Chalmers/GU
Mathematics

EXAM

TMA946/MAN280
APPLIED OPTIMIZATION

Date:	03–08–25
Time:	House V, morning
Aids:	Text memory-less calculator
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Erik Broman (0740-459022)
Result announced:	03–09–08 Short answers are also given at the end of the exam on the notice board for optimization in the MD building.

Exam instructions

When you answer the questions

State your methodology carefully.
Use generally valid methods and theory.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the linear programming problem to

$$
\begin{array}{llllll}
\text{minimize } z = & 3x_1 & +cx_2 & +x_3 & & \\
\text{subject to} & 2x_1 & & +x_3 & \geq 3, & \\
 & 2x_1 & +2x_2 & +x_3 & = 5, & \\
 & x_1, & x_2, & x_3 & \geq 0. &
\end{array}
$$

(2p) a) For $c = 1$, solve the problem by using the Simplex method using both Phase I and Phase II.

(1p) b) Find the interval for the values of the constant c such that the optimal basis from problem a) is optimal.

Hint: The interval should include the value $c = 1$.

Some matrix inverses that might come in handy for the problem in a) are

$$
\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0.5 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 0.5 & 0 \\ -0.5 & 0.5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -0.5 & 0.5 \end{pmatrix},
$$
$$
\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0.5 & 0 \\ -1 & 1 \end{pmatrix}.
$$

(3p)
Question 2

(linear programming theory)

Let A be a matrix of order $m \times n$ and c and b vectors of order $n \times 1$ and $m \times 1$, respectively. Prove that the following LP-problem is *either* infeasible *or* has an optimal objective value of zero.

$$
\begin{array}{llll}
\text{minimize} & z = c^T x & -b^T y & \\
\text{subject to} & Ax & & \geq b, \\
 & & -A^T y & \geq -c, \\
 & x, & y & \geq 0.
\end{array}
$$

(3p) Question 3

(modelling)

The company Kvartersbutiken runs a chain of small convenience stores with generous opening hours. They have now decided to establish themselves in Göteborg. In order to know where to place their stores, they have made an extensive market survey. In this survey, the population of Göteborg has been divided into m customer areas, with c_i potential customers in each area. The company has surveyed n possible store locations in their research, and the maximum customer capacity of a store in location j is given by s_j. A store is said to belong to a customer area's *primary* region if the store and the area are really close, and customers will always prefer stores in the primary area. The set P_i lists the stores in the primary region for customer area i. If there is no store in the primary area, or if all these stores are full, then some customers may choose to walk a bit if they really need to buy an item, whereas others will simply not shop. The company assumes that 50% of the potential customers who cannot be served within the primary region will go to a store within the secondary region, while the other 50% will go home without shopping. The stores in the secondary area of region i is given by the set S_i. The annual cost of running a store in location j is r_j, and each customer served will give an annual income of q.

Figure 1 illustrates the different customer areas.

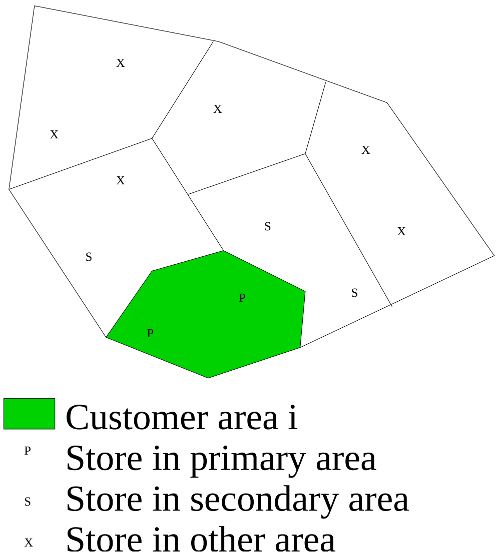

Figure 1: Customer areas.

Your task is to formulate an optimization problem to find where the stores should be located, given the customer behaviour described.

Important note: In order to be able to solve the problem using standard software, the formulated problem should be a *mixed integer linear problem*. In other words, all constraints and the objective function should be linear functions. Specifically, constraints of the type $x = \max\{y, 0\}$ are *not* allowed (although, if this type of constraint is needed, it may be rewritten into the allowed form). Furthermore, conditions written as "x is defined to be 0 if y is 0" is not allowed; this type of logical condition must be rewritten as linear constraints.

Question 4

(definitions)

(2p) a) For a function $f : \Re^n \mapsto \Re$, define (in mathematical notation!) the property of *convexity*.

Define also (in mathematical notation!) the property of *strict convexity* for the function f.

Provide an example function which is convex on $\Re^n$ but *not* strictly convex on $\Re^n$.

Provide, finally, an example function which is strictly convex and differentiable on $\Re^n$, but for which it is *not* true that the Hessian matrix is positive definite everywhere.

(1p) b) Define the term *degenerate basic feasible solution* in linear programming. Introduce all necessary notation.

Question 5

(sensitivity analysis and parametric optimization)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & f(x) := \sum_{j=1}^{n} c_j x_j^2, \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j = b, \end{aligned}$$

where c_j $(j = 1, \dots, n)$ and b all are positive constants.

(2p) a) Establish that a unique optimal solution exists to the above problem, and describe the optimal solution as an *explicit* function of the constants c_j $(j = 1, \dots, n)$ and b.

Hint: Utilize the KKT conditions for the problem. End up with an explicit formula of the form $x_j^* = p_j(c, b)$ for some function p_j.

Interpret the expression. Answer in particular the following question: For each $j = 1, \dots, n$, is $x_j^* = p_j(c, b)$ an increasing/decreasing function of c_j and/or b? Why is it natural?

(1p) b) We now let b be a parameter, that is, we allow the right-hand side of the constraint to vary. (The constants c_j remain constant, and we write $x_j^* = p_j(b)$ to reflect this.) We are interested in learning about the sensitivity of the optimal solution as a function of b, and will have good use for the explicit formula derived in (a).

Establish that the optimal solution x^* is differentiable at b. Determine the expression of the derivative of $x_j^* = p_j(b)$ as a function of b, that is, determine the expression of $\nabla_b x_j^* = p_j'(b)$.

Interpret the expression of the above derivative. In particular, answer the following questions: (i) What is the sign of the derivative? (ii) Which variable x_j increases/decreases the fastest when the value of b increases? Why?

(3p) Question 6

(steepest descent method)

Consider the following (rather simple!) optimization problem:

$$\text{minimize } \frac{1}{2}\|x\|^2 := \frac{1}{2}\sum_{i=1}^{n} x_i^2. \tag{1}$$

We would like to apply the steepest descent algorithm with a fixed stepsize (that is, the line search is replaced by a fixed step in each iteration) to solve this problem:

initialization: choose a starting point $x^0 \in \Re^n$, and a constant $s > 0$.

main step: $x^{t+1} := x^t - s\nabla f(x^t)$; let $t := t + 1$ and repeat until a convergence criterion is met.

[In the description of the algorithm, f is the objective function, $\|x\|^2$ in our case.]

(2p) a) Find the set of all pairs (x^0, s) in $\Re^n \times \Re$, for which the algorithm described converges towards the globally optimal solution to the problem (1).

(1p) b) It is known that the steepest descent algorithm is guaranteed to converge to stationary points of the continuously differentiable function $f : \Re^n \mapsto \Re$ from *any* starting point x^0 if $0 < s < 2/L$, where L is a *Lipschitz constant* for ∇f on $\Re^n$. (We note that ∇f is Lipschitz continuous on $\Re^n$ if there exists a constant $L \geq 0$ such that for every pair $x, y \in \Re^n$ it holds that

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|.)$$

Is the convergence set given by this inequality bigger, smaller, or equal to the one you have obtained in a)?

Hint: What is the Lipschitz constant L for the objective function in (1)?

Question 7

(convexity)

For any set $S \subset \Re^n$ the *convex hull* of S, denote $\operatorname{hull}(S)$, is the set of all possible convex combinations of points in S. An illustration of the convex hull is given in Figure 2.

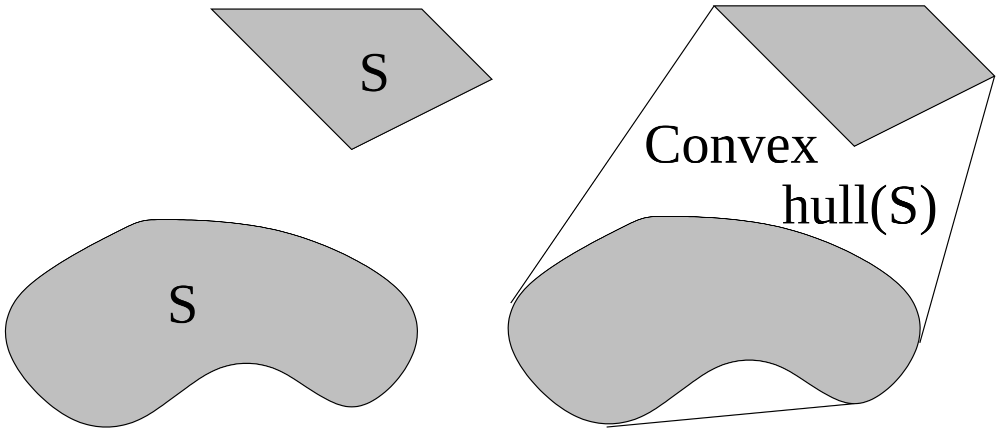

Figure 2: A set S consisting of two disjoint sets (left) and its convex hull(right).

(2p) a) Let S_1 and S_2 be arbitrary subsets of $\Re^n$. Show that $\operatorname{hull}(S_1 \cap S_2) \subseteq \operatorname{hull}(S_1) \cap \operatorname{hull}(S_2)$.

(1p) b) Prove by means of a counterexample that $\operatorname{hull}(S_1 \cap S_2) = \operatorname{hull}(S_1) \cap \operatorname{hull}(S_2)$ does *not* hold in general.

Good luck!