**(2p)** a) Determine all the stationary points as well as the global minima in the minimization problem $(P)$.

**(1p)** b) Show that the optimal objective value of the problem $(P)$ equals the optimal objective value of the problem to

$$(\hat{P}) \begin{cases} \underset{x \in \Re^n}{\text{minimize}} \; x^{\mathrm{T}} A x, \\ \text{subject to } x^{\mathrm{T}} x \le 1, \end{cases}$$

if the latter problem is non-convex.

---

## Question 7

(KKT)

Consider the nonlinear programming problem to

$$(P) \begin{cases} \underset{x \in \Re^n}{\text{minimize}} \; f(x), \\ \text{subject to } g_i(x) \le 0, \qquad i = 1, \dots, m, \end{cases}$$

where $f, g_i$ are continuously differentiable functions.

The following questions should be solved independently from each other.

**(1p)** a) For numerical purposes we would like to scale the variables, constraints, and the objective function of the problem, so that they have the similar magnitudes. Formally, we perform the following substitutions:

- *change of variables*: $y_j := \alpha_j x_j$, $\alpha_j > 0$, $j = 1, \dots, n$ (short-hand notation: $y = \boldsymbol{\alpha} * x$, $x = \boldsymbol{\alpha}^{-1} * y$);
- *scaling of the constraints*: $\tilde{g}_i(y) := \beta_i g_i(\boldsymbol{\alpha}^{-1} * y)$, $\beta_i > 0$, $i = 1, \dots, m$; and
- *scaling of the objective function*: $\tilde{f}(y) := \beta_0 f(\boldsymbol{\alpha}^{-1} * y)$, $\beta_0 > 0$;

and we consider the new (scaled) problem to

$$(\tilde{P}) \begin{cases} \underset{y \in \Re^n}{\text{minimize}} \; \tilde{f}(y), \\ \text{subject to } \tilde{g}_i(y) \le 0, \qquad i = 1, \dots, m. \end{cases}$$

Let $x_0 \in \Re^n$ be a KKT-point for the problem $(P)$, with KKT-multipliers $\lambda_0 \in \Re^m_+$. Show that the point $y_0 = \boldsymbol{\alpha} * x_0$ is a KKT-point for the scaled problem $(\tilde{P})$. What is the relation between the new KKT-multipliers [for $y_0$ in $(\tilde{P})$] with $\lambda_0$?

**(2p)** b) We consider now a specific instance of the problem $(P)$, where $f(x) := \|x - c\|^2 = \sum_{i=1}^n (x_i - c_i)^2$, for some vector $c \in \Re^n$.

This problem is usually solved as a substitute for the problem to

$$(\hat{P}) \begin{cases} \underset{x \in \Re^n}{\text{minimize}} \; \|x - c\| = \sqrt{f(x)}, \\ \text{subject to } g_i(x) \le 0, \qquad i = 1, \dots, m. \end{cases}$$

Assume that the point $x_0 \in \Re^n$ is a KKT-point for the problem $(P)$ with the associated multipliers $\lambda_0 \in \Re^m_+$. Is $x_0$ a KKT-point for $(\hat{P})$? If yes, relate the "new" multipliers with $\lambda_0$.

Similarly, assume that $x^* \in \Re^n$ is a *global* minimum for the problem $(P)$. Is $x^*$ a global minimum for the problem $(\hat{P})$?

What is the advantage of the problem $(P)$ over $(\hat{P})$?

---

*Good luck!*