Chalmers/GU
Mathematics

EXAM

TMA947/MAN280
APPLIED OPTIMIZATION

Date:	04–08–23
Time:	House M, morning
Aids:	Text memory-less calculator
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Anton Evgrafov (0740-459022)
Result announced:	04–09–06 Short answers are also given at the end of the exam on the notice board for optimization in the MD building.

Exam instructions

When you answer the questions

Use generally valid methods and theory.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(linear programming duality)

(2p) a) Consider the following linear program:

$$\begin{array}{lrcrcl} \text{maximize} & z = x_1 & + & 2x_2, & & \\ \text{subject to} & -2x_1 & + & x_2 & \leq & 2, \\ & -x_1 & + & 2x_2 & \leq & 7, \\ & x_1 & & & \leq & 3, \\ & x_1 & , & x_2 & \geq & 0. \end{array}$$

This problem has the optimal solution $\boldsymbol{x}^* = (3, 5)^{\mathrm{T}}$. By using duality and complementarity, determine the optimal dual solution, as well as confirm that the solution $\boldsymbol{x}^*$ given is indeed optimal in the primal problem.

(1p) b) Consider the standard LP problem to

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$.

Suppose that this problem has a feasible solution. Prove that if it has an unbounded solution, then the corresponding dual problem cannot have any feasible solution.

(3p) Question 2

(convexity)

Show the following result for convex sets, known as the *Separation Theorem*:

Suppose that the set $C \subseteq \mathbb{R}^n$ is closed and convex, and that the point $\boldsymbol{y}$ does not lie in C. Then there exist a real α and an $n \times 1$ vector $\boldsymbol{\pi} \neq \boldsymbol{0}^n$ such that $\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{y} > \alpha$ and $\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{x} \leq \alpha$ for all $\boldsymbol{x} \in C$.

Illustrate it also geometrically.

When showing this result, you may refer to any other theorems needed without proof, but you *must* state the ones you use *clearly*.

(3p) Question 3

(modelling)

An American oil company manufactures three types of gasoline (gas 1, gas 2, gas 3). Each type of gasoline is produced by mixing together three types of crude oil (crude 1, crude 2, crude 3). The sales price per barrel of gasoline and the purchase price per barrel of crude oil is given below.

The company can purchase up to 5,000 barrels of each type of crude oil daily. The three types of gasoline differ in their octane rating and lead content. The crude oil blended to form gas 1 must have an octane rating of at least 10 and contain at most 1% lead. The crude oil blended to form gas 2 must have an octane rating of at least 8 and contain at most 2% lead. The crude oil blended to form gas 3 must have an octane rating of at least 6 and contain at most 1% lead. The octane rating and the lead content of the three types of oil are given in a table below. It costs USD 4 to transform one barrel of oil into one barrel of gasoline. The company's refinery can process up to 14,000 barrels of crude oil daily.

The company's customers require the following amounts each gasoline: gas 1—3,000 barrels per day, gas 2—2,000 barrels per day, gas 3—3,000 barrels per day. The company considers it an obligation to meet these demands. The company also has the option of advertising to stimulate demand for its products. Each dollar spent daily in advertising a particular type of gas increases the daily demand for that type of gas by ten barrels. For example, if the company decides to spend USD 20 daily in advertising gas 2, the daily demand for gas 2 will increase by $20 \times 10 = 200$ barrels.

Formulate the linear program that will enable the company to maximize daily profits (profits = revenues − costs). To simplify matters, assume the company cannot store any extra gasoline. This implies that the daily amount of gas produced should equal the daily demand for each gas type.

Data:

Sales price per barrel (in USD): Gas 1: 70; Gas 2: 60; Gas 3: 50.

Purchase price per barrel (in USD): Crude 1: 45; Crude 2: 35; Crude 3: 25.

Octane rating: Crude 1: 12; Crude 2: 6; Crude 3: 8.

Lead content (in %): Crude 1: 0.5; Crude 2: 2.0; Crude 3: 3.0.

Question 4

(on the Armijo step length rule in unconstrained optimization)

Consider the unconstrained optimization problem to

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}), \\ &\text{subject to} \quad \boldsymbol{x} \in \mathbb{R}^n, \end{aligned} \tag{1a}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}$ is continuously differentiable and weakly coercive, hence lower bounded, on $\mathbb{R}^n$. Suppose that to this problem we apply the steepest descent algorithm with the Armijo step length rule, starting at some $\boldsymbol{x}_0 \in \mathbb{R}^n$. In the Armijo rule, we replace the exact line search, in which we

$$\underset{\alpha \geq 0}{\text{minimize}} \ \varphi(\alpha) := f(\boldsymbol{x}_k + \alpha \boldsymbol{p}_k),$$

by the following rule:

Let $\mu \in (0, 1)$. The step lengths accepted by the Armijo step length rule are the positive values α which satisfy the inequality

$$\varphi(\alpha) - \varphi(0) \leq \mu \alpha \varphi'(0), \tag{2a}$$

that is,

$$f(\boldsymbol{x}_k + \alpha \boldsymbol{p}_k) - f(\boldsymbol{x}_k) \leq \mu \alpha \nabla f(\boldsymbol{x}_k)^{\mathrm{T}} \boldsymbol{p}_k. \tag{2b}$$

Usually, the value of the step length α is taken to be of the form $\alpha := \alpha_0 \cdot \beta^i$, where $\alpha_0 > 0$ is the initial step taken, $\beta \in (0, 1)$ is a factor by which we multiply the initial step if it is not successful (usually, we set $\beta = 1/2$ so that the step length is halved repeatedly), and i is an integer which we first give the value 0, and then increase by one until $\alpha := \alpha_0 \cdot \beta^i$ is small enough to satisfy (2).

The purpose of this exercise is to point out that the Armijo rule is not always a good rule, because the initial step length α_0 may be hard to choose appropriately. (Especially, the initial step might be *too small* to provide fast convergence; since the Armijo rule is based on decreasing the trial step length and never to increase it, fast convergence can in such cases not occur.)

Recall the following on the issue of convergence rates: Suppose that $\{\boldsymbol{x}_k\} \to \boldsymbol{x}^*$. We say that the speed of convergence is *linear* if the quotients

$$q_k := \frac{\|\boldsymbol{x}_{k+1} - \boldsymbol{x}^*\|}{\|\boldsymbol{x}_k - \boldsymbol{x}^*\|}$$

satisfy that

$$\limsup_{k\to\infty} q_k < 1.$$

Suppose now that

$$f(x) := x^4/4, \qquad x \in \mathbb{R}. \tag{3}$$

(2p) a) Describe the iteration of the steepest descent method for the problem (1) when f is given by (3), that is, give the formula that provides x_{k+1} from x_k when the step length chosen is α_k.

Further, show that no matter how small or large (but finite and positive) the value of α_0 is chosen, linear convergence cannot be obtained for the steepest descent method using the Armijo step length rule, when applied to the given problem. Explain why!

(1p) b) Suppose we instead apply Newton's method with line searches. Show that linear convergence is guaranteed for any choice of $\alpha_0 > 0$ small enough, or for $\alpha_k = 1$ for all k and any value of $\mu > 0$ small enough.

Question 5

(nonlinear programming optimality)

Consider the optimization problem to

$$\begin{aligned}
&\underset{x}{\text{minimize}} && f(\boldsymbol{x}), && && \text{(1a)}\\
&\text{subject to} && g_i(\boldsymbol{x}) \leq 0, && i = 1, \dots, m, && \text{(1b)}\\
& && h_j(\boldsymbol{x}) = 0, && j = 1, \dots, \ell, && \text{(1c)}
\end{aligned}$$

where the functions f, g_i ($i = 1, \dots, m$), and h_j ($j = 1, \dots, \ell$) are continuously differentiable on $\mathbb{R}^n$.

(1p) a) Suppose that $\bar{\boldsymbol{x}}$ is feasible in the problem (1). Prove the following statement by using linear programming duality: $\bar{\boldsymbol{x}}$ satisfies the Karush–Kuhn–Tucker (KKT) conditions if and only if the following LP problem has the optimal value zero:

$$\underset{p}{\text{minimize}} \quad \nabla f(\bar{\boldsymbol{x}})^{\mathrm{T}} \boldsymbol{p},$$

$$\begin{aligned} \text{subject to} \quad & g_i(\bar{\boldsymbol{x}}) + \nabla g_i(\bar{\boldsymbol{x}})^{\mathrm{T}} \boldsymbol{p} \leq 0, \qquad & i = 1, \ldots, m, \\ & h_j(\bar{\boldsymbol{x}}) + \nabla h_j(\bar{\boldsymbol{x}})^{\mathrm{T}} \boldsymbol{p} = 0, \qquad & j = 1, \ldots, \ell. \end{aligned}$$

Describe briefly how this LP problem could be used to devise an iterative method for the problem (1).

(2p) b) Prove that if the problem (1) is convex then each one of its KKT points is globally optimal.

(3p) Question 6

(linear programming geometry)

Consider the non-empty polyhedron $X = \{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b};\ \boldsymbol{x} \geq \boldsymbol{0}^n \}$, where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. We say that a linear inequality of the form $\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x} \leq d_0$ is *redundant relative to the set* X if $X \cap \{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{d}^{\mathrm{T}}\boldsymbol{x} \leq d_0 \} = X$.

Show the following:

$$\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x} \leq d_0 \text{ is redundant relative to the set } X$$

$$\Longleftrightarrow$$

$$\exists \boldsymbol{\mu} \geq \boldsymbol{0}^m \text{ with } \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu} \geq \boldsymbol{d} \text{ and } \boldsymbol{b}^{\mathrm{T}}\boldsymbol{\mu} \leq d_0.$$

Hint: Use LP duality in one of the directions.

[This result implies a natural procedure for detecting unnecessary constraints in LP problems.]

Question 7

(Lagrangian duality)

Consider the following optimization (linear) problem:

$$\begin{aligned} \text{minimize} \quad & f(x, y) = x - 0.5y, \\ \text{subject to} \quad & -x + y \leq -1, \\ & -2x + y \leq -2, \\ & (x, y) \in \mathbb{R}^2_+. \end{aligned} \tag{1}$$

(2p) a) Show that the problem satisfies Slater's constraint qualification. Derive the Lagrangian dual problem corresponding to the Lagrangian relaxation of the two linear inequality constraints, and show that its set of optimal solutions is convex and bounded.

(1p) b) Calculate the set of subgradients of the Lagrangian dual function at the dual points $(1/4, 1/3)^{\mathrm{T}}$ and $(1, 0)^{\mathrm{T}}$.

Good luck!