

# Lecture 7: Linear programming models

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## Duality and optimality

- LP: linear objective, linear constraints
- LP problems can be given a “standard form”
- LP problems are convex problems with a CQ fulfilled (linear constraints  $\implies$  Abadie)
- Strong duality holds; Lagrangian dual same as LP dual
- KKT necessary and sufficient!
- Simplex method: always satisfies complementarity; always primal feasibility after finding the first feasible solution; searches for a dual feasible point

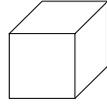
### Basic method and its foundations

- Know that if there exists an optimal solution, at least one of them is an extreme point (Thm. 8.10)
- Search only among extreme points
- Extreme points can be described in algebraic terms (Thm. 3.17). Find such a point
- Generate a descent direction; line search leads to the boundary! Choose direction so that the boundary point is an extreme point
- $\implies$  Move to a neighbouring extreme point such that the objective value improves—the Simplex method!
- Convergence *finite*

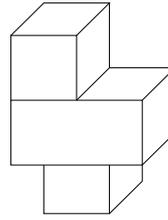
### An introductory problem—A DUPLO game

- A manufacturer produces two pieces of furniture: tables and chairs
- The production of furniture requires two different pieces of raw-material, large and small pieces
- One table is assembled from two pieces of each; one chair is assembled from one of the larger pieces and two of the smaller pieces

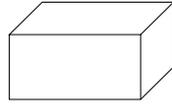
Small piece



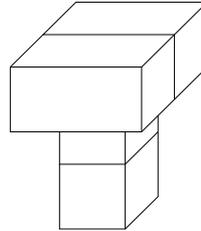
Chair,  $x_2$



Large piece



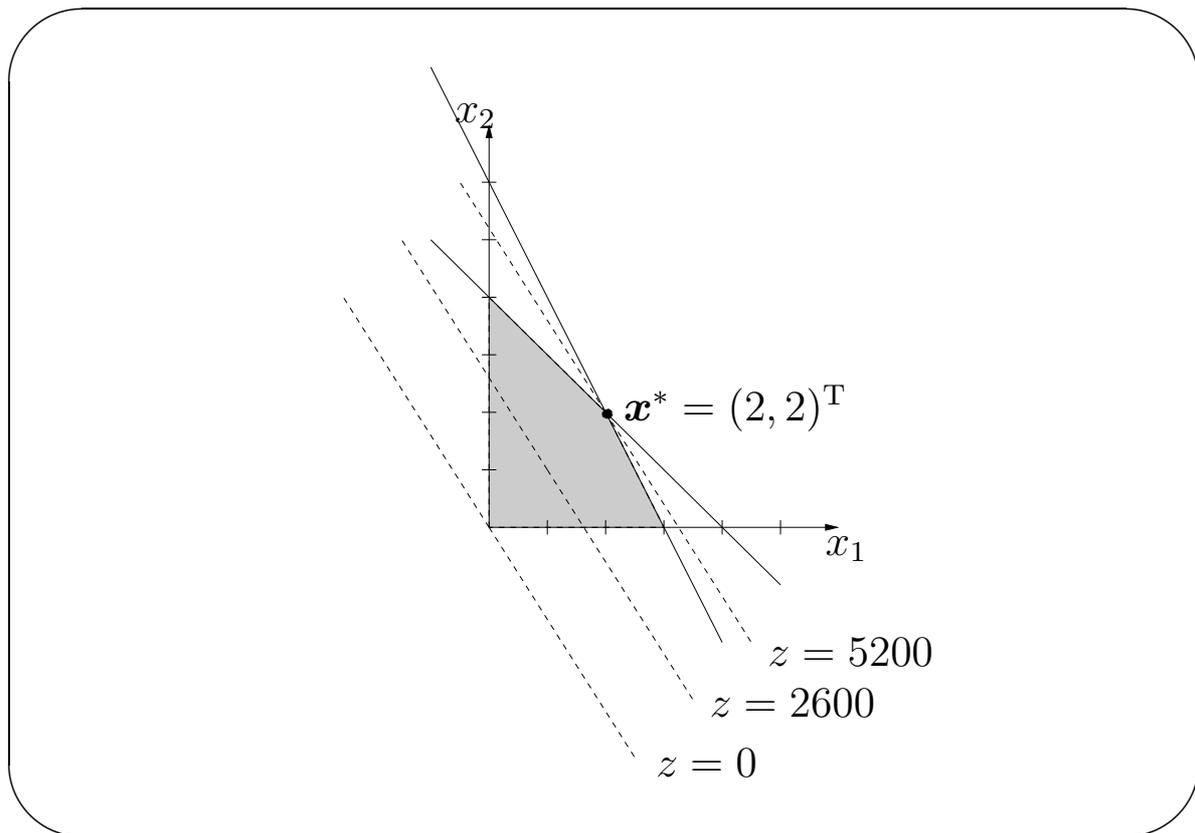
Table,  $x_1$



- Data: 6 large and 8 small pieces available. Selling a table gives 1600 SEK, a chair 1000 SEK
- Not trivial to choose an optimal production plan

- What is the problem and how do we solve it?
- Solution by (1) the DUPLO game; (2) graphically; (3) the Simplex method

$$\begin{aligned}
 &\text{maximize } z = 1600x_1 + 1000x_2 \\
 &\text{subject to } \quad \quad \quad 2x_1 \quad \quad + x_2 \leq 6, \\
 &\quad \quad \quad \quad \quad \quad 2x_1 \quad \quad + 2x_2 \leq 8, \\
 &\quad \quad \quad \quad \quad \quad x_1, \quad \quad x_2 \geq 0
 \end{aligned}$$



### Further topics

- Sensitivity analysis: What happens with  $z^*$ ,  $\mathbf{x}^*$  if ...?
- A dual problem: A manufacturer (Billy) produce book shelves with same raw material. Billy wish to expand their production; interested in acquiring our resources
- Two questions (with identical answers): (1) what is the lowest bid (price) for the total capacity at which we are willing to sell?; (2) what is the highest bid (price) that Billy are prepared to offer for the resources? The answer is a measure of the wealth of the company in terms of their resources (the shadow price)

### A dual problem

- To study the problem, we introduce the variables
  - $y_1$  = the price which Billy offers for each large piece,
  - $y_2$  = the price which Billy offers for each small piece,
  - $w$  = the total bid which Billy offers
- Example: Net income for a table is 1600 SEK; need to get at least price bid  $\mathbf{y}$  such that  $2y_1 + 2y_2 \geq 1600$

$$\begin{array}{ll} \text{minimize} & w = 6y_1 + 8y_2 \\ \text{subject to} & 2y_1 + 2y_2 \geq 1600, \\ & y_1 + 2y_2 \geq 1000, \\ & y_1, \quad y_2 \geq 0 \end{array}$$

- Why the sign?  $\mathbf{y}$  is a price!
- Optimal solution:  $\mathbf{y}^* = (600, 200)^T$ . The bid is  $w^* = 5200$  SEK
- Remarks: (1)  $z^* = w^*$ ! (Strong duality!) Our total income is the same as the value of our resources. (2) The price for a large piece equals its *shadow price*!

### Geometric $\iff$ Algebraic connections

- Must have equality constraints. Why? Inequalities cannot be manipulated while keeping the same solution set! Equalities can!
- Counter-example:

$$P = \{ \mathbf{x} \in \mathbb{R}_+^2 \mid x_1 + x_2 \geq 1; \quad 2x_1 + x_2 \leq 2 \}$$

- Good to know: Every polyhedron  $P$  can be described in the form

$$P = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}; \quad \mathbf{x} \geq \mathbf{0}^n \}$$

- We call this the *standard form*

- Slack variables: ( $\mathbf{x} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m, \mathbf{A} \in \mathbb{R}^{m \times n}$ )

$$\begin{array}{l} \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}^n \end{array} \iff \begin{array}{l} \mathbf{A}\mathbf{x} + \mathbf{I}^m \mathbf{s} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}^n, \\ \mathbf{s} \geq \mathbf{0}^m \end{array} \iff \begin{array}{l} [\mathbf{A} \ \mathbf{I}^m] \mathbf{v} = \mathbf{b}, \\ \mathbf{v} \geq \mathbf{0}^{n+m} \end{array}$$

- We assume even that  $\mathbf{b} \geq \mathbf{0}^m$ ; otherwise, multiply necessary rows by  $-1$
- Idea: We describe an extreme point through this characterization of the feasible set; we then prove that moving between “adjacent” extreme points is simple

- *Basic feasible solution* (Algebra)  $\iff$  *Extreme point* (Geometry)
- Note:  $\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{b} \implies$  Linear algebra
- $\mathbf{x} \geq \mathbf{0}^n : \mathbf{Ax} = \mathbf{b} \implies$  Polyhedra, convex analysis
- Sign restrictions? If  $x_j$  is free of sign, substitute it everywhere by

$$x_j = x_j^+ - x_j^-,$$

where  $x_j^+, x_j^- \geq 0$

### DUPLO example with slack variables

$$\begin{aligned} & \text{maximize } z = 1600x_1 + 1000x_2 \\ & \text{subject to} \quad 2x_1 + x_2 + s_1 = 6 \quad (1) \\ & \quad \quad \quad 2x_1 + 2x_2 + s_2 = 8 \quad (2) \\ & \quad \quad \quad x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

### Basic feasible solutions (BFS)

- Consider an LP in standard form:

$$\begin{aligned} & \text{minimize} && z = \mathbf{c}^T \mathbf{x} \\ & \text{subject to} && \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & && \mathbf{x} \geq \mathbf{0}^n, \end{aligned}$$

$\mathbf{A} \in \mathbb{R}^{m \times n}$  with  $\text{rank } \mathbf{A} = m$  (otherwise, delete rows),  
 $n > m$ , and  $\mathbf{b} \in \mathbb{R}_+^m$

- A point  $\tilde{\mathbf{x}}$  is a *basic solution* if
  1.  $\mathbf{A}\tilde{\mathbf{x}} = \mathbf{b}$ ; and
  2. the columns of  $\mathbf{A}$  corresponding to the non-zero components of  $\tilde{\mathbf{x}}$  are linearly independent

- A basic solution that satisfies non-negativity is called a *basic feasible solution* (BFS)
- Additional terms: degenerate, non-degenerate basic solutions
- Connection BFS–extreme points?
- A point  $\mathbf{x}$  is an *extreme point* of the set  $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}; \mathbf{x} \geq \mathbf{0}^n\}$  if and only if it is a *basic feasible solution*
- Proof by the fact that the rank of  $\mathbf{A}$  is full + Thm. 3.17 (algebraic char. of extreme points)

### The Representation Theorem revisited

Let  $P = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}; \mathbf{x} \geq \mathbf{0}^n \}$  and  $V = \{ \mathbf{v}^1, \dots, \mathbf{v}^k \}$  its set of extreme points. If and only if  $P$  is nonempty,  $V$  is nonempty (finite). Let  $C = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0}^m; \mathbf{x} \geq \mathbf{0}^n \}$  and  $D = \{ \mathbf{d}^1, \dots, \mathbf{d}^r \}$  be the set of extreme directions of  $C$ . If and only if  $P$  is unbounded  $D$  is nonempty (finite). Every  $\mathbf{x} \in P$  is the sum of a convex combination of points in  $V$  and a non-negative linear combination of points in  $D$ :

$$\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{v}^i + \sum_{j=1}^r \beta_j \mathbf{d}^j,$$

where  $\alpha_1, \dots, \alpha_k \geq 0$ :  $\sum_{i=1}^k \alpha_i = 1$ , and  $\beta_1, \dots, \beta_r \geq 0$

### Existence of optimal solutions to LP

- Let the sets  $P$ ,  $V$  and  $D$  be defined as in the above theorem and consider the LP

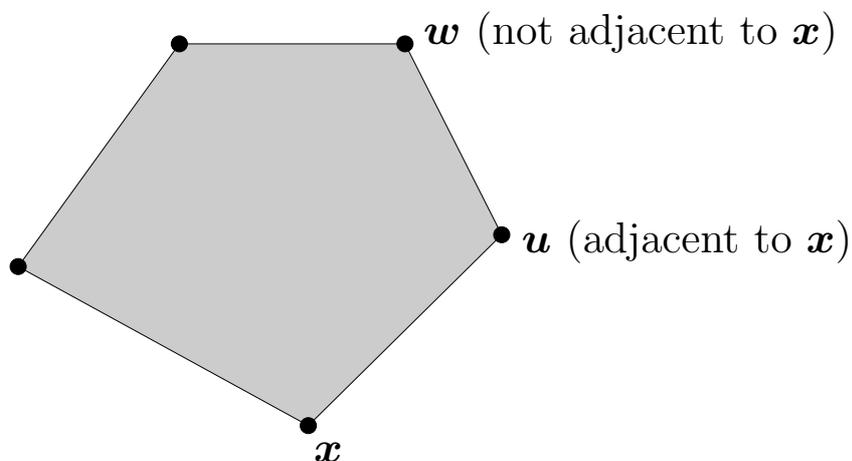
$$\begin{aligned} & \text{minimize} && z = \mathbf{c}^T \mathbf{x} \\ & \text{subject to} && \mathbf{x} \in P \end{aligned}$$

This problem has a finite optimal solution if and only if  $P$  is nonempty and  $z$  is lower bounded on  $P$ , that is, if  $\mathbf{c}^T \mathbf{d}^j \geq 0$  for all  $\mathbf{d}^j \in D$ . If the problem has a finite optimal solution, then there exists an optimal solution among the extreme points

- Proof.

## Adjacent extreme points

- Consider the following polytope:



- No point on the line segment joining  $x$  and  $u$  can be written as a convex combination of any pair of points that are not on this line segment. However, this is not true for the points on the line segment between the extreme points  $x$  and  $w$ . The extreme points  $x$  and  $u$  are said to be *adjacent* (while  $x$  and  $w$  are not adjacent)
- Two extreme points are adjacent if and only if there exist corresponding BFSs whose sets of basic variables differ in exactly one place*

