Chalmers/GU
Mathematics

**EXAM**

# TMA947/MAN280
# APPLIED OPTIMIZATION

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| **Date:** | 06–03–06 |
| **Time:** | House V, morning |
| **Aids:** | Text memory-less calculator |
| **Number of questions:** | 7; passed on one question requires 2 points of 3.<br>Questions are *not* numbered by difficulty.<br>To pass requires 10 points and three passed questions. |
| **Examiner:** | Michael Patriksson |
| **Teacher on duty:** | Peter Lindroth (0762-721860) |
| **Result announced:** | 06–03–24<br>Short answers are also given at the end of<br>the exam on the notice board for optimization<br>in the MV building. |

## Exam instructions

**When you answer the questions**

*Use generally valid methods and theory.*
*State your methodology carefully.*

*Only write on one page of each sheet. Do not use a red pen.*
*Do not answer more than one question per page.*

**At the end of the exam**

*Sort your solutions by the order of the questions.*
*Mark on the cover the questions you have answered.*
*Count the number of sheets you hand in and fill in the number on the cover.*

## Question 1

(the Simplex method)

Consider the following linear program:

$$\begin{array}{lrrrl} \text{minimize} & z = -x_1 & +2x_2 & +x_3, & \\ \text{subject to} & 2x_1 & +x_2 & -x_3 & \leq 7, \\ & -x_1 & +2x_2 & +3x_3 & \geq 3, \\ & x_1, & x_2, & x_3 & \geq 0. \end{array}$$

**(2p)** a) Solve this problem by using phase I and phase II of the simplex method. [*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

**(1p)** b) Using the information from your solution to the problem above, state whether there is an optimal solution to the corresponding dual problem or not. Motivate! Do not make any additional calculations!

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**(3p)** 

## Question 2

(The Karush–Kuhn–Tucker conditions)

Consider the problem to

$$\begin{aligned} &\text{minimize } f(\boldsymbol{x}), \\ &\text{subject to } g_i(\boldsymbol{x}) \leq 0, \qquad i = 1, \dots, m, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$, are given differentiable functions. Prove the following result.

*Assume that at a given point $\boldsymbol{x}^* \in S$ Abadie's constraint qualification holds. If $\boldsymbol{x}^* \in S$ is a local minimum of $f$ over $S$ then there exists a vector $\boldsymbol{\mu} \in \mathbb{R}^m$ such that*

$$\nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(\boldsymbol{x}^*) = \mathbf{0}^n, \tag{2a}$$

$$\mu_i g_i(\boldsymbol{x}^*) = 0, \qquad i = 1, \dots, m, \tag{2b}$$

$$\boldsymbol{\mu} \geq \mathbf{0}^m. \tag{2c}$$

*In other words,*

$$\left.\begin{array}{r} \boldsymbol{x}^* \textit{ local minimum of } f \textit{ over } S \\ \textit{Abadie's CQ holds at } \boldsymbol{x}^* \end{array}\right\} \implies \exists \boldsymbol{\mu} \in \mathbb{R}^m : \; (2) \textit{ holds.}$$

You may refer (simply by giving its name) to any additional theorems that you wish to utilize in your proof.

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## Question 3

(true or false claims in optimization)

For each of the following three claims, your task is to decide whether it is true or false. Motivate your answers.

**(1p)** a) The vector $\boldsymbol{p} := (1, -1)^{\mathrm{T}}$ is a descent direction for $f(\boldsymbol{x}) := (x_1 + x_2^2)^2$ at $\boldsymbol{x} = (1, 0)^{\mathrm{T}}$.

**(1p)** b) The vector $\boldsymbol{x} := (0, 1)^{\mathrm{T}}$ satisfies the KKT conditions for the problem to

$$\begin{aligned} \underset{x \in \mathbb{R}^2}{\text{maximize}} \quad & f(x) := \ln(x_1 + x_2), \\ \text{subject to} \quad & x_1 + 2x_2 \leq 2, \\ & x_2 \geq 0. \end{aligned}$$

**(1p)** c) Consider the linear program defined by

$$\begin{aligned} f(\boldsymbol{b}) := \underset{x \in \mathbb{R}^n}{\text{maximum}} \quad & \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \qquad (1) \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

where $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, and $\boldsymbol{b} \in \mathbb{R}^m$, and let $\boldsymbol{y} \in \mathbb{R}^m$ be the vector of dual variables associated with the constraints (1). Assume that for the given value of $\boldsymbol{b}$ there exists a non-degenerate primal optimal BFS. Hence, there exists a unique optimal dual solution $\boldsymbol{y}^* \in \mathbb{R}^m$. Suppose that $y_1^* > 0$ and let $\overline{\boldsymbol{b}} \in \mathbb{R}^m$ be such that $b_1 < \overline{b}_1 < b_1 + \varepsilon$ and $\overline{b}_i = b_i$, $i = 2, \ldots, m$. For small values of $\varepsilon > 0$ it holds that $f(\overline{\boldsymbol{b}}) > f(\boldsymbol{b})$.

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## Question 4

(nonlinear programming)

Consider the nonlinear program to

$$\begin{aligned} &\text{minimize } f(\boldsymbol{x}), \\ &\text{subject to } h_j(\boldsymbol{x}) = 0, \qquad j = 1, \dots, \ell, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \dots, \ell$, are given differentiable functions.

**(1p)** a) When is the problem (1) a convex problem?

**(2p)** b) Suppose that having utilized a nonlinear programming solver the printout is a vector $\boldsymbol{x}^*$. Suppose that the solver prints that $\boldsymbol{x}^*$ is a local minimum, and you can confirm (based on your experience with the problem) that this is true. But you can also confirm that the vector $\boldsymbol{\lambda}^* \in \mathbb{R}^\ell$ which is also printed does not satisfy the KKT conditions together with $\boldsymbol{x}^*$. Explain how this can be the case.

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## (3p) Question 5

(modelling)

Sudoku is an old Japanese number puzzle that has grown in popularity recently. Some claim that it deals with numbers but has nothing to do with mathematics; however, this is obviously not true. The rules for a Sudoku puzzle are simple. We start with a $9 \times 9$-matrix $\boldsymbol{A}$ which is partitioned into nine $3 \times 3$-matrices $\boldsymbol{A}_1, \dots, \boldsymbol{A}_9$. Certain entries of $\boldsymbol{A}$ contain numbers from the set $\{1, \dots, 9\}$. An example of such a pre-assignment is shown below.

The Sudoku problem is now to:
Fill the empty entries of the matrix $\boldsymbol{A}$ such that
– each row of $\boldsymbol{A}$
– each column of $\boldsymbol{A}$
– each $3 \times 3$-submatrix $\boldsymbol{A}_1, \dots, \boldsymbol{A}_9$
contains every number of $\{1, \dots, 9\}$ exactly once.

Your task is to model an *integer linear program*, that is, a program that if all integral requirements would be relaxed is a linear program (the objective function and the constraints are all described by linear functions). This model shall, given

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| | | | | 4 | 6 | | | |
| | | 7 | | | | | | 2 |
| | 1 | | | | 3 | | 6 | 9 |
| | 4 | | 6 | | 9 | | 5 | |
| 9 | 8 | | 2 | | | | 7 | |
| 2 | | | | | | 9 | | |
| | | | 8 | 1 | | | | |
| 6 | | | | | | 4 | | |

Figure 1: Example of a Sudoku problem.

a set of pre-assigned numbers, give a feasible solution to the Sudoku problem, and if this is not possible it should give a solution that minimizes the number of pre-assignments that cannot be fulfilled for the Sudoku problem to be solved.

*Hint:* Assume that the pre-assigned entries are collected in a set $N$ whose size is $n$ and is of the form $N = \{\, (i,j,k) \mid \text{the number } k \text{ is pre-assigned to row } i,\ \text{column } j \,\}$.

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## Question 6

(definitions)

**(1p)** a) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a given function. Define the property that $f$ is a *strictly convex* function on $\mathbb{R}^n$.

**(1p)** b) Define the *termination criterion* that determines an *unbounded* LP solution.

**(1p)** c) Define the concept of an *extreme point* of a convex set.

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## Question 7

(Lagrangian duality for equality constrained problems)

Consider the primal problem

$$\begin{aligned} f^* := \operatorname*{infimum}_{x}\ & f(\boldsymbol{x}), && (1a)\\ \text{subject to } & \boldsymbol{x} \in X, && (1b)\\ & h_j(\boldsymbol{x}) = 0, \quad j = 1, \dots, \ell, && (1c) \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$ $(j = 1, \dots, \ell)$ are continuous functions, and $X \subseteq \mathbb{R}^n$ is closed.

For an arbitrary vector $\boldsymbol{\lambda} \in \mathbb{R}^\ell$, we define the *Lagrange function*

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) := f(\boldsymbol{x}) + \sum_{j=1}^{\ell} \lambda_j h_j(\boldsymbol{x}) = f(\boldsymbol{x}) + \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{h}(\boldsymbol{x}). \tag{2}$$

Let

$$q(\boldsymbol{\lambda}) := \operatorname*{infimum}_{x \in X}\ L(\boldsymbol{x}, \boldsymbol{\lambda}) \tag{3}$$

be the *Lagrangian dual function*, defined by the infimum value of the Lagrange function over $X$ when we have Lagrangian relaxed the explicit constraints with multiplier values $\boldsymbol{\lambda}$; the *Lagrangian dual problem* is to

$$\begin{aligned} \operatorname*{maximize}_{\boldsymbol{\lambda}}\ & q(\boldsymbol{\lambda}), && (4a)\\ \text{subject to } & \boldsymbol{\lambda} \in \mathbb{R}^\ell. && (4b) \end{aligned}$$

We suppose that $X$ is non-empty, closed and bounded so that the "infimum" can be replaced by "minimum" in (1), (3), and both the primal and dual objective functions in (1) and (4) attain their optimal values. Moreover, then $q$ is finite, continuous, and concave on the whole set $\mathbb{R}^\ell$, so that (4) is a "well-behaved" convex problem.

**(1p)** a) Let $\bar{\boldsymbol{\lambda}} \in \mathbb{R}^\ell$. Define a subgradient, $\bar{\gamma}$, to $q$ at $\bar{\boldsymbol{\lambda}}$. Also define the entire subdifferential, $\partial q(\bar{\boldsymbol{\lambda}})$ to $q$ at $\bar{\boldsymbol{\lambda}}$.

**(1p)** b) Let $\boldsymbol{x}(\bar{\boldsymbol{\lambda}})$ be an optimal solution to the Lagrangian subproblem defining the value of $q(\bar{\boldsymbol{\lambda}})$. Show that the vector $\boldsymbol{h}(\boldsymbol{x}(\bar{\boldsymbol{\lambda}})) \in \partial q(\bar{\boldsymbol{\lambda}})$; that is, show that the vector $\boldsymbol{h}(\boldsymbol{x}(\bar{\boldsymbol{\lambda}}))$ of constraint function values at the subproblem solution $\boldsymbol{x}(\bar{\boldsymbol{\lambda}})$ is a subgradient of $q$ at $\bar{\boldsymbol{\lambda}}$.

**(1p)** c) Show that if, for some $\boldsymbol{\lambda}^* \in \mathbb{R}^\ell$, it holds that $\mathbf{0}^\ell \in \partial q(\boldsymbol{\lambda}^*)$ then $\boldsymbol{\lambda}^*$ is globally optimal in the Lagrangian dual problem (4).

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*Good luck!*