EXERCISE 3: OPTIMALITY CONDITIONS FOR UNCONSTRAINTED AND CONVEXLY CONSTRAINTED OPTIMIZATION

NICLAS ANDRÉASSON

Exercise 1. Find the rectangular parallelepiped of unit volume that has the minimum surface area. □

Exercise 2. Consider the parametric minimization problem to

$$\underset{x_1, x_2}{\text{minimize}} \ \frac{3}{2}(x_1^2 + x_2^2) + (1+a)x_1x_2 - (x_1 + x_2) + b, \tag{1}$$

where a and b are some unknown real-valued parameters.

Find all possible values of a and b such that the problem (1) possesses a unique globally optimal solution. Write down this solution (in terms of the parameters a and b). □

Exercise 3. Let $\boldsymbol{A}$ be a symmetric $n \times n$ matrix. For $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{x} \neq \boldsymbol{0}^n$, consider the function

$$\rho(\boldsymbol{x}) = \frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}},$$

and the related optimization problem to

$$\underset{x \neq 0^n}{\text{minimize}} \ \rho(\boldsymbol{x}). \tag{2}$$

Determine all the stationary points as well as the global minima in the minimization problem (2). □

Exercise 4 (the variational inequality). Among all rectangles contained in a given circle, show that the one that has maximal area must be a square. □

Exercise 5 (the variational inequality). Consider the positive orthant

$$S = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq \boldsymbol{0}^n \,\}.$$

Derive necessary optimality conditions for the problem to

$$\begin{aligned} &\text{minimize} && f(\boldsymbol{x}) \\ &\text{subject to} && \boldsymbol{x} \in S, \end{aligned}$$

where $f \in C^1$. □

Date: January 26, 2005.

EXERCISE 6 (the variational inequality). Consider the problem to

$$\begin{aligned} \text{maximize} \quad & x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \\ \text{subject to} \quad & \sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, \quad i = 1, \ldots, n, \end{aligned}$$

where a_i are given positive scalars. Find a global maximum and show that it is unique. □