EXERCISE 9-10: LINEAR PROGRAMMING DUALITY

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Rules for the construction of the LP dual problem

primal/dual constraint		**dual/primal variable**
canonical inequality	$\Longleftrightarrow$	≥ 0
non-canonical inequality	$\Longleftrightarrow$	≤ 0
equality	$\Longleftrightarrow$	unrestricted

EXERCISE 1 (constructing the LP dual). Consider the linear program

$$\begin{array}{lrl} \text{maximize} & z = 6x_1 - 3x_2 - 2x_3 + 5x_4 & \\ \text{subject to} & 4x_1 + 3x_2 - 8x_3 + 7x_4 = 11, & (1) \\ & 3x_1 + 2x_2 + 7x_3 + 6x_4 \geq 23, & (2) \\ & 7x_1 + 4x_2 + 3x_3 + 2x_4 \leq 12, & (3) \\ & x_1, \quad x_2 \qquad\qquad \geq 0, & \\ & x_3 \qquad \leq 0, & \\ & x_4 \quad \text{free}. & \end{array}$$

Construct the linear programming dual. □

EXERCISE 2 (constructing the LP dual). Consider the linear program

$$\begin{array}{ll} \text{minimize} & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}. \end{array}$$

(a) Construct the linear programming dual.
(b) Show that the dual problem is always feasible (independent of $\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{l}$, and $\boldsymbol{u}$).

□

EXERCISE 3 (constructing an optimal dual solution from an optimal BFS). Consider the linear program in standard form

$$\begin{array}{ll} \text{minimize} & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n. \end{array} \qquad \text{(P)}$$

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Assume that an optimal BFS, $\boldsymbol{x}^* = (\boldsymbol{x}_B^{\mathrm{T}}, \boldsymbol{x}_N^{\mathrm{T}})^{\mathrm{T}}$, is given by the partition $\boldsymbol{A} = (\boldsymbol{B}, \boldsymbol{N})$. Show that

$$\boldsymbol{y} = (\boldsymbol{B}^{-1})^{\mathrm{T}} \boldsymbol{c}_B$$

is an optimal solution to the LP dual problem. □

EXERCISE 4 (application of the Weak and Strong Duality Theorems). Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned} \tag{P}$$

and the perturbed problem to

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \tilde{\boldsymbol{b}}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n. \end{aligned} \tag{P'}$$

Show that if (P) has an optimal solution, then the perturbed problem (P') cannot be unbounded (independent of $\tilde{\boldsymbol{b}}$). □

EXERCISE 5 (application of the Weak and Strong Duality Theorems). Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}. \end{aligned} \tag{P}$$

Assume that the objective function vector $\boldsymbol{c}$ cannot be written as a linear combination of the rows of $\boldsymbol{A}$. Show that (P) cannot have an optimal solution. □

EXERCISE 6 (application of the Weak and Strong Duality Theorems). Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n. \end{aligned} \tag{P}$$

Construct a polyhedron that equals the set of optimal solutions to (P). □

EXERCISE 7 (application of the Weak and Strong Duality Theorems). Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n. \end{aligned} \tag{P}$$

Let $\boldsymbol{x}^*$ be an optimal solution to (P) with the optimal objective function value z^*, and let $\boldsymbol{y}^*$ be an optimal solution to the LP dual of (P). Show that

$$z^* = (\boldsymbol{y}^*)^{\mathrm{T}} \boldsymbol{A}\boldsymbol{x}^*.$$

□

EXERCISE 8 (linear programming primal-dual optimality conditions). Consider the linear program

$$
\begin{array}{llrrrrrl}
\text{maximize} & z = & & -4x_2 & +3x_3 & +2x_4 & -8x_5 & \\
\text{subject to} & & 3x_1 & +x_2 & +2x_3 & +x_4 & & = 3, \\
& & x_1 & -x_2 & & +x_4 & -x_5 & \geq 2, \\
& & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0.
\end{array}
$$

Find an optimal solution by using the LP primal-dual optimality conditions. □

EXERCISE 9 (linear programming primal-dual optimality conditions). Consider the linear program (the continuous knapsack problem)

$$
\begin{aligned}
\text{maximize} \quad & z = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{a}^{\mathrm{T}} \boldsymbol{x} \leq b, \\
& \boldsymbol{x} \leq \mathbf{1}^n, \\
& \boldsymbol{x} \geq \mathbf{0}^n,
\end{aligned} \tag{P}
$$

where $\boldsymbol{c} > \mathbf{0}^n$, $\boldsymbol{a} > \mathbf{0}^n$, $b > 0$, and

$$\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \cdots \geq \frac{c_n}{a_n}.$$

Show that the feasible solution $\boldsymbol{x}$ given by

$$x_j = 1,\ j = 1, \ldots, r-1, \quad x_r = \frac{b - \sum_{j=1}^{r-1} a_j}{a_r}, \quad x_j = 0,\ j = r+1, \ldots, n,$$

where r is such that $\sum_{j=1}^{r-1} a_j < b$ and $\sum_{j=1}^{r} a_j > b$, is an optimal solution. □

EXERCISE 10 (KKT versus LP primal-dual optimality conditions). Consider the linear program

$$
\begin{aligned}
\text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b},
\end{aligned} \tag{P}
$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{c} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$. Show that the KKT optimality conditions are equivalent to the LP primal-dual optimality conditions. □

EXERCISE 11 (Lagrangian primal-dual versus LP primal-dual). Consider the linear program

$$
\begin{aligned}
\text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}.
\end{aligned}
$$

Show that the Lagrangian primal-dual optimality conditions are equivalent to the LP primal-dual optimality conditions. □

EXERCISE 12 (sensitivity analysis: perturbations in the right-hand side). Consider the linear program

$$
\begin{array}{llrrrl}
\text{minimize} & z = & -x_1 & +2x_2 & +x_3 & \\
\text{subject to} & & 2x_1 & +x_2 & -x_3 & \leq 7, \\
& & -x_1 & +2x_2 & +3x_3 & \geq 3 + \delta, \\
& & x_1, & x_2, & x_3 & \geq 0.
\end{array}
$$

(a) Show that the basic solution that corresponds to the partition $\boldsymbol{x}_B = (x_1, x_3)^{\mathrm{T}}$ is an optimal solution to the problem when $\delta = 0$.

(b) Find the values of the perturbation $\delta \in \mathbb{R}$ such that the above BFS is optimal.

(c) Find an optimal solution when $\delta = -7$. □