EXERCISE 7: THE GEOMETRY OF LINEAR PROGRAMMING

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EXERCISE 1 (LP modelling). Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Formulate the following problem as a linear programming problem:

$$\begin{aligned} \text{minimize} \quad & \|\boldsymbol{Ax}-\boldsymbol{b}\|_1 := \sum_{i=1}^{m} |(\boldsymbol{Ax}-\boldsymbol{b})_i| \\ \text{subject to} \quad & \|\boldsymbol{x}\|_\infty := \max_{i=1,\dots,n} |x_i| \leq 1. \end{aligned}$$

□

EXERCISE 2 (LP modelling). Consider the sets $V = \{\boldsymbol{v}^1, \dots, \boldsymbol{v}^k\} \subset \mathbb{R}^n$ and $W = \{\boldsymbol{w}^1, \dots, \boldsymbol{w}^l\} \subset \mathbb{R}^n$. Formulate the following problem as a linear programming problem: Construct, if possible, a sphere that separates the sets V and W, that is, find a center $\boldsymbol{x}^c \in \mathbb{R}^n$ and a radius $R \geq 0$ such that

$$\begin{aligned} \|\boldsymbol{v}-\boldsymbol{x}^c\|_2 &\leq R, \quad \text{for all } \boldsymbol{v} \in V, \\ \|\boldsymbol{w}-\boldsymbol{x}^c\|_2 &\geq R, \quad \text{for all } \boldsymbol{w} \in W. \end{aligned}$$

□

EXERCISE 3 (linear-fractional programming). Consider the linear-fractional program

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) = (\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}+\alpha)/(\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}+\beta) \\ \text{subject to} \quad & \boldsymbol{Ax} \leq \boldsymbol{b}, \end{aligned} \tag{1}$$

where $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, and $\boldsymbol{b} \in \mathbb{R}^m$. Further, assume that the polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{Ax} \leq \boldsymbol{b}\}$ is bounded and that $\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}+\beta > 0$ for all $\boldsymbol{x} \in P$. Show that (1) can be solved by solving the linear program

$$\begin{aligned} \text{minimize} \quad & g(\boldsymbol{y}, z) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} + \alpha z \\ \text{subject to} \quad & \boldsymbol{Ay} - z\boldsymbol{b} \leq \boldsymbol{0}^m, \\ & \boldsymbol{d}^{\mathrm{T}}\boldsymbol{y} + \beta z = 1, \\ & z \geq 0. \end{aligned} \tag{2}$$

□

EXERCISE 4 (standard form). Transform the linear program

$$\begin{aligned} \text{minimize} \quad z = \;& x_1 - 5x_2 - 7x_3 \\ \text{subject to} \quad & 5x_1 - 2x_2 + 6x_3 \geq 5, && (1) \\ & 3x_1 + 4x_2 - 9x_3 = 3, && (2) \\ & 7x_1 + 3x_2 + 5x_3 \leq 9, && (3) \\ & x_1 \geq -2, \end{aligned}$$

into standard form! □

Date: February 16, 2005.

EXERCISE 5 (standard form). Consider the linear program

$$\begin{array}{lrl} \text{minimize} & z = 5x_1 + 3x_2 - 7x_3 & \\ \text{subject to} & 2x_1 + 4x_2 + 6x_3 \phantom{{}+x_4} & = 11, \\ & 3x_1 - 5x_2 + 3x_3 + x_4 & = 11, \\ & x_1, \quad x_2, \quad\quad\quad x_4 & \geq 0. \end{array}$$

(a) Show how to transform this problem into standard form by eliminating the unrestricted variable x_3.

(b) Why cannot this technique be used to eliminate variables with non-negativity restrictions?

□

EXERCISE 6 (basic feasible solutions). Suppose that a linear program includes a free variable x_j. When transforming this problem into standard form, x_j is replaced by

$$\begin{aligned} x_j &= x_j^+ - x_j^-, \\ x_j^+, x_j^- &\geq 0. \end{aligned}$$

Show that no basic feasible solution can include both x_j^+ and x_j^- as non-zero basic variables. □