EXERCISE 2: CONVEXITY

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EXERCISE 1 (convex functions). Suppose that the function $g : \mathbb{R}^n \to \mathbb{R}$ is convex. Show the following:

a) The function
$$f(\boldsymbol{x}) := -\ln(-g(\boldsymbol{x}))$$
is convex on the set
$$S := \{\, \boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) < 0 \,\}.$$

b) The function
$$f(\boldsymbol{x}) := 1/\ln(-g(\boldsymbol{x}))$$
is convex on the set
$$S := \{\, \boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) < -1 \,\}.$$

□

EXERCISE 2 (convex problem). Suppose that $g : \mathbb{R}^n \to \mathbb{R}$ is convex and $\boldsymbol{d} \in \mathbb{R}^n$. Is the problem to

$$\begin{aligned} \text{maximize} \quad & -(x_1^2 + \cdots + x_n^2) \\ \text{subject to} \quad & -\frac{1}{\ln(-g(\boldsymbol{x}))} \geq 0, \\ & \boldsymbol{d}^{\mathrm{T}}\boldsymbol{x} = 2, \\ & g(\boldsymbol{x}) \leq -2, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

convex? □

EXERCISE 3 (convex problem). Is the problem to

$$\begin{aligned} \text{minimize} \quad & x_1 \ln x_1 \\ \text{subject to} \quad & x_1^2 + x_2^2 \geq 1, \\ & 2x_1 \geq 1, \\ & (x_1 - 2)^2 + (x_2 - 2)^2 \leq 1, \\ & \boldsymbol{x} \geq \boldsymbol{0}^2, \end{aligned}$$

convex? □

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EXERCISE 4 (convexity of polyhedra). Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{b}$ an $m \times 1$ vector. Show that the polyhedron

$$P = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \,\},$$

is a convex set. □

EXERCISE 5 (application of Farkas' Lemma). In a paper submitted for publication in an operations research journal, the author considered the set

$$P = \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{R}^{n+m} \;\middle|\; \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} \geq \boldsymbol{c}; \quad \boldsymbol{x} \geq \boldsymbol{0}^n; \quad \boldsymbol{y} \geq \boldsymbol{0}^m \right\},$$

where $\boldsymbol{A}$ is an $m \times n$ matrix, $\boldsymbol{B}$ a positive semi-definite $m \times m$ matrix and $\boldsymbol{c} \in \mathbb{R}^m$. The author explicitly assumed that the set P is compact in $\mathbb{R}^{n+m}$. A reviewer of the paper pointed out that the only compact set of the above form is the empty set. Prove the reviewer's assertion. □

EXERCISE 6 (extreme points). Consider the polyhedron P defined by

$$\begin{aligned} x_1 + x_2 &\leq 2, \\ x_2 &\leq 1, \\ x_3 &\leq 2, \\ x_2 + x_3 &\leq 2. \end{aligned}$$

a) Is $\boldsymbol{x}^1 = (1, 1, 0)^{\mathrm{T}}$ an extreme point to P?
b) Is $\boldsymbol{x}^2 = (1, 1, 1)^{\mathrm{T}}$ an extreme point to P?

□

EXERCISE 7 (existence of extreme points in LPs). Let $\boldsymbol{A}$ be an $m \times n$ matrix such that $\operatorname{rank} \boldsymbol{A} = m$ and $\boldsymbol{b}$ an $m \times 1$ vector. Show that if the polyhedron

$$P = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}; \quad \boldsymbol{x} \geq \boldsymbol{0}^n \,\}$$

has a feasible solution, then it has an extreme point. □

EXERCISE 8 (separation). Show that each closed convex set A in $\mathbb{R}^n$ is the intersection of all the closed halfspaces in $\mathbb{R}^n$ containing A, that is, a set of the form

$$B = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{x} \leq b_i, \quad i \in \mathcal{K} \,\},$$

where $b_i \in \mathbb{R}$ and $\boldsymbol{a}_i \in \mathbb{R}^n$ for each $i \in \mathcal{K}$. Is this a polyhedron, and hence, is every closed convex set a polyhedron? □