Chalmers/GU
Mathematics

EXAM

TMA947/MAN280
APPLIED OPTIMIZATION

Date:	05–03–14
Time:	House V, morning
Aids:	Text memory-less calculator
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Niclas Andréasson (0740-459022)
Result announced:	05–03–29
	Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid methods and theory.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the Simplex method and sensitivity analysis in linear programming)

Consider the following linear program:

$$\begin{array}{llrrrrl} \text{minimize} & z = & -2x_1 & +(5+c)x_2 & -2x_3 & & \\ \text{subject to} & & x_1 & -3x_2 & +4x_3 & \leq & 2, \\ & & -3x_1 & +x_2 & +3x_3 & \geq & -3+b, \\ & & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$

(1p) a) Let $b = c = 0$. Show that the basis $\boldsymbol{x}_B = (x_1, x_3)^{\mathrm{T}}$ corresponds to the unique optimal solution.

(1p) b) Let $c = 0$ and find all values of b such that $\boldsymbol{x}_B = (x_1, x_3)^{\mathrm{T}}$ is optimal. Then, let $b = 0$ and find all values of c such that $\boldsymbol{x}_B = (x_1, x_3)^{\mathrm{T}}$ is optimal.

(1p) c) Let $c = 0$ and $b = -4$. The basis $\boldsymbol{x}_B = (x_1, x_3)^{\mathrm{T}}$ is then primal infeasible but dual feasible. Starting with this basis, use the dual simplex method to find an optimal solution.

In order to calculate necessary matrix inverses the following identity is useful:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(3p)
Question 2

(Newton's method)

Consider the unconstrained problem to

$$\begin{array}{ll} \text{minimize} & f(x, y) := \dfrac{x^3}{6} + \dfrac{y^2}{2} \\ \text{subject to} & (x, y)^{\mathrm{T}} \in \mathbb{R}^2. \end{array}$$

Let $(x_0, y_0)^{\mathrm{T}}$ be the starting point and assume that $x_0 \neq 0$. Show that if Newton's method with a unit step length is applied to this problem, then for $k = 1, 2, \ldots,$ the kth iteration point is given by

$$(x_k, y_k)^{\mathrm{T}} = \left(\frac{x_0}{2^k}, 0 \right)^{\mathrm{T}}.$$

Will the method converge to an optimal solution?

Question 3

(Farkas' Lemma and other theorems of the alternative)

Farkas' Lemma can be stated as follows:

Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ *and* $\boldsymbol{b} \in \mathbb{R}^m$*. Then, exactly one of the systems*

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}^n, \end{aligned} \tag{I}$$

and

$$\begin{aligned} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} &\leq \boldsymbol{0}^n, \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} &> 0, \end{aligned} \tag{II}$$

has a feasible solution, and the other system is inconsistent.

(2p) a) Prove Farkas' Lemma.

(1p) b) Consider the following version of Farkas' Lemma:

Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ *and* $\boldsymbol{b} \in \mathbb{R}^m$*. Then, exactly one of the systems*

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &\geq \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}^n, \end{aligned} \tag{I'}$$

and

$$\begin{aligned} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} &\leq \boldsymbol{0}^n, \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} &> 0, \\ \boldsymbol{y} &\geq \boldsymbol{0}^m \end{aligned} \tag{II'}$$

has a feasible solution, and the other system is inconsistent.

Prove this result by utilizing Farkas' Lemma.

[*Note:* The latter result is one of many versions of Farkas' Lemma; they are often referred to as *Theorems of the alternative.*]

Question 4

(optimality)

Consider the problem to

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(\boldsymbol{x}) := x_1 \log x_2 + \mathrm{e}^{x_1}, \\ \text{subject to} \quad & 1 \leq x_j \leq 2, \qquad j = 1, 2. \end{aligned}$$

Suppose that you have downloaded a MATLAB based solver on the web, and have run it with default settings on this problem. It prints out:

`Optimal solution:` $x_1^* = x_2^* = 1$.

The main question that concerns us is whether the solver has found an optimal solution.

(1p) a) Investigate whether $\boldsymbol{x}^*$ satisfies the KKT conditions or not. Is Abadie's CQ fulfilled for this problem?

(1p) b) Investigate whether the problem is convex or not. As a consequence of your answer to this question, can you draw any conclusions regarding the global optimality of $\boldsymbol{x}^*$? If not, can you verify that $\boldsymbol{x}^*$ is globally optimal by any other means?

(1p) c) Consider a general problem:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(\boldsymbol{x}), \\ \text{subject to} \quad & g_i(\boldsymbol{x}) \leq 0, \qquad i = 1, \dots, m, \\ & h_j(\boldsymbol{x}) = 0, \qquad j = 1, \dots, \ell, \end{aligned}$$

where the functions f, g_i $(i = 1, \dots, m)$, and h_j $(j = 1, \dots, \ell)$ are continuously differentiable on $\mathbb{R}^n$.

Suppose that your solver has solved an instance in $\mathbb{R}^3$ of this general problem and reports:

$\boldsymbol{x}^* = (1, 2.3, 4.5)^{\mathrm{T}}$ `is a KKT point.`

In your investigation of your problem you have noticed that no familiar CQ is fulfilled, and yet you know that the problem is convex. What conclusions can you draw regarding the optimality of $\boldsymbol{x}^*$?

(3p) Question 5

(the variational inequality)

Consider the problem to

$$\begin{aligned}
\text{maximize} \quad & f(\boldsymbol{x}) := \left(\sum_{i=1}^{n} c_i x_i \right) \left(\sum_{i=1}^{n} \frac{1}{c_i} x_i \right) \\
\text{subject to} \quad & \sum_{i=1}^{n} x_i = 1, \\
& x_i \geq 0, \quad i = 1, \dots, n,
\end{aligned}$$

where $0 < c_1 < c_2 \leq \cdots \leq c_{n-1} < c_n$, and $n \geq 3$. Use the variational inequality to find an optimal solution. Show that the solution obtained is unique.

Hint: Recall that for a C^1 function f minimized over a closed and convex set $S \subset \mathbb{R}^n$ the variational inequality states that

$$\nabla f(\boldsymbol{x}^*)^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}^*) \geq 0, \qquad \boldsymbol{x} \in S,$$

and that it characterizes $\boldsymbol{x}^*$ as a stationary point in the problem at hand.

Assume that $x_i > 0$ for some $i = 2, \dots, n-1$, and use the variational inequality to derive a contradiction.

(3p) Question 6

(modelling)

You are asked to plan a one week (7 days) golf trip. The number of participants is 20. Each day they will play in groups of 4 (that is, in total there are 5 groups each day). If two players, say A and B, belong to the same group some of the days (perhaps more than one day) then we say that there has been a *meeting* between A and B.

Your task is to formulate an *integer linear program* for finding a schedule that maximizes the total number of (unique) meetings during the week. (If the two players A and B meet more than once their meeting should still not be counted more than once.) An ideal schedule would, of course, be such that each pair of players meet at least once during the week; this may, however, not be possible.

Question 7

(convex analysis)

Consider the function $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that f is convex but not differentiable. We are interested in characterizing a global minimum by some kind of derivative condition.

According to convex analysis, the function f is characterized by a condition similar to that in the C^1 convex case, namely that the epigraph of f is supported by a Taylor-like expansion: *f is convex on $\mathbb{R}^n$ if and only if it holds that for every $\boldsymbol{x} \in \mathbb{R}^n$ there exists at least one vector $\boldsymbol{g} \in \mathbb{R}^n$ for which*

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}), \qquad \forall \boldsymbol{y} \in \mathbb{R}^n. \tag{1}$$

In the C^1 convex case, the vector $\boldsymbol{g} \equiv \nabla f(\boldsymbol{x})$, and we refer to the vector $\boldsymbol{g}$ as a *subgradient* to f at $\boldsymbol{x}$. It holds that the function f is differentiable at $\boldsymbol{x}$ if and only if this vector is unique, in which case the above inequality reduces to the classic C^1 case for the given value of $\boldsymbol{x}$. Further, the set of vectors $\boldsymbol{g}$ satisfying the above inequality,

$$\partial f(\boldsymbol{x}) := \{\, \boldsymbol{g} \in \mathbb{R}^n \mid f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \boldsymbol{g}^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}), \quad \forall \boldsymbol{y} \in \mathbb{R}^n \,\}, \tag{2}$$

is referred to as the *subdifferential* of f at $\boldsymbol{x}$. This set is nonempty, convex and compact for every $\boldsymbol{x} \in \mathbb{R}^n$. Last, we note that the directional derivative of f in the direction of $\boldsymbol{p} \in \mathbb{R}^n$ in the C^1 case equals $f'(\boldsymbol{x}; \boldsymbol{p}) = \nabla f(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{p}$, while it is in the current case extended to the following:

$$f'(\boldsymbol{x}; \boldsymbol{p}) = \underset{g \in \partial f(x)}{\operatorname{maximum}}\ \boldsymbol{g}^{\mathrm{T}}\boldsymbol{p}. \tag{3}$$

This result follows from an equivalent way of expressing $\partial f(\boldsymbol{x})$, based on directional derivatives:

$$\partial f(\boldsymbol{x}) := \{\, \boldsymbol{g} \in \mathbb{R}^n \mid \boldsymbol{g}^{\mathrm{T}}\boldsymbol{p} \leq f'(\boldsymbol{x}; \boldsymbol{p}), \quad \forall \boldsymbol{y} \in \mathbb{R}^n \,\}. \tag{4}$$

(Recall that the original definition is that $f'(\boldsymbol{x}; \boldsymbol{p}) = \lim_{\alpha \to 0_+}[f(\boldsymbol{x} + \alpha\boldsymbol{p}) - f(\boldsymbol{x})]/\alpha$.)

(2p) a) Your first task is to prove the following extension of the first-order optimality conditions in the C^1 convex case to the C^0 convex case:

The following three statements are equivalent:

1. *f is globally minimized at $\boldsymbol{x}^* \in \mathbb{R}^n$;*
2. *$\mathbf{0}^n \in \partial f(\boldsymbol{x}^*)$;*
3. *$f'(\boldsymbol{x}^*; \boldsymbol{p}) \geq 0$ for all $\boldsymbol{p} \in \mathbb{R}^n$.*

(1p) b) Recall the definition of a direction of descent: *the vector $\boldsymbol{p} \in \mathbb{R}^n$ is a* direction of descent *with respect to f at $\boldsymbol{x}$ if*

$$\exists \delta > 0 \;\; \textit{such that} \;\; f(\boldsymbol{x} + \alpha \boldsymbol{p}) < f(\boldsymbol{x}) \;\; \textit{for every} \;\; \alpha \in (0, \delta].$$

According to the result in a), for convex functions this implies that $f'(\boldsymbol{x}; \boldsymbol{p}) < 0$. Does this result hold true also for non-convex functions?

Good luck!